A sharp lower bound for the distribution of a max-stable process in terms of its extremal coefficients

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(joint work with Martin Schlather)

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Preliminaries
from
Extreme Value Theory

Univariate limits

- for sums of i.i.d. random variables $\{X_n\}_{n=1}^{\infty}$ (with finite second moments)

$$\frac{\sum_{k=1}^{n} X_k - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as } n \to \infty$$

- for maxima of i.i.d. random variables $\{X_n\}_{n=1}^{\infty}$ (if the limit exists and is non-degenerate)

$$\frac{\max_{k=1}^{n} X_k - b_n}{a_n} \xrightarrow{\mathcal{D}} G_\gamma(x) = \exp(-(1+\gamma x)_+^{-1/\gamma})$$

$\gamma < 0$ Weibull $\quad$ $\gamma = 0$ Gumbel $\quad$ $\gamma > 0$ Fréchet

Multivariate limits

- for sums of i.i.d. random vectors $\{X^{(n)}\}_{n=1}^{\infty}$ in $\mathbb{R}^m$ (with finite second moments)

$$\frac{\sum_{k=1}^{n} X^{(k)} - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, (\rho_{ij})_{i,j=1}^{m}\right) \qquad \text{as } n \to \infty$$

- for maxima of i.i.d. random vectors $\{X^{(n)}\}_{n=1}^{\infty}$ in $\mathbb{R}^m$ (if the limit exists and has std. Fréchet margins)

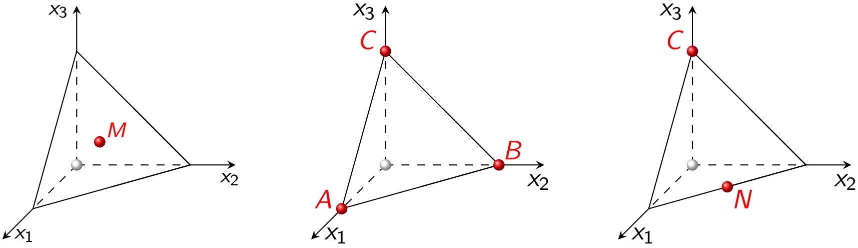
$$\frac{\max_{k=1}^{n} X_k - b_n}{a_n} \xrightarrow{\mathcal{D}} G_H(x) = \exp\left[-\int_{S_+} \max_{i=1,\ldots,m}\left(\frac{a_i}{x_i}\right) H(\mathrm{d}a)\right]$$

for some positive Radon measure H on $S_+ = \{a \in \mathbb{R}_+^m : \|a\| = 1\}$ (= the **spectral measure**)

Example: Spectral measures for $m = 3$

$$X = (X_1, X_2, X_3) \sim G_H(x) = \exp\left[-\int_{S_+} \max_{i=1,\ldots,m}\left(\frac{a_i}{x_i}\right) H(\mathrm{d}a)\right]$$

$X_1 = X_2 = X_3$	X_1, X_2, X_3 independent	$X_1 = X_2$, but X_3 independent
$H = 3\delta_M$	$H = \delta_A + \delta_B + \delta_C$	$H = 2\delta_N + \delta_C$



Processes

Centered Gaussian processes $\{X_t\}_{t \in T}$

$$\forall\, t_1, \ldots, t_m \in T \qquad (X_{t_1}, \ldots, X_{t_m}) \sim \mathcal{N}(0, (C(t_i, t_j))_{i,j=1}^m)$$

- can arise as distributional limits of normalized sums of stochastic processes

Simple max-stable processes $\{X_t\}_{t \in T}$

$$\forall\, t_1, \ldots, t_m \in T \qquad (X_{t_1}, \ldots, X_{t_m}) \sim G_{H_{(t_1, \ldots, t_m)}}$$

- arise as distributional limits of normalized maxima of stochastic processes

Extremal coefficients
and
Tawn-Molchanov processes

Summary statistics in extreme value analysis

Problem: Correlation function and higher moments

- do not always exist.
- are not appropriate for an extreme value context.

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Appropriate summary statististics:

- extreme value index
- extremal index (time series)
- extremal coefficient function (ECF)
- extremogram
- tail correlation function
- Pickands dependence function
- mean excess function
- ...

Extremal coefficient function (ECF)

The **extremal coefficient function (ECF)** $\theta : \mathcal{F}(T) \to \mathbb{R}$ of a simple max-stable process $\{X_t\}_{t \in T}$ is given by

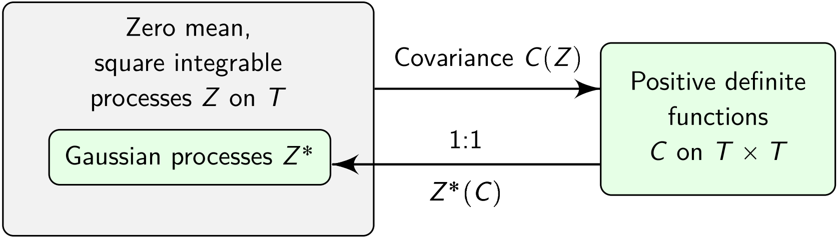
$$\mathbb{P}\left(\max_{t \in M} X_t \leqslant x\right) = \mathbb{P}(X_t \leqslant x)^{\theta(M)}$$

- arguments M = finite subsets of T
- $\mathcal{F}(T)$ = set of finite subsets of T

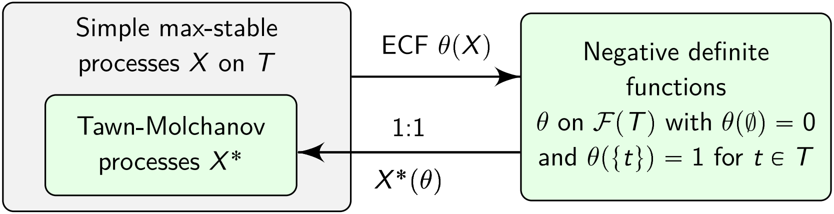
Interpretation

$\theta(M)$ = effective number of independent variables among $\{X_t\}_{t \in M}$

Motivation/Analogy:



Situation for ECFs



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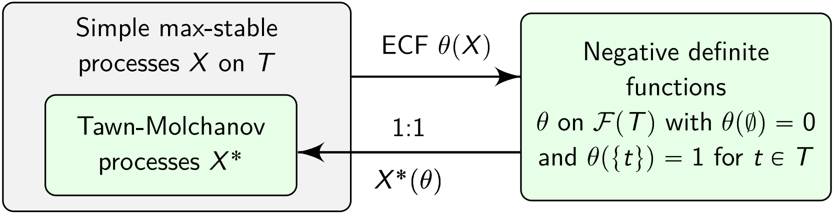


Definition

A function $\psi : \mathcal{F}(T) \to \mathbb{R}$ is **negative definite**, if for all $n \geqslant 1$, $\{K_1, \ldots, K_n\} \subset \mathcal{F}(T)$ and $\{a_1, \ldots, a_n\} \subset \mathbb{R}$ with $\sum_{j=1}^n a_j = 0$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \psi(K_j \cup K_k) \leqslant 0.$$

Tawn-Molchanov processes

Theorem (S. and Schlather '13)

a) Let $\theta : \mathcal{F}(T) \to \mathbb{R}$. Then

$$\theta \text{ is an ECF} \iff \begin{cases} \theta \text{ is negative definite,} \\ \theta(\emptyset) = 0 \text{ and } \theta(\{t\}) = 1 \text{ for } t \in T. \end{cases}$$

b) If θ is an ECF, then there exists a simple max-stable process $X^* = \{X_t^*\}_{t \in T}$ on T with ECF θ. X^* has f.d.d.

$$-\log \mathbb{P}(X_t^* \leqslant x_t,\ t \in M) = \sum_{\emptyset \neq L \subset M} \tau_L^M(\theta) \max_{t \in L} \frac{1}{x_t},$$

where

$$\tau_L^M(\theta) = \sum_{I \subset L} (-1)^{|I|+1} \theta((M \setminus L) \cup I).$$

$X^* = \{X_t^*\}_{t \in T}$ = **Tawn-Molchanov process** associated to the ECF θ

[Coles and Tawn '96, Schlather and Tawn '02, Molchanov '08]

Continuity

The Tawn-Molchanov process $X^* = \{X^*_t\}_{t \in T}$ with ECF θ satisfies

$$\mathbb{P}(|X^*_s - X^*_t| > \varepsilon) \leqslant 2\left[1 - \exp\left(-\frac{\theta(\{s,t\}) - 1}{\varepsilon}\right)\right] \leqslant \frac{2}{\varepsilon}\left[\theta(\{s,t\}) - 1\right].$$

Theorem (S. and Schlather '13)

Let $X^* = \{X^*_t\}_{t \in T}$ be a Tawn-Molchanov process on a metric space T. Then the following statements are equivalent:

(i) X^* is stochastically continuous.

(ii) θ is continuous.

(iii) The bivariate map $(s,t) \mapsto \theta(\{s,t\})$ is continuous.

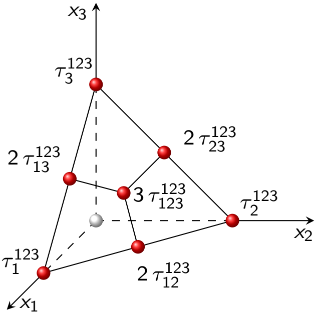
(iv) The bivariate map $(s,t) \mapsto \theta(\{s,t\})$ is continuous on the diagonal.

Spectral measure of the f.d.d.

Reference sphere

$S_+ = \{a \in \mathbb{R}^m_+ : \|a\|_1 = 1\}$

$$H = \sum_{\emptyset \neq L \subset M} |L| \tau_L^M(\theta)\, \delta_{e_L/|L|}$$



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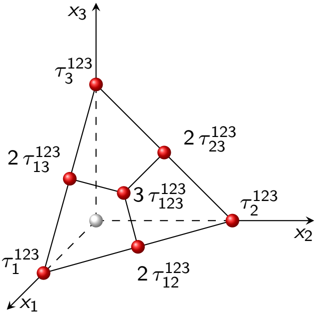
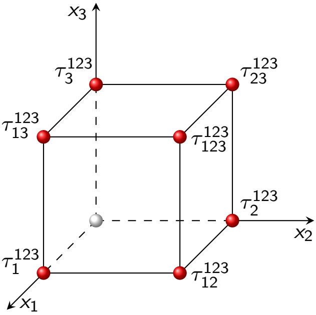




Spectral representation of the Tawn-Molchanov process

Theorem (S. and Schlather '13)

Any Tawn-Molchanov process $X^* = \{X_t^*\}_{t \in T}$ has a spectral representation $(\Omega, \mathcal{A}, \nu, V)$, where

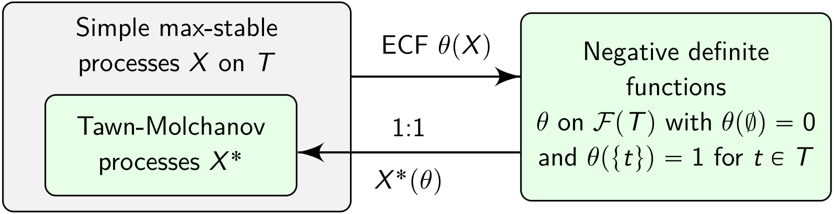
- $(\Omega, \mathcal{A}, \nu)$ is a Radon measure on the Cantor cube

  $$\Omega = \{0,1\}^T \backslash \{\mathbf{1}_\emptyset\}$$

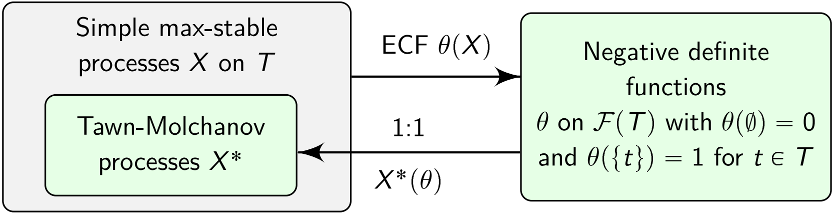
  and its Borel-σ-algebra $\mathcal{A}$.
- $V_t(\omega) = \omega(t)$.

A sharp lower bound

A sharp inequality



A sharp inequality



Theorem (S. and Schlather '13)

Among all simple max-stable processes X with fixed ECF θ the Tawn-Molchanov processe X^* gives a sharp lower bound for the f.d.d.

$$\mathbb{P}(X_{t_1} \leqslant x_1, \dots, X_{t_m} \leqslant x_m) \geqslant \underbrace{\mathbb{P}(X^*_{t_1} \leqslant x_1, \dots, X^*_{t_m} \leqslant x_m)}_{\text{depending only on } x \in \mathbb{R}^m_+ \text{ and } \theta}$$

A sharp inequality

Example:

The following inequality holds trivially for (X_s, X_t) simple max-stable:

$$\begin{aligned}\mathbb{P}(X_s \leqslant x,\ X_t \leqslant y) &\geqslant \mathbb{P}(\max(X_s, X_t) \leqslant \min(x, y)) \\ &= \exp\left(-\frac{\theta(\{s,t\})}{\min(x,y)}\right).\end{aligned}$$

A sharp inequality

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The following inequality holds trivially for (X_s, X_t) simple max-stable:

$$\mathbb{P}(X_s \leqslant x,\, X_t \leqslant y) \geqslant \mathbb{P}(\max(X_s, X_t) \leqslant \min(x, y)) \\ = \exp\left(-\frac{\theta(\{s,t\})}{\min(x,y)}\right).$$

The sharp inequality gives an additional factor:

$$\mathbb{P}(X_s \leqslant x,\, X_t \leqslant y) \geqslant \exp\left(-\frac{\theta(\{s,t\})}{\min(x,y)}\right) \exp\left((\theta(\{s,t\}) - 1)\left|\frac{1}{x} - \frac{1}{y}\right|\right).$$

A sharp inequality: Geometric idea of proof

[Molchanov '08]:

- For each simple max-stable distribution function G_H, the function

$$\ell(x) = -\log G_H(1/x) = \int_{S^+} \max_{i=1,\dots,m} x_i \, H(da), \qquad x \in [0,\infty)^m$$

is sublinear and homogeneous.

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- Therefore, the function ℓ may be expressed as support function of a compact convex set $\mathcal{K} \subset [0,\infty)^m$

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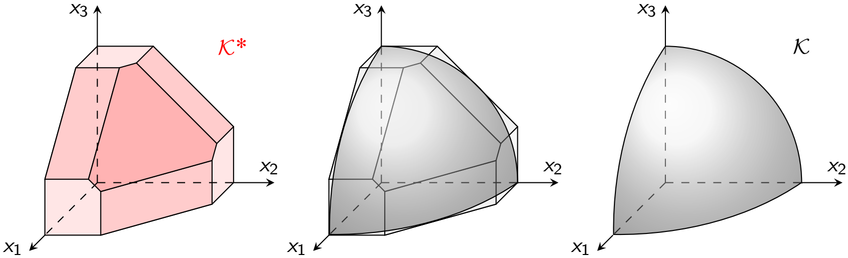
$$\ell(x) = \sup\{\langle x, y\rangle : \, y \in \mathcal{K}\} \qquad x \in [0,\infty)^m.$$

- $\mathcal{K}$ is called **dependency set** $\mathcal{K}$ of G_H.

A sharp inequality: Geometric idea of proof

The *dependency set* $\mathcal{K}^*$ of the Tawn-Molchanov process X^* is *maximal* w.r.t. inclusion when the ECF is fixed.

$$\mathcal{K}^* = \bigcup_{\substack{\mathcal{K} \text{ dependency set} \\ \text{with the same ECF as } \mathcal{K}^*}} \mathcal{K}.$$



Summary

Max-stable processes

- arise as distributional limits of suitably normalized maxima of stochastic processes

Subclass: Tawn-Molchanov processes

- are in a 1:1 correspondence with extremal coefficient functions
- yield a sharp lower bound for the f.d.d. of max-stable processes

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Thank you!