Functional Analysis

WS 2015/2016
Prof. Dr. M. Disertori
P. Gladbach; R. Schubert

universität bonn

Problem Sheet 11.

Due 29.1.2016.

Problem 1. (Dual space of L^∞) (7+3 Points)

a) Let X be a normed space. Prove that if X' is separable then so is X.
Hint: Take a countable dense subset $\{T_n\}$*of* X'. *Observe that for every* T_n *there is* $x_n \in X$ *with* $\|x_n\| \leq 1$ *and* $|T_n(x_n)| \geq \frac{1}{2}\|T_n\|$. *Set* $Y \coloneqq span\{x_n\}$ *and prove by contradiction that* $\bar{Y} = X$. *Since* $L^1(U)$ *is separable and* $L^\infty(U)$ *is not, this implies that* $L^\infty(U)' \neq L^1(U)$.

b) Prove that a Banach space X is reflexive if and only if its dual X' is reflexive.
Remember that a Banach space X *is reflexive if the natural embedding* $J_X : X \to X''$ *into its double dual is surjective.*

Problem 2. (Positive functionals) (10 Points)

Let $U \subset \mathbb{R}^n$ and let X be a closed subspace of $B(U, \mathbb{R})$ with the property that $f \in X$ implies $|f| \in X$. Let T be a linear Funktional on X and for all $f \in X$

$$|T(f)| \leq \|f\| = \sup_{x\in U} |f(x)| \text{ and } f \geq 0 \implies T(f) \geq 0.$$

Show that there is a linear functional $\bar{T}$ on $B(U, \mathbb{R})$ such that $\bar{T} = T$ on X and such that for all $f \in B(U, R)$

$$\left|\bar{T}(f)\right| \leq \|f\| = \sup_{x\in U} |f(x)| \text{ and } f \geq 0 \implies \bar{T}(f) \geq 0.$$

Hint: Consider $f_+ = \max(f, 0)$ *and introduce an appropriate sublinear function* p.

Problem 3. (Convex separation) (5+5+5+5 Points)

Let X be a real Banach space. Let $U \subset X$ be open, convex, with $0 \in U$.

a) Define $p(x) = \inf\{t > 0 \,:\, x/t \in U\}$. Show that $p : X \to \mathbb{R}$ is subadditive, with $x \in U$ if and only if $p(x) < 1$.

b) Let $x \in X \setminus U$. Show that there is $F \in X'$ with $F(x) \geq 1$ and $F < 1$ in U. Draw a picture.

c) Let $A, B \subset X$ be two nonempty convex disjoint subsets, A open. Let $x_0 \in A$, $y_0 \in B$, and define $V = (A - x_0) - (B - y_0) = \{(x - x_0) - (y - y_0) \,:\, x \in A, y \in B\}$. Show that V is open, convex, with $0 \in V$, $y_0 - x_0 \in X \setminus V$.

d) Show that there exist $F \in X'$ and $\lambda \in \mathbb{R}$ with $F < \lambda$ in A and $F \geq \lambda$ in B. Draw a picture. Can we pick $\lambda = 1$?