



Stochastic Processes Sheet 1

For discussion in the tutorials in the 3rd week of the lecture period

General information

- Important announcements and course material will be provided via **ecampus**
 - Please sign in for exactly ONE of the four exercise class on **ecampus**. The registration will on Thursday, 15th October, at 10 am.
 - The lectures and the exercise classes will start in the second week of the lecture period.
 - Beginning with sheet 2, you have to hand in your solutions to the problems, in groups of at most three people and electronically via ecampus. More about this in your first tutorial class.
 - If you have any questions left regarding the organisation of the lecture, please write an email to cmanai@uni-bonn.de.
-

Exercise 1

Let $\mathbb{P}$ be a probability measure on Ω , endowed with a σ -field $\mathcal{A}$.

- (i) What is the meaning of the following events, where all $A_n \in \mathcal{A}$?

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \rightarrow \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.$$

- (ii) Prove that $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$ are in $\mathcal{A}$.

- (iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

$$A_{2p} = \left[-1, 2 + \frac{1}{2p} \right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1 \right].$$

What are $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$?

(iv) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n), \quad \mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n).$$

Exercise 2

Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\mathbb{R}; \mathcal{B}(\mathbb{R}))$ that agree on all intervals of the form $(x, y]$, where $-\infty \leq x \leq y < \infty$. Prove that the two probability measures are equal.

Exercise 3

Prove the Bonferroni (inclusion–exclusion) inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i)

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i),$$

(ii)

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j),$$

(iii)

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k).$$

Exercise 4

Let $(\Omega, \mathcal{F}, \mu)$ be a σ -finite measure space, λ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ and $f : \Omega \rightarrow \mathbb{R}_+$ be a non-negative measurable function. Use Fubini's theorem to prove that

$$\int f d\mu = \int_0^\infty \mu(f \geq t) \lambda(dt).$$

Conclude that for all $p \geq 1$:

$$\int f^p d\mu = \int_0^\infty p t^{p-1} \mu(f \geq t) \lambda(dt).$$