

Probability Theory II (Spring 2026)

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Primary reference: Varadhan, Stochastic Processes

Prerequisites: Varadhan, Probability Theory

Other references:

Norris, Advanced Probability

Stroock, Probability Theory, An Analytic View

Le Gall, Brownian Motion, Martingales, and Stochastic Calculus

1. Continuous-time processes & martingales

From now on, $(\Omega, \mathcal{F}, P)$ denotes a probability space on which all random variables are defined.

1.1. Continuous-time processes

Defn. Given an index set Π , a stochastic process $X = \{X_t : t \in \Pi\}$ is a collection of random variables $X_t = X(t, \omega)$, $t \in \Pi$, $\omega \in \Omega$, taking values in some measurable space.

- If Π is a countable set, X is a discrete-time stochastic process.
- If $\Pi \subseteq \mathbb{R}$ is a finite or infinite interval, X is a continuous-time stochastic process.

Careful: For Π uncountable, events like

$$\{\omega : X_t(\omega) \in [a, b] \ \forall t \in \Pi\}$$

or $\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\}$

are not measurable in general.

Soln: Build regularity into Ω .

Defn. $C[a,b]$ is the Banach space of continuous functions $f: [a,b] \rightarrow \mathbb{R}$ with norm

$$\|f\| = \sup_{t \in [a,b]} |f(t)|.$$

Defn. $D[a,b]$ is the Polish space (complete sep metric) of left-continuous functions with right limits (càdlàg functions) $f: [a,b] \rightarrow \mathbb{R}$ with metric

$$d(f,g) = \inf \left\{ r > 0 : \exists \lambda: [a,b] \rightarrow [a,b] \text{ increasing, invertible, and continuous s.t. } \sup_t |\lambda(t) - t| \leq r, \sup_t |f(t) - g(\lambda(t))| \leq r \right\}.$$

Prop. $D[a,b]$ is indeed a Polish space.

[Billingsley, Convergence of Prob. Measures, Chapter 3].

Prop. The σ -fields of Borel sets of $C[a,b]$ or $D[a,b]$ are generated by the cylinder sets, i.e., by

$$\{x \in X : \pi_t x = x(t) \in I\}, \quad I \in \mathcal{B}(\mathbb{R}), \quad t \in [a,b]$$

Sketch. Let $\mathcal{F}$ denote the cylinder σ -field. Since $\{x \in X : x(t) \in I\}$ is a Borel set, $\mathcal{F} \subset \mathcal{B}$. Conversely,

$$\{x \in C : \|x\| \leq r\} = \bigcap_{t \in [a,b] \cap \mathbb{Q}} \pi_t^{-1}([-r, r]) \in \mathcal{F}.$$

Exercise: $X = D$

We will always use the cylinder σ -field (= Borel σ -field) on $C[a,b]$ or $D[a,b]$. The space $C[0,\infty)$ is defined by $x \in C[0,\infty)$ if $x|_{[0,T]} \in C[0,T]$ and likewise for $D[0,\infty)$ with unif. on compacts metric.

Defn. Let $(\Omega, \mathcal{F})$ be a measurable space. Then a family $\{\mathcal{F}_t : t \geq 0\}$ of sub- σ -fields $\mathcal{F}_t \subset \mathcal{F}$ is

- a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$.
- a right-cont. filtration if $\mathcal{F}_t \supset \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$

Defn. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ be a filtered measurable space. Then a stochastic process (X_t) is

- adapted if $\forall t: X_t \in \mathcal{F}_t$, i.e., X_t is $\mathcal{F}_t$ -meas.
- integrable if $\forall t: X_t \in L^1$, i.e., $E|X_t| < \infty$.

Example. If Ω is a space of functions on $\Pi C[0,\infty)$,
 $\mathcal{F}_t = \sigma(\omega(s) : s \leq t)$

is the canonical filtration. The canonical process
 $X_t(\omega) = \omega(t)$, $\omega \in \Omega$ is $(\mathcal{F}_t)$ -adapted.

If Ω is a space of right-cont. functions, it is also adapted to $(\mathcal{F}_{t+})$.

Prop. Let (X_t) be a $D[0, \infty)$ -valued adapted process. Then (X_t) is progressively measurable: $\forall T > 0$

$$(\omega, t) \mapsto X_t(\omega)$$

on $\Omega \times [0, T]$ is measurable w.r.t. $\mathcal{F}_T \otimes \mathcal{B}([0, T])$.

Sketch. By right-cont. can approximate by

$$X_t^n(\omega) = X_{2^{-n} \lceil 2^n t \rceil}(\omega)$$

For every $A \in \mathcal{B}(\mathbb{R})$,

$$\{(\omega, t) \in \Omega \times [0, T] : X_t^n(\omega) \in A\}$$

$$= \bigcup_{\substack{k=1 \\ 2^{-n}k < T}}^{\infty} \underbrace{\{X_{2^{-n}k} \in A\}}_{\in \mathcal{F}_T} \times \underbrace{[2^{-n}(k-1), 2^{-n}k)}_{\in \mathcal{B}([0, T])}.$$

Thus X^n is $\mathcal{F}_T \otimes \mathcal{B}([0, T])$ -measurable. Take limits.

Sometimes it is also convenient to assume that the $\mathcal{F}_t$ contain all null sets, i.e. is complete.

Defn. A filtration satisfies the usual conditions if it is right-continuous and complete (w.r.t. a given probability measure).

1.2. Functions of bounded variation

Defn. The total variation of $x: [a, b] \rightarrow \mathbb{R}$ is

$$V_x[a, b] = \sup \left\{ \sum_{i=1}^n |x(t_i) - x(t_{i-1})| : \underbrace{a \leq t_0 < \dots < t_n \leq b}_{\text{partition of } [a, b]} \right\}.$$

The space $BV[a, b]$ of functions of bounded variation consists of $x: [a, b] \rightarrow \mathbb{R}$ with $V_x[a, b] < \infty$.

Recall: A signed measure μ on $[a, b]$ is the difference $\mu = \mu_+ - \mu_-$ (Hahn-Jordan decomposition) of two mutually singular finite positive measures μ_+ and μ_- . The total variation measure of μ is

$$|\mu| = \mu_+ + \mu_-$$

Hahn-Jordan thm: Given any finite positive measures μ_1 and μ_2 there are μ_+ and μ_- s.t. $\mu = \mu_1 - \mu_2 = \mu_+ - \mu_-$.

Prop. (i) Given a signed measure μ on $[a, b]$, its CDF $x(t) = \mu([a, t])$ is in $BV[a, b]$, càdlàg, and

$$V_x[a, b] = |\mu|([a, b]).$$

(ii) Given $x \in BV[a, b]$ that is right-continuous, there is μ s.t. $x(t) = \mu([a, t])$.

Proof. (i) Let $\mu = \mu_+ - \mu_-$ be a signed measure and x its CDF. Let $a \leq t_0 < t_1 < \dots < t_n \leq b$. Then

$$\sum_{i=1}^n |x(t_i) - x(t_{i-1})| = \sum_{i=1}^n |\mu(t_{i-1}, t_i]| \leq |\mu|(a, b]$$

$$\Rightarrow V_x[a, b] \leq |\mu|(a, b].$$

For the other direction, let $(t_i^{(m)})_{i=0}^{n_m}$ be a sequence of nested partitions with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \xrightarrow{m \rightarrow \infty} 0$. It suffices to show that

$$|\mu|(a, b] = \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} |x(t_i^{(m)}) - x(t_{i-1}^{(m)})|.$$

To see this, define the probability on $(a, b]$ by

$$\nu(dt) = \frac{|\mu|(dt)}{|\mu|(a, b]}.$$

Let $\mathcal{F}_m = \sigma((t_{i-1}^{(m)}, t_i^{(m)}] : 1 \leq i \leq n_m) \subset \mathcal{F}_{m+1}$

$$X = \frac{d\mu}{d|\mu|} = 1_{\text{supp } \mu_+} - 1_{\text{supp } \mu_-}$$

$$X_m = E[X | \mathcal{F}_m].$$

For $s \in (t_{i-1}^{(m)}, t_i^{(m)}]$ then

$$X_m(s) = \frac{\mu((t_{i-1}^{(m)}, t_i^{(m)}])}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])} = \frac{x(t_i^{(m)}) - x(t_{i-1}^{(m)})}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])}$$

$$\Rightarrow E|X_m| = \frac{1}{\mu([a,b])} \sum_{i=1}^{n_m} |x(t_i^{(m)}) - x(t_{i-1}^{(m)})|$$

and the claim is $E|X_m| \rightarrow 1$. But (X_m) is a bd martingale, so there is Y s.t. $X_m \rightarrow Y$ in L^1

$$\Rightarrow E|X_m| \rightarrow E|Y|.$$

Since $\sigma(\cup \mathcal{F}_n) = \mathcal{B}([a,b])$ in fact $X=Y$ a.s., so

$$E|Y| = E|X| = 1.$$

(ii) Now let $x \in BV[a,b]$. Assume $x(a)=0$ and set

$$x_{\pm}(t) = \frac{1}{2}(V_x[a,t] \pm x(t)).$$

Claim: $x_{\pm}$ are increasing, i.e. $x_{\pm}(s) \geq x_{\pm}(t)$ if $s > t$.

Let $a \leq t_0 < \dots < t_n \leq t$ be a partition of $[0,t]$,
s.t. $a \leq t_0 < \dots < t_n \leq t \leq s$ is a partition of $[0,s]$.

$$\Rightarrow 2x_{\pm}(s) = V_x[a,s] \pm x(s)$$

$$\geq \underbrace{\sum |x(t_i) - x(t_{i-1})|}_{\geq V_x[a,t] - \varepsilon} + \underbrace{|x(s) - x(t)|}_{\geq \pm x(t)} \pm x(s)$$

$$\Rightarrow x_{\pm}(s) \geq x_{\pm}(t).$$

Claim: If x is right-continuous, so are $x_{\pm}$.

It suffices to show $v(t) = V_x[a, t]$ is right-continuous.
Let $\beta = v(t+) - v(t)$. Need to show $\beta = 0$. Let $h > 0$ s.t.

$$v(t+h) - v(t) < \beta + \varepsilon$$
$$|x(s) - x(t)| < \varepsilon \quad \text{for } s \in [t, t+h].$$

There is a partition of $[t, t+h]$ s.t.

$$\sum_{i=1}^n |x(t_i) - x(t_{i-1})| \geq \frac{3}{4} V_x[t, t+h] \geq \frac{3}{4} \beta$$
$$\Rightarrow \sum_{i=2}^n |x(t_i) - x(t_{i-1})| \geq \frac{3}{4} \beta - |x(t_1) - x(t_0)| \geq \frac{3}{4} \beta - \varepsilon$$

Since $\beta \leq v(t_1) - v(t_0)$ there is a part of $[t_0, t_1]$ s.t.

$$\sum |x(t'_i) - x(t'_{i-1})| \geq \frac{3}{4} \beta$$

Thus there is a partition of $[t, t+h]$ s.t.

$$V_x[t, t+h] \geq \frac{3}{2} \beta - \varepsilon$$

On the other hand,

$$V_x[t, t+h] \leq \beta + \varepsilon$$

$$\Rightarrow \beta \leq 4\varepsilon \quad \forall \varepsilon > 0 \Rightarrow \beta = 0.$$

Defn. Let $x \in BV[a, b]$ be right-continuous with associated signed measure μ . For $f \in L^1(|\mu|)$ the Lebesgue-Stieltjes integral is defined by

$$\int_s^t f(u) dx(u) = \int_{(s, t]} f(u) \mu(du) \quad (a \leq s < t \leq b)$$

$$\int_s^t f(u) |dx(u)| = \int_{(s, t]} f(u) |\mu|(du)$$

and set $f \circ x(t) = \int_0^t f(u) dx(u)$.

Exercise. Let $x \in BV[a, b]$ be right-continuous. Then

$$\left| \int_a^b f(t) dx(t) \right| \leq \int_a^b |f(t)| |dx(t)|$$

and $f \circ x$ is in BV and right-cont. with signed measure $f(t) dx(t)$ and $TV \quad |f(t)| |dx(t)|$.

Prop. Let $x \in BV$ be right-cont. and f bounded and left-cont. Then for any sequence of partitions $(t_i^{(m)})_{i=1}^{n_m}$ of $[a, b]$ with step size $\xrightarrow{m \rightarrow \infty} 0$:

$$\int_a^b f(t) dx(t) = \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)}) (x(t_i) - x(t_{i-1}))$$

$$\int_a^b f(t) |dx(t)| = \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)}) |x(t_i) - x(t_{i-1})|.$$

Proof. Let $f_m(a)=0$, $f_m(t)=f(t_{i-1}^{(m)})$ if $t \in (t_{i-1}^{(m)}, t_i^{(m)}]$.

$\Rightarrow f(t) = \lim_{m \rightarrow \infty} f_m(t)$ by left-continuity.

$$\Rightarrow \sum_{i=1}^{n_m} f(t_{i-1}^{(m)}) (x(t_i^{(m)}) - x(t_{i-1}^{(m)})) = \int_{(a,b]} f_m(t) dx(t)$$

$$\xrightarrow{\text{DCT}} \int f(t) dx(t).$$

The second claim is similar.

Exercise. Let $x, y \in BV[a, b]$ be right-cont. Then

$$x(b)y(b) - x(a)y(a) = \int_a^b x(t) dy(t) + \int_a^b y(t-) dx(t)$$

1.3. Continuous-time martingales

Defn. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a martingale is a family of random variables $M_t, t \in \mathbb{T}$ such that

(i) $(t \mapsto M_t) \in D(\mathbb{T})$, P -a.s. [if $\mathbb{T}$ is continuous]

(ii) $\forall t \geq 0: M_t \in \mathcal{F}_t$ and $E|M_t| < \infty$.

(iii) $\forall 0 \leq s \leq t: E[M_t | \mathcal{F}_s] = M_s$, P -a.s.

If $(t_k) \subset \mathbb{T}$ then $(M(t_k))_k$ is a discrete-time martingale w.r.t. $(\mathcal{F}_{t_k})$. By taking $t_k = 2^{-n}k$ and using right-continuity, most properties extend from discrete to continuous.

Doob's inequality. If (M_t) is a martingale and

$$E_\ell = \left\{ \omega: \sup_{0 \leq s \leq t} |M_s(\omega)| \geq \ell \right\}$$

then

$$P[E_\ell] \leq \frac{1}{\ell} E[1_{E_\ell} |M_t|] \leq \frac{1}{\ell} E[|M_t|]$$

$$P[E_\ell] \leq \frac{1}{\ell^2} E[1_{E_\ell} |M_t|^2] \leq \frac{1}{\ell^2} E[|M_t|^2].$$

Proof. The discrete-time versions of Doob's inequalities imply the statement with E_e replaced by

$$E_e^n = \left\{ \sup_{\substack{0 \leq s \leq t \\ s \in 2^{-n}\mathbb{N}}} |M_s| \geq e \right\}.$$

By right-continuity, $P(E_e^n) \uparrow P(E_e)$.

Defn. (X_t) is a submartingale if (i) & (ii) hold,

(iii') $\forall 0 \leq s \leq t: E[X_t | \mathcal{F}_s] \geq X_s$, P -a.s.

and a supermartingale if $E[X_t | \mathcal{F}_s] \leq X_s$.

Exercise. Let (X_t) be a martingale, $\phi: \mathbb{R} \rightarrow \mathbb{R}$ convex, and assume $E[|\phi(X_T)|] < \infty$. Then $\phi(X_t)$ is integrable for any $t \in T$ and a submartingale.

Defn. A stopping time is a random variable τ with values in $[0, \infty]$ s.t.

$$\forall t \geq 0: E_t = \{\tau \leq t\} \in \mathcal{F}_t.$$

Define $\mathcal{F}_\tau = \{A \in \mathcal{F}: A \cap E_t \in \mathcal{F}_t\}$.

Exercise. • If τ_1, τ_2 are stopping times, so are $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$.

- If $\tau_1 \leq \tau_2$ are stopping times, $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.
- If f is a measurable function s.t. $f(t) \geq t$ for all t then $f(\tau)$ is a stopping time if τ is one. Thus $\tau_n = 2^{-n} \lceil 2^n \tau \rceil$ is a stopping time s.t. $\tau_n \geq \tau$ and $\tau_n \rightarrow \tau$.

Exercise: Let X be a random variable in L^1 i.e. $E|X| < \infty$. Then the set of random variables $E[X | \mathcal{G}]$ where $\mathcal{G} \subset \mathcal{F}$ ranges over sub- σ -fields is uniformly integrable: $\forall \varepsilon > 0 \exists \lambda > 0$ s.t. $\forall \mathcal{G}$:

$$E[|E[X | \mathcal{G}]| \mathbf{1}_{|E[X | \mathcal{G}]| > \lambda}] \leq \varepsilon.$$

Vitali's convergence theorem: Equivalent:

- $X_j \in L^1$ and $X_j \rightarrow X$ in L^1
- $X_j \uparrow I$ and $X_j \rightarrow X$ in probability.

Thm. (OST). Let (M_t) be a martingale and $\tau_1 \leq \tau_2$ be two bounded stopping times. Then $M_{\tau_i} \in L^1$ and

$$E[M_{\tau_2} | \mathcal{F}_{\tau_1}] = M_{\tau_1} \quad \text{a.s.}$$

Proof. Assume $\tau_1, \tau_2 \leq N$. Let $\tau_i^n = 2^{-n} \lceil 2^n \tau_i \rceil \leq N+1$ where N is a deterministic constant. The discrete-time OST implies that

$$M_{\tau_2^n} = E[M_{N+1} | \mathcal{F}_{\tau_2^n}]$$

In particular, $M_{\tau_2^n}$ is UI. Since $M_{\tau_2^n} \rightarrow M_{\tau_2}$ a.s. therefore $M_{\tau_2^n} \rightarrow M_{\tau_2}$ in L^1 and likewise for τ_1 . The discrete OST also implies

$$E[M_{\tau_2^n} | \mathcal{F}_{\tau_1^n}] = M_{\tau_1^n}$$

In particular, for any $A \in \mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_1^n}$,

$$E[M_{\tau_2^n} 1_A] = E[M_{\tau_1^n} 1_A]$$

Since $M_{\tau_i^n} \rightarrow M_{\tau_i}$ in L^1 the claim follows.

Cor. For any stopping time, $M_{t \wedge \tau}$ is a martingale.

Exercise. Extend to (M_t) a submartingale.

1.4. Semimartingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space.

Example Let (X_n) be an adapted discrete-time process. Define

$$A_n = \sum_{j=1}^n E[X_j - X_{j-1} | \mathcal{F}_{j-1}], \quad A_0 = 0.$$

Then $Y_n = X_n - A_n$ is a martingale: for $m \leq n$,

$$\begin{aligned} E[Y_n | \mathcal{F}_m] &= E[X_n | \mathcal{F}_m] - \sum_{j=1}^n \underbrace{E[E[X_j - X_{j-1} | \mathcal{F}_{j-1}] | \mathcal{F}_m]}_{= \begin{cases} E[X_j - X_{j-1} | \mathcal{F}_m] & (j > m) \\ E[X_j - X_{j-1} | \mathcal{F}_{j-1}] & (j \leq m) \end{cases}} \\ &= X_m - \sum_{j=1}^m E[X_j - X_{j-1} | \mathcal{F}_{j-1}]. \end{aligned}$$

and $A_n \in \mathcal{F}_{n-1}$. The decomposition $X_n = Y_n + A_n$ with these properties is the Doob or semimartingale decomposition of X_n . The process Y_n is the martingale part of X_n and A_n the compensator.

Exercise: The semimartingale decomposition is unique.

The semimartingale decomposition is more subtle in the continuous case.

Prop. Let (A_t) be a continuous-time martingale. If A is continuous and BV, P-a.s., then A is const.

Proof. Assume $A_0 = 0$ and let A_t^* be the variation of A on $[0, t]$. It suffices to consider $A^*(t) \leq C$.

Otherwise, consider $A_{t \wedge T}$ where $T = \inf\{t: A_t^* \geq C\}$.

Let $t_j = (j/n)T$. Then

$$\begin{aligned} \sum_{j=1}^n |A_{t_j} - A_{t_{j-1}}|^2 &\leq \sup_j |A_{t_j} - A_{t_{j-1}}| \times \sum_j |A_{t_j} - A_{t_{j-1}}| \\ &\leq C \sup_j |A_{t_j} - A_{t_{j-1}}|. \end{aligned}$$

Since $|A_{t_j}| \leq |A_0| + C$ and A is continuous on $[0, T]$, thus uniformly continuous,

$$E\left[\sum_{j=1}^n |A_{t_j} - A_{t_{j-1}}|^2\right] \rightarrow 0.$$

On the other hand, since A is a martingale,

$$E\left[\sum_{j=1}^n |A_{t_j} - A_{t_{j-1}}|^2\right] = E[|A_T - A_0|^2]$$

Thus $E[|A_T - A_0|^2] = 0$, i.e. $A_T = A_0$ a.s.

Prop. Let (M_t) be a bounded martingale and (A_t) be an adapted continuous process of finite variation with $A_t^* \leq C_t$ for every t . Then

$$X_t = M_t A_t - M_0 A_0 - \int_0^t M_s dA_s$$

is a martingale.

Rk. Formally, $X_t = \int A_s dM_s$.

Proof. Since A is continuous,

$$\begin{aligned} \int_0^t M_s dA_s &= \lim_{n \rightarrow \infty} \sum_{j=1}^n M_{t_j} (A_{t_j} - A_{t_{j-1}}) \\ &= M_t A_t - M_0 A_0 + \lim_{n \rightarrow \infty} \sum_{j=1}^n A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) \\ \Rightarrow X_t &= \lim_{n \rightarrow \infty} \sum_{j=1}^n A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) = \lim_{n \rightarrow \infty} X_t^n \end{aligned}$$

Since $A_{t_{j-1}} \in \mathcal{F}_{t_{j-1}}$ and M is a martingale,

$$E[A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}] = 0.$$

and similarly X^n is a martingale.

If M is bd, $X_t^n \rightarrow X_t$ in L^1 . It follows that X is a martingale.

2. Brownian motion

In fact, suffices to have t in dense set.

2.1. Kolmogorov's continuity criterion

Thm. (Kolmogorov's criterion). Assume $(X_t)_{t \in \mathbb{R}^d}$ is a stochastic process s.t. for some $p \geq 1$, $\alpha > 0$,

$$\mathbb{E}[|X_t - X_s|^p] \leq A |t - s|^{d+\alpha}, \quad 0 \leq t, s \leq 1.$$

Then there is a version $\tilde{X}$ of X , i.e. $P(X_t = \tilde{X}_t) = 1 \ \forall t$, that is continuous, in fact γ -Hölder continuous:

$$P\left[\sup_{t \neq s} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\gamma} \geq \lambda\right] \leq C \frac{A}{\lambda^p}$$

provided that $\gamma < \alpha/p$ and with $C = C(\alpha, \gamma, p)$.

The essence is really a real analysis estimate. Formally, the estimate follows taking expectation over the following inequality ("Besov space embedding").

Prop. For $x: \mathbb{R}^d \rightarrow \mathbb{R}^n$ continuous, $p \geq 1$ and $\gamma > 0$,

$$\underbrace{\sup_{\substack{t \neq s \\ t, s \in B}} \frac{|x(t) - x(s)|}{|t - s|^\gamma}}_{[x]_\gamma} \leq C_{\gamma, p} \underbrace{\left(\iint_{B \times B} \frac{|x(t) - x(s)|^p}{|t - s|^{p(\gamma + 2d/p)}} dt ds \right)^{1/p}}_{[x]_{\gamma + 2d/p, p}} \quad (*)$$

where $B = B_1(0) \subset \mathbb{R}^d$.

Garsia-Rodemich-Rumsey inequality. Let p and ψ be strictly increasing continuous functions on $(0, \infty)$ s.t. $p(0) = 0$ and $\psi(t) \rightarrow \infty$ ($t \rightarrow \infty$). Then if $x : [0, T] \rightarrow \mathbb{R}^n$ is continuous and

$$\int_0^T \int_0^T \psi\left(\frac{|x(t) - x(s)|}{p(t-s)}\right) ds dt \leq B < \infty$$

then

$$|x(t) - x(s)| \leq \int_0^{t-s} \psi^{-1}\left(\frac{4B}{u^2}\right) p'(u) du$$

Proof of (f_k) when $d=1$. Let $\psi(u) = u^p$, $p(u) = u^{\delta+2/p}$,

$$B = B(x) = \int_0^1 \int_0^1 \frac{|x(t) - x(s)|^p}{|t-s|^{\delta p+2}} ds dt.$$

$$\begin{aligned} \Rightarrow |x(t) - x(s)| &\leq C_p \int_0^{t-s} \left(\frac{4B}{u^2}\right)^{1/p} u^{\delta+2/p-1} du \\ &= C'_p B^{1/p} \int_0^{t-s} u^{\delta-1} du \leq C_{p,\delta} B^{1/p} |t-s|^\delta \end{aligned}$$

Rk. If x is only in L_{loc} and the RHS is finite, there is $\tilde{x}$ s.t. $\tilde{x} = x$ a.e. s.t. the estimate holds: Set

$$x^\varepsilon(t) = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} x(t+u) du$$

Then $[x^\varepsilon]_x$ is bounded. By Arzela-Ascoli there is $\tilde{x}$ s.t. $\|x^\varepsilon - \tilde{x}\|_\infty \rightarrow 0$ along some $\varepsilon \downarrow 0$. But $x^\varepsilon \rightarrow x$ a.e. by Lebesgue diff. thm., so $\tilde{x} = x$ a.e.

Proof of Kolmogorov's criterion. If we already knew that X was continuous, the estimate would follow from the GRR inequality. But we have not even assumed that $t \mapsto X_t$ is Borel measurable!

Let

$$X_t^n = X_{2^{-n}[2^n t]}$$

Then X_t^n is measurable. Set

$$\hat{X}_t(\omega) = \begin{cases} \lim_{n \rightarrow \infty} X_t^n(\omega) & \text{for } (\omega, t) \text{ s.t. the limit exists} \\ 0 & \text{else.} \end{cases}$$

Since $P[|X_t^n - X_t| \geq 2^{-n}] \leq A 2^{+n} (2^{-n})^{1+\beta}$ is summable the limit exist a.s. for every fixed t .

Let

$$Y = \|\hat{X}\| = \left(\int_0^1 \int_0^1 \frac{|\hat{X}_t - \hat{X}_s|^p}{|t-s|^{\alpha p + 2}} ds dt \right)^{1/p}$$

$$\Rightarrow E[Y^p] = A \int_0^1 \int_0^1 |t-s|^{1+\alpha - p\alpha - 2} ds dt < \infty \text{ if } \alpha > \alpha p.$$

Thus, for every ω , $\hat{X}$ coincides for a.e. t with a $\tilde{X}$ s.t.

$$[\hat{X}]_\alpha \leq CY \quad \text{for all } \omega.$$

It remains to see that $P[X_t = \tilde{X}_t] = 1$ for every t .
It suffices to show this for $t = 2^{-n}k$. Then

$$P[X_t = \tilde{X}_t] = P[\hat{X}_t = \tilde{X}_t].$$

But $P[|\hat{X}_t - \hat{X}_t^\varepsilon| > \delta] \lesssim \delta^{-1} \varepsilon^{1+\beta} \rightarrow 0$, so as before
 $\hat{X}_t^\varepsilon \rightarrow \hat{X}_t$ a.s. as $\varepsilon \rightarrow 0$, so $\hat{X}_t = \tilde{X}_t$ a.s.

Proof (GRR). Let

$$I(t) = \int_0^T \psi\left(\frac{|x(t) - x(s)|}{p(t-s)}\right) ds$$

so that $\int_0^T I(t) dt = B$. By the mean-value theorem,
there is $t_0 \in (0, T)$ s.t. $I(t_0) \leq B/T$.

Given t_{n-1} , define d_{n-1} by

$$p(d_{n-1}) = \frac{1}{2} p(t_{n-1}) \quad (\text{so } 0 < d_{n-1} < t_{n-1})$$

and $t_n \in (0, d_{n-1})$ s.t.

$$I(t_n) \leq \frac{2B}{d_{n-1}} \quad \& \quad \psi\left(\frac{|x(t_n) - x(t_{n-1})|}{p(t_n - t_{n-1})}\right) \leq 2 \frac{I(t_{n-1})}{d_{n-1}}.$$

Each can fail on set of measure $< \frac{d_{n-1}}{2}$ of t_n .

Thus both can fail only on a set of meas. $< d_{n-1}$.

Thus

$$t_0 > d_0 > t_1 > d_1 > \dots$$

Then:

$$p(d_{n+1}) = \frac{1}{2} p(t_{n+1}) \leq \frac{1}{2} p(d_n) \Rightarrow d_n, t_n \downarrow 0.$$

Also:

$$\begin{aligned} p(t_n - t_{n+1}) &\leq p(t_n) = 2p(d_n) = 4(p(d_n) - \frac{1}{2}p(d_n)) \\ &\leq 4(p(d_n) - p(d_{n+1})) \end{aligned}$$

$$\begin{aligned} \Rightarrow |x(t_{n+1}) - x(t_n)| &\leq p(t_n - t_{n+1}) \psi^{-1}\left(\frac{2I(t_n)}{d_n}\right) \\ &\leq 4(p(d_n) - p(d_{n+1})) \psi^{-1}\left(\frac{4B}{d_n d_{n-1}}\right) \\ &\leq 4 \int_{d_{n+1}}^{d_n} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \leq \frac{4B}{d_n^2} \end{aligned}$$

$$\Rightarrow |x(t_0) - x(0)| \leq 4 \int_0^T \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

By an analogous argument,

$$|x(T) - x(t_0)| \leq 4 \int_0^T \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

$$\Rightarrow |x(T) - x(0)| \leq 8 \int_0^T \psi^{-1}\left(\frac{4B}{u^2}\right) p(du).$$

Given $0 \leq s < t \leq T$ set

$$\bar{x}(u) = x(s + \frac{(t-s)}{T} u)$$

$$\bar{p}(u) = p(\frac{(t-s)}{T} u)$$

The argument above with $B \rightsquigarrow \left(\frac{T}{t-s}\right)^2 B = \bar{B}$ gives

$$\begin{aligned} |x(t) - x(s)| &\leq 8 \int_0^T \psi^{-1}\left(\frac{4B}{u^2}\right) \bar{p}(du) \\ &= 8 \int_0^{t-s} \psi^{-1}\left(\frac{4\bar{B}}{u^2}\right) p(du). \end{aligned}$$

2.2. Definition of Brownian motion

Defn. A stochastic process $(B_t)_{t \geq 0}$ is called a Brownian motion or Wiener process if it is continuous and:

- (i) $\forall t < s: B_t - B_s \sim N(0, t-s)$.
- (ii) $\forall t < s: B_t - B_s$ is independent of $\sigma(B_u: u \leq s)$.

Thm. There exists a unique probability measure on $C[0, \infty)$ such that the canonical process is a Wiener process. This is the Wiener measure.

Lemma. If (i) holds for a stochastic process on some probability space then there is a version that is in fact Hölder continuous for any Hölder exponent strictly less than $1/2$.

Proof. Since $W_t - W_s \sim N(0, t-s)$,
$$E[|W_t - W_s|^p] \leq C_p |t-s|^{p/2} \text{ for any } p \geq 1.$$

Thus Kolmogorov applies $\gamma = \frac{1}{2} - \varepsilon$ when $p \geq \frac{1}{\varepsilon}$:
$$E[|W_t - W_s|^p] \leq C_p |t-s|^{p/2} \leq C_p |t-s|^{\gamma p + 1}$$

Proof (construction of Wiener measure). Given any probability space $(\Omega, \mathcal{F}, P)$ on which a Brownian motion (B_t) is defined, consider the map

$$I: \Omega \rightarrow C[0, \infty), \quad \omega \mapsto (t \mapsto B_t(\omega)).$$

This map is measurable. Thus $P \circ I^{-1}$ defines a Borel measure on $C[0, \infty)$.

Using a monotone class argument one checks it is uniquely defined by its fin-dim. distributions.

Prop. Brownian motion is not in $C^{1/2}$:

$$P\left[\sup_{\substack{t \neq s \\ 0 \leq t, s \leq 1}} \frac{|X_t - X_s|}{|t-s|^{1/2}} = \infty\right] = 1.$$

Proof. For each $t > s$, $(B_t - B_s)/|t-s|^{1/2}$ is $\sim N(0, 1)$.

$$\sup_{t \neq s} \frac{|X_t - X_s|}{|t-s|^{1/2}} \geq \sup_{0 \leq k \leq 2^n} \frac{|X_{2^{-n}(k+1)} - X_{2^{-n}k}|}{2^{-n/2}}$$

$$\Rightarrow P\left[\sup_{t \neq s} \frac{|X_t - X_s|}{|t-s|^{1/2}} > \lambda\right] \leq 1 - P[Z < \lambda]^{2^n} \rightarrow 0.$$

where $Z \sim N(0, 1)$.

There are many ways of constructing (B_t) .

Defn. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then $S \subset L^2(\Omega, \mathcal{F}, P)$ is a Gaussian space if S is a closed linear subspace and any X is a Gaussian random variable.

Example. Let (X_i) be i.i.d $N(0, 1)$ on some probability space. Then $\text{span}\{X_i\}$ is a Gaussian space. The X_i are an orthonormal system:

$$E[X_i X_j] = \delta_{ij}$$

Note that limits of Gaussian random variables are Gaussian.

Prop. Let H be a separable Hilbert space and S as in the example. Then there is an isometry $I: H \rightarrow S$. Thus:

- $\forall f \in H: I(f) \sim N(0, \|f\|_H^2)$
- $\forall f, g \in H: E I(f) I(g) = (f, g)_H$

In fact, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.

Proof. Let (e_i) be an orthonormal basis for H . Set

$$I(f) = \sum_{i=1}^{\infty} (f, e_i) X_i.$$

The limit exists in L^2 and a.s. since

$$k \mapsto \sum_{i=1}^k (f, e_i) X_i$$

is a martingale bounded in L^2 .

Defn. Let $\dot{W}$ be an isometry from $L^2(\mathbb{R}_+)$ into some Gaussian space. Then $\dot{W}$ is called white noise.

For $t \geq 0$, define

$$W_t = \dot{W}(1_{[0,t]})$$

Rk. Would like to think of $\dot{W}(f)$ as $\int f dW$ but W has infinite variation almost surely.

Exercise. (i) For $A \subset \mathbb{R}_+$ Borel, $|A| < \infty$, $\dot{W}(A) \sim N(0, |A|)$

(ii) For $A, B \subset \mathbb{R}_+$ with $A \cap B = \emptyset$, $\dot{W}(A)$ and $\dot{W}(B)$ are independent.

(iii) For $A = \bigcup_{i=1}^{\infty} A_i$ with A_i disjoint, $\dot{W}(A) = \sum_{i=1}^{\infty} \dot{W}(A_i)$ in L^2 and a.s. depends on the $(A_i)_i$.

$\dot{W}$ looks like a random measure but it is not.

Prop. For any $t_1, \dots, t_n$ the vector $(W_{t_i})_{i=1}^n$ is jointly Gaussian with covariance

$$E[W_t W_s] = s \wedge t \quad \text{for } s, t \geq 0.$$

Moreover, $W_0 = 0$ a.s. and

$W_t - W_s$ is independent of $\sigma(W_u : u < s)$

$$W_t - W_s \sim N(0, t-s).$$

Exercise: Let B be a Brownian motion. Then

- $-B$ is a Brownian motion (symmetry)
- $\frac{1}{\lambda} B_{\lambda^2 t}$ is a Brownian motion (scale invariance)
- $B_{t+s} - B_s$ is a Brownian motion that is indep. of $(B_u : u \leq s)$ (Markov prop.).

Exercise. Understand how Brownian motion is the Gaussian measure with covariance $(-\Delta)^{-1}$ on $[0, \infty)$ with 0-boundary condition at 0.

Defn. A stochastic process taking values in $\mathbb{R}^d$ is a d -dim. Brownian motion if each component is a Brownian motion.

2.3. Heat equation and Feynman-Kac formula

Let $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}}$ be the heat kernel on $\mathbb{R}$.

Defn. The semigroup of Brownian motion is

$$T_t f(x) = E_x[f(B_t)]$$

where E_x is the expectation of a BM starting at $x \in \mathbb{R}$.

Fact. Since $B_t - x \sim N(0,t)$,

$$\begin{aligned} T_t f(x) &= \int p_t(x,y) f(y) dy = \int f(x+y) p_t(0,y) dy \\ &= \int f(x+y\sqrt{t}) p_1(0,y) dy. \end{aligned}$$

Defn. The infinitesimal generator is

$$Lf(x) = \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t}$$

for f such that the limit exists.

Fact. For $f \in C^2$ the limit exists and

$$Lf = \frac{1}{2} f''.$$

Proof. If $f \in C^3$,

$$\lim_{t \rightarrow 0} \int \frac{f(x+y\sqrt{t}) - f(x)}{t} p_1(0,y) dy$$

$$= \lim_{t \rightarrow 0} \int \frac{f'(x)y\sqrt{t} + \frac{1}{2}f''(x)y^2t + t^{3/2}e_t(y)}{t} p_1(0,y) dy$$

with $|e_t(y)| \leq C|y|^2$. Thus

$$Lf(x) = \frac{1}{2}f''(x).$$

If $f \in C^2$, one still has $t^{3/2}|e_t(y)| = o(t)$.

Thus $u(t,x) = T_t f(x)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad u(t,x) \rightarrow f(x) \quad (t \rightarrow 0).$$

Exercise. Let $X_t = x + \sigma B_t + mt$ be Brownian motion with variance σ^2 and drift m . Show

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + m f'(x).$$

Fact. Given $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a continuous adapted process (B_t) is a Brownian motion iff

$$E[f(B_t) | \mathcal{F}_s] = T_{t-s} f(B_s) \quad \forall f \text{ continuous.}$$

Feynman-Kac formula. Let V be bounded and continuous. Then

$$u(t, x) = E_x \left[\exp \left(\int_0^t V(B_s) ds \right) f(X_t) \right]$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + V u, \quad u(0, x) = f(x).$$

Proof. Expand exponential:

$$\begin{aligned} u(t, x) &= \sum_{n=0}^{\infty} \frac{1}{n!} E_x \left[\int_0^t \cdots \int_0^t V(B_{t_1}) \cdots V(B_{t_n}) f(X_t) dt_1 \cdots dt_n \right] \\ &= \sum_{n=0}^{\infty} E_x \left[\int_{0 < t_1 < \cdots < t_n < t} V(B_{t_1}) \cdots V(B_{t_n}) f(X_t) dt_1 \cdots dt_n \right] \\ &= \sum_{n=0}^{\infty} E_x \left[\int_{0 < t_n < \cdots < t_1 < t} V(B_{t_{t_1}}) \cdots V(B_{t_{t_n}}) f(X_t) dt_1 \cdots dt_n \right] \\ &= \sum_{n=0}^{\infty} \int_{0 < t_n < \cdots < t_1 < t} T_{t-t_1} V T_{t_1-t_2} \cdots V T_{t_{n-1}-t_n} V T_{t_n} f dt_1 \cdots dt_n \end{aligned}$$

Differentiate term by term.

Example: Arcsine law.

$$\text{Let } \tilde{z}_t = \frac{1}{t} \int_0^t 1_{B_s > 0} ds.$$

$$\Rightarrow E[\tilde{z}_t] = \frac{1}{2}.$$

What is the distribution of $\tilde{z}_t$?

$$\tilde{z}_t = \int_0^1 1_{B_{ts} > 0} ds \stackrel{(d)}{=} \int_0^1 1_{\sqrt{t}B_s > 0} ds = \int_0^1 1_{B_s > 0} ds$$

$$\Rightarrow \tilde{z}_t \sim \tilde{z}_1.$$

Prop. $P[\tilde{z}_1 \leq x] = \int_0^x \frac{1}{\pi \sqrt{y(1-y)}} dy = \frac{2}{\pi} \arcsin(\sqrt{x}).$

Proof. Let $V(x) = 1_{x > 0}$. This V is not cont. and a more careful argument would involve a limiting argument to apply Feynman-Kac. Anyway,

$$u(t, x) = E_x \left[\exp \left(-\sigma \int_0^t V(B_s) ds \right) \right]$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \sigma V u, \quad u(0, x) = 1.$$

Let $g(\lambda, x) = \int_0^\infty e^{-\lambda t} u(t, x) dt$. Then

$$\lambda g + \sigma V g - \frac{1}{2} \frac{\partial^2 g}{\partial x^2} = 0.$$

Thus

$$(\lambda + \sigma)g - \frac{1}{2}g'' = 1 \quad (x > 0)$$

$$\lambda g - \frac{1}{2}g'' = 0 \quad (x < 0)$$

and g should be bounded at $\pm\infty$ and continuously differentiable at 0. The solution is

$$g(\lambda, x) = \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{\lambda + \sigma}}.$$

Thus

$$\begin{aligned} & E\left[\int_0^\infty \exp(-\lambda t - \sigma \int_0^t V(B_s) ds) dt\right] \\ &= E\left[\int_0^\infty \exp(-\lambda t - \sigma t \xi_t) dt\right] \\ &= E\left[\int_0^\infty \exp(-\lambda t - \sigma t \xi_1) dt\right] \\ &= E\left[\frac{1}{\lambda + \sigma \xi_1}\right] = \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{\lambda + \sigma}}. \end{aligned}$$

Thus $E\left[\frac{1}{1 + \sigma \xi_1}\right] = \frac{1}{\sqrt{1 + \sigma}}$. Differentiating:

$$E[\xi_1^n] = \frac{1}{\Gamma(n)} \int_0^1 x^{n-\frac{1}{2}} (1-x)^{-\frac{1}{2}} dx.$$

Since the support is bd. moments characterize the distribution.

2.4. Strong Markov and reflection property

Markov property. Let $t > s$. Then

$$\begin{aligned}\forall A: \quad P[B_t \in A | \mathcal{F}_s] &= p_{t-s}(B_s, A) \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{(y-B_s)^2}{2(t-s)}} dy\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\forall f \text{ bd}: \quad P[f(B_t) | \mathcal{F}_s] &= p_{t-s} f(B_s) \\ &= \int p_{t-s}(B_s, y) f(y) dy.\end{aligned}$$

Proof. Immediate from independent increments.

Strong Markov property. Let τ be a stopping time. Then $B_{\tau+t} - B_\tau$ is again a BM independent of $\mathcal{F}_\tau$ conditional on $\tau < \infty$.

Proof. let $\tau_n = 2^{-n} \lceil 2^n \tau \rceil$ and $t_k = 2^{-n} k$.

Given $A \in \mathcal{F}_{\tau+s}$ set $A_k = A \cap \{\tau_n = t_k\} \in \mathcal{F}_{t_k+s}$. Then

$$\begin{aligned}E[f(B_{\tau_n+t}) 1_{A_k}] &= E[f(B_{t_k+t}) 1_{A_k}] \\ &= E[\bar{T}_{t-s} f(B_{t_k+s}) 1_{A_k}] \\ &= E[\bar{T}_{t-s} f(B_{\tau_n+s}) 1_{A_k}]\end{aligned}$$

$$\sum_{n=1}^{\infty} E[f(B_{\tau_n+t}) 1_{A \cap \{\tau < \infty\}}] = E[\sum_{n=1}^{\infty} f(B_{\tau_n+t}) 1_{A \cap \{\tau < \infty\}}]$$

For f bounded, can take $n \rightarrow \infty$:

$$E[f(B_{\tau+t}) 1_{A \cap \{\tau < \infty\}}] = E[\sum_{n=1}^{\infty} f(B_{\tau_n+t}) 1_{A \cap \{\tau < \infty\}}]$$

$$\Rightarrow E[f(B_{\tau+t}) | \mathcal{F}_{\tau+s}] = E[\sum_{n=1}^{\infty} f(B_{\tau_n+t}) | \mathcal{F}_{\tau+s}] \text{ a.s. on } \tau < \infty$$

Rk. Let $\theta_s : C[0, \infty) \rightarrow C[0, \infty)$, $\theta_s \omega(t) = \omega(t+s)$.

Then for fin. stopping times

$$P_x[\theta_\tau \omega \in A | \mathcal{F}_\tau] = P_{x(\tau)}[A], \text{ } P_x\text{-a.s.}$$

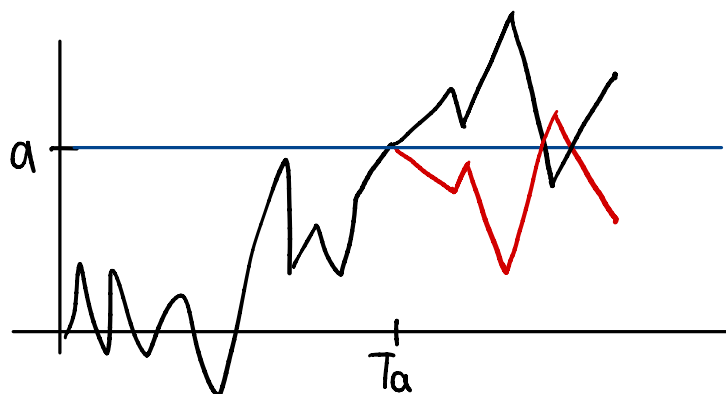
For a stopping time that is not a.s. finite,

$$P_x[\theta_\tau \omega \in A | \mathcal{F}_\tau] = P_{x(\tau)}[A], \text{ } P_x\text{-a.s. on } \tau < \infty.$$

Reflection principle. Let $a > 0$, $T_a = \inf\{t \geq 0 : B_t = a\}$, and

$$X_t = \begin{cases} B_t & (t \leq T_a) \\ 2a - B_t & (t \geq T_a) \end{cases}$$

Then (X_t) is also a Brownian motion.



Proof. On the event $T_a < \infty$ set

$$\tilde{B}_t = B_{T_a+t} - B_{T_a}.$$

By the strong Markov property, $\tilde{B}$ is a BM cond. on $T_a < \infty$ indep. of $\mathcal{F}_{T_a}$. Same for $-\tilde{B}$.

$$\Rightarrow B_t = B_{T_a \wedge t} + \tilde{B}_{(t-T_a)^+} 1_{T_a < \infty}$$

$$X_t = B_{T_a \wedge t} - \tilde{B}_{(t-T_a)^+} 1_{T_a < \infty}.$$

Thus B and X have the same distribution.

Cor. For $a \geq 0$ and $b \leq a$,

$$P\left[\sup_{0 \leq s \leq t} B_s \geq a, B_t \leq b\right] = P[B_t \geq 2a - b]$$

In particular, $\sup_{0 \leq s \leq t} B_s \sim |B_t|$ and

$$P\left[\sup_{0 \leq s \leq t} |B_s| \geq a\right] \leq 4 P[B_t \geq a]$$

Proof. For the first claim notice:

$$\text{LHS} = P[T_a \leq t, B_t \leq b]$$

$$\text{RHS} = P[2a - B_t \leq b]$$

$$= P[T_a \leq t, X_t \leq b] + \underbrace{P[T_a > t, B_t \geq 2a - b]}_0$$

For the second one:

$$P\left[\sup B_s \geq a\right] = \underbrace{P\left[\sup B_s \geq a, B_t \geq a\right]}_{P[B_t \geq a] \text{ trivially}}$$

$$+ \underbrace{P\left[\sup B_s \geq a, B_s \leq a\right]}_{P[B_t \geq a] \text{ by reflection}}$$

$$= 2 P[B_t \geq a] = P[|B_t| \geq a]$$

2.5. Martingales

Exercise. Let B be a standard BM with $B_0 = 0$. Then the following are continuous martingales:
 $(B_t), \quad B_t^2 - t, \quad e^{\lambda B_t - \frac{1}{2}\lambda^2 t}.$

Prop. Let $f: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be C_b^1 in t and C_b^2 in x . Then
$$M_t = f(t, B_t) - f(0, B_0) - \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2} \Delta \right) f(s, B_s) ds$$
is a continuous martingale.

Proof. It suffices to show $E[M_{s+t} - M_s | \mathcal{F}_s] = 0$ a.s. By the Markov property, suffices to consider $s=0$.

Claim: for any $t > s$, $E_x[M_t - M_s] = 0$.

$$\begin{aligned} E_x[M_t - M_s] &= E_x \left[f(t, B_t) - f(s, B_s) - \int_s^t \left(\frac{\partial}{\partial u} + \frac{1}{2} \Delta \right) f(u, B_u) du \right] \\ &= E_x[f(t, B_t)] - E_x[f(s, B_s)] \\ &\quad - \int_s^t E_x \left[\left(\frac{\partial}{\partial u} + \frac{1}{2} \Delta \right) f(u, B_u) \right] du \end{aligned}$$

$$\begin{aligned}
&= \int p_t(x,y) f(t,y) dy - \int p_s(x,y) f(s,y) dy \\
&\quad - \int_s^t \underbrace{\int p_u(x,y) \left(\frac{\partial}{\partial u} + \frac{1}{2} \Delta \right) f(u,y) dy}_{\int p_u(x,y) \frac{\partial f}{\partial u}(u,y) dy} du \\
&\quad + \int \frac{1}{2} \Delta p_u(x,y) f(u,y) dy
\end{aligned}$$

Using $\frac{1}{2} \Delta p_u = \frac{\partial}{\partial u} p_u$ Thus

$$\begin{aligned}
E_x[M_t - M_s] &= \int p_t(x,y) f(t,y) dy - \int p_s(x,y) f(s,y) dy \\
&\quad - \int_s^t \frac{\partial}{\partial u} (p_u(x,y) f(u,y)) dy du \\
&= 0.
\end{aligned}$$

Thus $E_x[M_t] = \lim_{s \rightarrow 0} E_x[M_t - M_s] = 0$.

Example. Let $T_x = \inf\{t \geq 0 : B_t = x\}$. Then

$$P_0[T_{-a} < T_b] = \frac{b}{a+b} \quad E_0[T_{-a} \wedge T_b] = ab$$

Proof. By OST (note $B_{T_{-a} \wedge T_b} \in [-a, b]$ is bounded)

$$E_0[B_{T_{-a} \wedge T_b}] = E_0[B_0] = 0.$$

On the other hand,

$$E_0[B_{T_{-a} \wedge T_b}] = P_0[T_{-a} < T_b](-a) + (1 - P_0[T_{-a} < T_b])b$$

$$\Rightarrow (a+b)P_0[T_{-a} < T_b] = b.$$

To compute $E[T_{-a} \wedge T_b]$ use that $B_t^2 - t$ is a martingale. Thus again by OST:

$$E_0[B_{T_{-a} \wedge T_b}^2 - T_{-a} \wedge T_b] = 0$$

$$\begin{aligned} \Rightarrow E[T_{-a} \wedge T_b] &= E_0[B_{T_{-a} \wedge T_b}^2] \\ &= a^2 \cdot \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab. \end{aligned}$$

2.6. Recurrence and transience.

Thm. Let (B_t) be a d -dim. Brownian motion. Then

- $d=1$: $P[\#\{t \geq 0: B_t=0\} = \infty] = 1$ recurrent
- $d=2$: $P[\#\{t \geq 0: B_t=0\} \geq 2] = 0$ point trans.
 $P[\{t \geq 0: |B_t| < \varepsilon\} \text{ is unbd.}] = 1$ nbhd recurr.
- $d \geq 3$: $P[|B_t| \rightarrow \infty] = 1$. transient

Proof ($d=1$).

$$\begin{aligned} P[T_a \leq t] &= 2P[B_t \geq a] = \sqrt{\frac{2}{\pi t}} \int_a^\infty e^{-\frac{y^2}{2t}} dy. \\ &= \sqrt{\frac{2}{\pi}} \int_{a/\sqrt{t}}^\infty e^{-\frac{y^2}{2}} dy \end{aligned}$$

$$\Rightarrow f_a(t) = \frac{\partial}{\partial t} P[T_a \leq t] = \frac{1}{\sqrt{2\pi}} \frac{a}{t^{3/2}} e^{-\frac{a^2}{2t}}$$

In particular, $P[T_a < \infty] = \lim_{t \rightarrow \infty} P[T_a \leq t] = 1$.

The claim follows from the strong Markov prop.

Note even though $P[T_a < \infty] = 1$ one has $E[T_a] = \infty$.
Brownian motion is recurrent but not positive recurrent.

Proof. (d=2) Let $0 < r < R$. Let $f \in C_b^2(\mathbb{R}^2)$ s.t.

$$f(x) = \log |x| \quad (r \leq |x| \leq R).$$

Then

$$M_t = f(B_t) - f(B_0) - \frac{1}{2} \int_0^t \Delta f(B_s) ds$$

is a martingale. Set

$$T_r = \inf \{t \geq 0 : |B_t| = r\}, \quad T_R = \inf \{t \geq 0 : |B_t| = R\}.$$

Then by OST and using $\Delta f = 0$ on $r \leq |x| \leq R$:

$$0 = E[M_0] = E[M_T] = E[f(B_T) - f(B_0)].$$

Assume for now $|B_0| = |x| \in (r, R)$. Then

$$E_x[\log |B_T|] = E_x[f(B_T)] = E_x[f(B_0)] = \log |x|$$

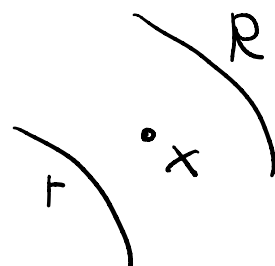
$$\Rightarrow P_x[T_r < T_R] \cdot \log \frac{r}{|x|} + (1 - P[T_r < T_R]) \log \frac{R}{|x|} = 0.$$

$$\Rightarrow P_x[T_r < T_R] = \frac{\log(R/|x|)}{\log(R/r)}.$$

Thus $P_x[T_r < T_R] \rightarrow 0$ as $r \rightarrow 0$ when $|x| < R$.

$$\Rightarrow P_x[B_t = 0 \text{ for some } t > 0]$$

$$\leq \lim_{R \rightarrow \infty} \lim_{r \rightarrow 0} P_x[T_r < T_R] = 0.$$



For $x=0$,

$$\begin{aligned} P_0[B_t=0 \text{ for some } t>\varepsilon] \\ = \int P_\varepsilon(0,y) P_y[B_t=0 \text{ for some } t>0] dy = 0 \end{aligned}$$

$$\stackrel{\varepsilon \rightarrow 0}{\Rightarrow} P_0[B_t=0 \text{ for some } t>0] = 0$$

On the other hand, for $|x| \neq 0$,

$$P_x[T_r < T_R] \rightarrow 1 \text{ as } R \rightarrow \infty.$$

$$\Rightarrow P_x[|B_t| < \varepsilon \text{ for some } t>0] = 1$$

which also holds for $x=0$ trivially. By Markov,

$$P_x[|B_t| < \varepsilon \text{ for some } t>n] = 1.$$

$$\Rightarrow P[\#\{t \geq 0 : |B_t| < \varepsilon\} = \infty] = 1.$$

3. Stochastic integrals

3.1. Wiener's integral

We have already seen Wiener's integral. Let

$$f(t) = \sum_{k=0}^n f_k 1_{(t_k, t_{k+1}]}(t).$$

be a deterministic step function. Then

$$\begin{aligned} \dot{W}(f) &= \sum_{k=0}^n f_k \dot{W}(1_{(t_k, t_{k+1}]}) \\ &= \sum_{k=0}^n f_k (B_{t_{k+1}} - B_{t_k}) \stackrel{\text{def}}{=} \int_0^\infty f dB. \end{aligned}$$

Given $f \in L^2(0, \infty)$ there are f_n as above s.t. $f_n \xrightarrow{L^2} f$.
Then $\dot{W}(f_n)$ converges in L^2 to $\dot{W}(f)$:

$$E[(\dot{W}(f_n) - \dot{W}(f))^2] = \|f_n - f\|_{L^2}^2 \rightarrow 0.$$

More generally, $\int_0^t f dB \stackrel{\text{def}}{=} \dot{W}(f 1_{[0, t]})$.

Exercise. The Wiener integral satisfies:

- $\int_0^t f dB$ can be chosen continuous in t a.s.
- $\int_0^t f dB$ is a martingale
- $(\int_0^t f dB)^2 - \int_0^t f^2 dt$ is a martingale

We will extend this to random f and general mart.

3.2. Martingales in L^2 .

Defn. Given $(\Omega, (\mathcal{F}_t), P)$, let

$$\text{Mart}^2 = \{ M: \Omega \times [0, \infty) \rightarrow \mathbb{R}: M \text{ is a right-cont. martingale, } \sup_{t \geq 0} E[|M_t|^2] < \infty \} / \sim$$

$\text{Mart}_c^2 = \{ M \in \text{Mart}^2 : M(\omega, \cdot) \text{ is cont. for all } \omega \} / \sim$
where $\sim$ means identification up to P -null sets.

Fact. Let $M \in \text{Mart}^2$. Then:

- Martingale convergence: $M_t \rightarrow M_\infty$ a.s. and in L^2 ;
- (M_t^2) is a submartingale, so $E[M_t^2]$ is increasing;
- Doob's L^2 inequality: $E[\sup_t M_t^2] \leq 4 E[M_\infty^2]$.

Prop. Mart^2 is a Hilbert space with norm

$$\|M\|_{\text{Mart}^2} = \left(\sup_{t \geq 0} E[M_t^2] \right)^{1/2} = E[M_\infty^2]^{1/2}$$

and inner product $E[M_\infty N_\infty]$. The space Mart_c^2 is a closed subspace.

Proof. The only nontrivial property is completeness.
 Let $(M^n) \subset \text{Mart}^2$ be Cauchy:

$$E[(M_\infty^n - M_\infty^m)^2] \rightarrow 0 \quad (n, m \rightarrow \infty).$$

By passing to a subsequence, can assume

$$E[(M_\infty^n - M_\infty^{n-1})^2] \leq 2^{-n}.$$

Then

$$\begin{aligned} E\left[\sum_{n=1}^{\infty} \sup_{t \geq 0} |M_t^n - M_t^{n-1}|\right] &\stackrel{(CS)}{\leq} \sum_{n=1}^{\infty} E\left[\sup |M_t^n - M_t^{n-1}|^2\right]^{1/2} \\ &\stackrel{(Doob)}{\leq} 2 \sum_{n=1}^{\infty} E\left[|M_\infty^n - M_\infty^{n-1}|^2\right]^{1/2} \\ &\leq 2 \sum_{n=1}^{\infty} 2^{-n/2} < \infty. \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \sup_t |M_t^n - M_t^{n-1}| < \infty \quad \text{a.s.}$$

Thus almost surely (M^n) is a Cauchy sequence in the closed subspace of $D[0, \infty)$ of bounded functions with uniform topology. Thus

$$\|M^n - M\|_\infty \rightarrow 0 \quad \text{for some } M \in D[0, \infty) \quad \text{a.s.}$$

Set $M=0$ outside this event.

It remains to show that:

(i) $E\left[\sup_{t \geq 0} |M_t^n - M_t|^2\right] \rightarrow 0$

(ii) M is a martingale.

To show (i), note:

$$\begin{aligned} E\left[\sup_t |M_t^n - M_t|^2\right] &= E\left[\lim_{m \rightarrow \infty} \sup_t |M_t^n - M_t^m|^2\right] \\ &\stackrel{\text{(Fatou)}}{\leq} \liminf_{m \rightarrow \infty} E\left[\sup_t |M_t^n - M_t^m|^2\right] \\ &\stackrel{\text{(Doob)}}{\leq} 4 \liminf_{m \rightarrow \infty} E[|M_\infty^n - M_\infty^m|^2] \rightarrow 0. \end{aligned}$$

To show (ii):

(Δ -ineq. + (M^n) is mart.)

$$\begin{aligned} \|E[M_t | \mathcal{F}_s] - M_s\|_{L^2} &\stackrel{(\Delta\text{-ineq.})}{\leq} \|E[M_t - M_t^n | \mathcal{F}_s]\|_{L^2} + \|M_s^n - M_s\|_{L^2} \\ &\stackrel{(\text{Jensen})}{\leq} \|M_t - M_t^n\|_{L^2} + \|M_s^n - M_s\|_{L^2} \\ &\leq 2 E\left[\sup_t |M_t - M_t^n|^2\right]^{1/2} \rightarrow 0. \end{aligned}$$

The same argument applies to $M \in \text{Mart}_c^2$, so Mart_c^2 is also complete (and obviously it is a subspace).

3.3. Quadratic variation

Thm. (Doob-Meyer). Let $M \in \text{Mart}_c^2$. Then a.s. there is a unique continuous, increasing, progressively measurable process A s.t. $A_0 = 0$ and $(M_t^2 - A_t)$ is a martingale.

Defn. The process A is denoted $\langle M \rangle = (\langle M \rangle_t)_{t \geq 0}$ and called the quadratic variation of M .

Example. Let B be a std. Brownian motion: $\langle B \rangle_t = t$.
(to be precise: $\langle B^T \rangle_t = t \wedge T$)

Proof. Uniqueness: Suppose A and A' satisfy the conditions. Then $A - A'$ is a martingale of bounded variation, so is constant a.s.

Finite case: Suppose there are deterministic constants C and T s.t.

$$|M_t - M_0| \leq C \text{ and } M_t = M_{t \wedge T}.$$

Then for any partition $P = (t_i)_{i=1}^n$ of $[0, T]$ set

$$A_t^{(P)} = \sum_{i=1}^n (M_{t_i \wedge t} - M_{t_{i-1} \wedge t})^2$$

$$\chi_t^{(P)} = \sum_{i=1}^n 2 M_{t_{i-1}} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$$

Then clearly $X^{(P)}$ is a bounded martingale and $A_{t_k}^{(P)}$ is increasing in k .

Claim: $M_{t_k}^2 - M_{t_0}^2 = X_{t_k}^{(P)} + A_{t_k}^{(P)}$.

$$\begin{aligned} A_{t_k}^{(P)} &= \sum_{i=1}^k (M_{t_i} - M_{t_{i-1}})^2 \\ &= \sum_{i=1}^k M_{t_i} (M_{t_i} - M_{t_{i-1}}) - \sum_{i=1}^k M_{t_{i-1}} (M_{t_i} - M_{t_{i-1}}) \\ &= -2 \sum_{i=1}^k M_{t_{i-1}} (M_{t_i} - M_{t_{i-1}}) + \sum_{i=1}^k \underbrace{(M_{t_i} + M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}})}_{M_{t_i}^2 - M_{t_{i-1}}^2} \\ &= -X_{t_k} + M_{t_k}^2 - M_{t_0}^2. \end{aligned}$$

Claim: $E[(A_{\infty}^{(P)})^2] \leq 12C^4$.

$$\begin{aligned} E[(A_{\infty}^{(P)})^2] &= E\left[\left(\sum_{i=1}^{\infty} (M_{t_i} - M_{t_{i-1}})^2\right)^2\right] \\ &= \sum_{i=1}^{\infty} E[(M_{t_i} - M_{t_{i-1}})^4] \\ &\quad + 2 \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} E[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_k} - M_{t_{k-1}})^2] \\ &\quad \underbrace{E[(M_{t_i} - M_{t_{i-1}})^2 \sum_{k=i+1}^{\infty} (M_{t_k} - M_{t_{k-1}})^2]}_{= E[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_{\infty}} - M_{t_i})^2]} \end{aligned}$$

orthogonal increm.

$$\begin{aligned} &E[(M_{t_k} - M_{t_{k-1}})^2] \\ &= E[M_{t_k}^2 - 2M_{t_k}M_{t_{k-1}} + M_{t_{k-1}}^2] \\ &= E[M_{t_k}^2 - M_{t_{k-1}}^2 - 2M_{t_{k-1}}(M_{t_k} - M_{t_{k-1}})] \\ &= E[M_{t_k}^2 - M_{t_{k-1}}^2] \end{aligned}$$

$$\leq (4+8)C^2 \sum_{i=1}^n E[(M_{t_i} - M_{t_{i-1}})^2]$$

$$= 12C^2 E[(M_{t_n} - M_{t_0})^2] \leq 12C^4.$$

Claim: Let $P_m = (t_k)$ with $t_k = k2^{-m}$, $k \leq n_m = \lfloor 2^m T \rfloor$.
Then $X^m = X^{(P_m)}$ converges in Mart_c^2 .

It suffices to show X^m is a Cauchy sequence:

$$E[(X_\infty^m - X_\infty^{m'})^2] \rightarrow 0$$

Indeed, for $m' > m$,

$$X_\infty^{m'} - X_\infty^m = \sum_{i=1}^{n_{m'}} (M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^m - m' \rfloor 2^{-m}}) (M_{i2^{-m'}} - M_{(i-1)2^{-m'}})$$

Since the terms are orthogonal, therefore

$$E[(X_\infty^{m'} - X_\infty^m)^2] = \sum_{i=1}^{n_{m'}} E[(M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^m - m' \rfloor 2^{-m}})^2 \times (M_{i2^{-m'}} - M_{(i-1)2^{-m'}})^2]$$

$$\leq E\left[\sup_{|s-t| \leq 2^{-m}} |M_t - M_s|^2 \times A_\infty^{m'} \right]$$

$$\stackrel{(CS)}{\leq} (12C^4)^{1/2} E\left[\underbrace{\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^4}_{\rightarrow 0 \text{ by uniform cont.}} \right]$$

$\rightarrow 0$ by uniform cont.

Let $X \in \text{Mart}_c^2$ be the limit of X^m and set

$$A = M^2 - X.$$

Then A is continuous since M and X are.

A is increasing since A^m is increasing on $k2^{-m}$ and the convergence is uniform on compact intervals. Indeed, by Doob's inequality,

$$\|\sup_t |X_t^n - X_t| \|_{L^2} \rightarrow 0.$$

By uniform continuity also

$$\sup_t |X_{[2^m t]2^{-m}} - X_t| \rightarrow 0$$

$$\sup_t |M_{[2^m t]2^{-m}} - M_t| \rightarrow 0$$

Finally, if M is not bounded, apply above to M^{τ_n} ,

$$\tau_n = \inf\{t \geq 0 : |M_t - M_0| > n\} \wedge n.$$

Thus $\langle M^{\tau_n} \rangle$ exists as above. By uniqueness,

$$\langle M^{\tau_n} \rangle_t = \langle M^{\tau_{n+1}} \rangle_{t \wedge \tau_n}$$

Thus one can define a consistent $A = \langle M \rangle$.

Thus $M^{\tau_n} - A$ is a martingale for every n .
 To see the limit is a martingale, need uniform integrability to get convergence in L^1 .

Claim: $E[A_\infty] = E[M_\infty^2 - M_0^2] < \infty$.

Indeed, by definition,

$$E[A_{t \wedge \tau_n}] = E[M_{t \wedge \tau_n}^2 - M_0^2]$$

and as $t \rightarrow \infty$:

$E[A_{t \wedge \tau_n}] \rightarrow E[A_{\tau_n}]$ by monotone conv.

$E[M_{t \wedge \tau_n}^2] \rightarrow E[M_{\tau_n}^2 - M_0^2]$ by dominated conv.

Then as $n \rightarrow \infty$

$E[A_{\tau_n}] \rightarrow E[A_\infty]$ by monotone conv.

$E[M_{\tau_n}^2] \rightarrow E[M_\infty^2 - M_0^2]$ by dominated conv.

This gives $A_\infty \in L^1$ and thus $A_t \leq A_\infty$ is unif. int.

Defn. For $M, N \in \text{Mart}_c^2$ also set (covariation)

$$\langle M, N \rangle = \frac{1}{4}(\langle M+N \rangle - \langle M-N \rangle).$$

3.4. Itô integral

The Wiener integral is an isometry

$$I: L^2(0, \infty) \rightarrow \text{Mart}_c^2$$
$$f \mapsto \int_0^t f \, dB$$

For example,

$$\|I(f)\|_{\text{Mart}^2}^2 = E[I(f)_\infty^2] = E[\dot{W}(f)^2] = \|f\|_{L^2}^2.$$

The Itô integral will be defined similarly.

Defn. Let A be right-continuous, increasing, progressively measurable. Then

$$L^2(A) = L^2(A, (\mathcal{F}_t), P)$$

denotes the Hilbert space of progressively measurable processes α s.t.

$$E\left[\int_0^\infty \alpha_t^2 \, dA_t\right] < \infty.$$

Prop. The space $\mathcal{E}$ of simple processes

$$\sum_{i=1}^n \alpha_{i-1} 1_{(t_{i-1}, t_i]}, \quad \alpha_{i-1} \in \mathcal{F}_{t_{i-1}}$$

is dense in $L^2(A)$.

Proof. Since $L^2(A)$ is a Hilbert space (complete!), it suffices to show that $\alpha = 0$ whenever

$$(*) (\alpha, \beta)_{L^2(A)} = 0 \quad \forall \beta \in \mathcal{E}$$

Assume (*). Set

$$X_t = \int_0^t \alpha_s dA_s.$$

Claim: X is a continuous martingale

Let $s < t$, $A \in \mathcal{F}_s$, $\beta = 1_A 1_{(s,t]} \in \mathcal{E}$. Then

$$\begin{aligned} 0 = (\alpha, \beta) &= E\left[1_A \int_s^t \alpha_u dA_u\right] \\ &= E\left[1_A (X_t - X_s)\right] = E\left[1_A E[X_t - X_s | \mathcal{F}_s]\right] \end{aligned}$$

$\Rightarrow E[X_t - X_s | \mathcal{F}_s] = 0$ a.s. $\Rightarrow X$ is a martingale.

Claim: X is also of finite variation

$$\int_0^t |\alpha_s| dA_s \leq \left(\int_0^t \alpha_s^2 dA_s \right)^{1/2} A_t$$

Therefore $X = 0$ a.s.

$\Rightarrow \alpha_s = 0$ for dA -a.e. s a.s.

$\Rightarrow \alpha = 0$ in $L^2(A)$.

Prop. Let $M \in \text{Mart}_c^2$. For $\alpha = \sum \alpha_{i-1} 1_{(t_{i-1}, t_i]} \in \mathcal{E}$ define

$$\int_0^t \alpha_s dM_s = \sum_{i=1}^n \alpha_{i-1} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

Then:

- $\int_0^t \alpha_s dM_s$ is a martingale in L^2
- $\langle \int_0^t \alpha_s dM_s \rangle_t = \int_0^t \alpha_s^2 d\langle M \rangle_s$.

In particular $E[(\int_0^\infty \alpha_s dM_s)^2] = \int_0^\infty \alpha_s^2 d\langle M \rangle_s$ and

$$\alpha \in \mathcal{E} \subset L^2(\langle M \rangle) \rightarrow \int_0^\cdot \alpha_s dM_s \in \text{Mart}_c^2$$

extends uniquely to an isometry $L^2(\langle M \rangle) \rightarrow \text{Mart}_c^2$.

Proof. Let $X_t^i = \alpha_{i-1} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$. Then for $t \geq s$:

$$s \geq t_{i-1} \Rightarrow E[X_t^i | \mathcal{F}_s] = \alpha_{i-1} (\underbrace{E[M_{t_i \wedge t} | \mathcal{F}_s]}_{M_{t_i \wedge s}} - M_{t_{i-1}}) = X_s^i$$

$$\begin{aligned} s \leq t_{i-1} \Rightarrow E[X_t^i | \mathcal{F}_s] &= E[\alpha_{i-1} \underbrace{E[M_{t_i \wedge t} - M_{t_{i-1} \wedge t} | \mathcal{F}_{t_{i-1}}]}_0 | \mathcal{F}_s] \\ &= 0 = X_s^i \end{aligned}$$

Thus X^i is a martingale and so is $\sum_{i=1}^n X^i$.

Also,

$$\|X^i\|_{\text{Mart}^2} = \sup_{t \geq 0} E[|X_t^i|^2] \leq 2 \|\alpha\|_\infty \|M\|_{\text{Mart}^2} < \infty$$

and thus $\sum X^i \in \text{Mart}_c^2$.

Claim: $E[X_\infty^i X_\infty^j] = 0$ if $i < j$

$$E[X_\infty^i X_\infty^j] = \underbrace{E[\alpha_{i-1} (M_{t_i} - M_{t_{i-1}}) \alpha_{j-1} E[M_{t_j} - M_{t_{j-1}} | \mathcal{F}_{t_{j-1}}]]}_{0} \quad t_{j-1} \geq t_i$$

Claim: $E[(X_\infty^i)^2] = E\left[\int_{t_{i-1}}^{t_i} \alpha_s^2 d\langle M \rangle_s\right]$.

$$\begin{aligned} E[(X_\infty^i)^2] &= E\left[\alpha_{i-1}^2 \underbrace{E[(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]}_{E[M_{t_i}^2 - M_{t_{i-1}}^2 | \mathcal{F}_{t_{i-1}}]}\right] \\ &= E[\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}} | \mathcal{F}_{t_{i-1}}] \\ &= E[\alpha_{i-1}^2 (\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}})] \\ &= E\left[\int_{t_{i-1}}^{t_i} \alpha_s^2 d\langle M \rangle_s\right] \end{aligned}$$

Thus $E\left[\left(\int_0^\infty \alpha_s dM_s\right)^2\right] = \sum_{i=1}^n E[(X_\infty^i)^2] = E\left[\int_0^\infty \alpha_s^2 d\langle M \rangle_s\right]$.

3.5. Kunita-Watanabe identity

Defn. For $M, N \in \text{Mart}_c^2$ set (covariation)

$$\langle M, N \rangle = \frac{1}{4}(\langle M+N \rangle - \langle M-N \rangle).$$

Note : $\langle M \rangle = \langle M, M \rangle$.

Exercise.

(i) $\langle M, N \rangle$ is the unique continuous progressively measurable process s.t. $MN - \langle M, N \rangle$ is a martingale and

$$E[(M_\infty - M_0)(N_\infty - N_0)] = E[\langle M, N \rangle_\infty]$$

(ii) For every stopping time τ ,
 $\langle M^\tau, N^\tau \rangle_t = \langle M^\tau, N \rangle_t = \langle M, N \rangle_{t \wedge \tau}.$

(iii) The map $(M, N) \mapsto \langle M, N \rangle$ is bilinear, symmetric.

Example. Let B and B' be indep std. BMs. Then:

- $\langle B, B' \rangle = 0$ since BB' is a martingale.
- Then $B'' = \rho B + (1-\rho^2)^{1/2} B'$ is also a std. BM and
 $\langle B, B'' \rangle_t = \rho t$

Prop. Let $M \in \text{Mart}_c^2$ and $\alpha \in L^2(\langle M \rangle)$. Then $\int_0^t \alpha dM$ is the unique martingale in Mart_c^2 s.t.

$$(*) \quad \underbrace{\left\langle \int_0^t \alpha dM, N \right\rangle_t}_{\text{It\^o integral}} = \underbrace{\int_0^t \alpha d\langle M, N \rangle}_{\text{Lebesgue-Stieltjes integral}} \quad \forall N \in \text{Mart}_c^2.$$

Proof. Let $\alpha = \sum_{i=1}^n \alpha_{i-1} 1_{(t_{i-1}, t_i]}$ be a simple process and

$$X_t^i = \alpha_{i-1} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$$

s.t. $\int_0^t \alpha dM = \sum_{i=1}^n X_t^i.$

Then

$$\begin{aligned} \langle X^i, N \rangle_t &= \alpha_{i-1} \langle M_{t_i \wedge \cdot} - M_{t_{i-1} \wedge \cdot}, N \rangle_t \\ &= \alpha_{i-1} (\langle M, N \rangle_{t_i \wedge t} - \langle M, N \rangle_{t_{i-1} \wedge t}) = \int_{t_{i-1} \wedge t}^{t_i \wedge t} \alpha_s d\langle M, N \rangle_s. \end{aligned}$$

$$\Rightarrow \left\langle \int_0^t \alpha dM, N \right\rangle = \sum \langle X^i, N \rangle_t = \int_0^t \alpha_s d\langle M, N \rangle_s.$$

Thus $(*)$ holds for $\alpha \in \mathcal{E}$. For $\alpha \in L^2(\langle M \rangle)$, let $\alpha_k \in \mathcal{E}$ be s.t. $\alpha_k \rightarrow \alpha$ in $L^2(\langle M \rangle)$. Then

$$\int \alpha_k dM \rightarrow \int \alpha dM \quad \text{in } \text{Mart}_c^2$$

It suffices to justify the following limits:

$$\begin{aligned} \langle \int \alpha dM, N \rangle_\infty &\stackrel{(+)}{=} \lim \langle \int \alpha_k dM, N \rangle_\infty \quad \text{in } L^1 \\ &= \lim \int_0^\infty \alpha_k d\langle M, N \rangle \quad \text{above} \\ &\stackrel{(+)}{=} \int_0^\infty \alpha d\langle M, N \rangle \quad \text{in } L^1 \end{aligned}$$

Then existence for $t < \infty$ follows by replacing N by N^t .

$$\begin{aligned} \text{Re } (+): \quad &E \left[\left| \int \alpha dM - \int \alpha_k dM, N \rangle_\infty \right|^2 \right] \\ &\leq E \left[\left(\int (\alpha - \alpha_k) dM \right)_\infty^2 \right]^{1/2} E \left[\langle N \rangle_\infty \right]^{1/2} \quad \text{KW ineq.} \\ &= E \left[\int_0^\infty (\alpha - \alpha_k)^2 d\langle M \rangle \right]^{1/2} \|N\|_{\text{Mart}^2} \quad \text{It\^o isometry} \\ &\rightarrow 0 \end{aligned}$$

and also by the KW inequality

$$\int_0^\infty (\alpha_k - \alpha) d\langle M, N \rangle \leq \|\alpha_k - \alpha\|_{L^2(\langle M \rangle)} \|N\|_{\text{Mart}^2} \rightarrow 0$$

Uniqueness: Suppose $X \in \text{Mart}_c^2$ satisfies $\langle X, N \rangle = \int \alpha d\langle M, N \rangle$

$$\Rightarrow \langle X - \int \alpha dM, N \rangle = 0 \quad \forall N$$

$$\Rightarrow \langle X - \int \alpha dM \rangle = 0 \Rightarrow X = \int \alpha dM.$$

Kunita-Watanabe inequality:

$$\int_0^\infty |\alpha_s| |\beta_s| |d\langle M, N \rangle_s| \leq \left(\int_0^\infty \alpha_s^2 d\langle M \rangle_s \right)^{1/2} \left(\int_0^\infty \beta_s^2 d\langle N \rangle_s \right)^{1/2} \text{ a.s.}$$

Idea: Approximate integrals by sums and apply CS.

Cor. Let τ be a stopping time. Then

$$\int_0^\infty \alpha 1_{[0, \tau]} dM = \int_0^\tau \alpha dM = \int_0^\infty \alpha dM^\tau.$$

Proof. Exercise.

Cor.

$$\left\langle \int_0^\cdot \alpha dM, \int_0^\cdot \beta dN \right\rangle_t = \int_0^t \alpha \beta d\langle M, N \rangle_s.$$

$$E\left[\int_0^t \alpha dM \int_0^t \beta dN\right] = E\left[\int_0^t \alpha \beta d\langle M, N \rangle_s\right].$$

Proof.

$$\begin{aligned} \left\langle \int_0^\cdot \alpha dM, \int_0^\cdot \beta dN \right\rangle_t &\stackrel{(KW)}{=} \int_0^t \alpha d\langle M, \underbrace{\int_0^\cdot \beta dN}_{\stackrel{(KW)}{=} \int_0^\cdot \beta d\langle M, N \rangle} \rangle_s \stackrel{(LS)}{=} \int_0^t \alpha \beta d\langle M, N \rangle_s \end{aligned}$$

3.6. Itô integral for local and semimartingales

Defn. A process M is a local martingale if there are stopping times τ_n s.t. $\tau_n \uparrow \infty$ a.s. and M^{τ_n} is a (uniformly integrable) martingale.

Example. (i) Every martingale is a local martingale: with $\tau_n = n \wedge \inf\{t : |M_t| \geq n\}$.

(ii) Let (B_t) be a 3-dim. BM. Then $1/|B_t|$ is a local martingale but not a martingale.

Sketch. Recall that for $f \in C_b^2$,

$$M_t^f = f(B_t) - f(B_1) - \frac{1}{2} \int_1^t \Delta f(B_s) ds$$

is a martingale, but $f(x) = 1/|x|$ is not C_b^2 .

But if $\tau_n = \inf\{t \geq 1 : |B_t| \leq \frac{1}{n}\}$ we can replace f by $f_n \in C_b^2$ s.t. $f(B_{t \wedge \tau_n}) = f_n(B_{t \wedge \tau_n})$ and thus $f(B_{t \wedge \tau_n})$ is a martingale.

That $\tau_n \rightarrow \infty$ is essentially the same argument as in the proof of transience.

To see M^f is not a martingale, note $\sup_t E[f(B_t)^2] < \infty$ but $E[f(B_t)] \rightarrow 0$.

Exercise. Let M be a continuous local martingale. Then there is an a.s. unique continuous increasing adapted process $\langle M \rangle$ s.t. $M^2 - \langle M \rangle$ is a cont. local martingale.

Idea. Use L^2 case and uniqueness.

Defn. Given an increasing process A define $L^2_{\text{loc}}(A)$ as the space of progressively measurable α s.t.

$$\forall T > 0: \int_0^T \alpha^2 dA < \infty \text{ a.s.}$$

A progressively measurable process α is locally bd if

$$\forall T > 0: \sup_{t \leq T} |\alpha(t)| < \infty \text{ a.s.}$$

Thm. Let M be a continuous local martingale.

Then for every $\alpha \in L^2_{\text{loc}}(\langle M \rangle)$ there is a unique continuous local martingale $X_t = \int_0^t \alpha dM$ s.t. $X_0 = 0$

$$\langle X, N \rangle = \int \alpha d\langle M, N \rangle \quad \text{for all cont. loc. mart } N.$$

Proof. Assume $M_0 = 0$ and (possibly changing on null set)
 $\int_0^t \alpha^2 d\langle M \rangle < \infty$ for all $t > 0$, $\omega \in \Omega$.

Set

$$\tau_n = \inf \{ t \geq 0 : \int_0^t (1 + \alpha^2) d\langle M \rangle > n \}.$$

Then τ_n is a stopping time, $\tau_n \uparrow \infty$ a.s.,
 $\langle M^{\tau_n} \rangle = \langle M \rangle_{\tau_n} \leq n$.

Thus $M^{\tau_n} \in \text{Mart}_c^2$ and $\int_0^\infty \alpha^2 d\langle M^{\tau_n} \rangle \leq n$.

$\Rightarrow \alpha \in L^2(\langle M^{\tau_n} \rangle)$ and $\int_0^\cdot \alpha dM^{\tau_n}$ is already defined

Check: $\int \alpha dM^{\tau_n} = (\int \alpha dM^{\tau_n})^{\tau_n}$ for $m > n$.

Thus one can define a consistent process $\int \alpha dM$.
and it is by construction a local martingale.

The Kunita-Watanabe characterization follows
similarly (apply L^2 version to M^{τ_n}, N^{τ_n}).

Defn. A (right-)continuous adapted process X is a (right-)continuous semimartingale if

$$X = X_0 + M + A$$

with $X_0 \in \mathcal{F}_0$, M a (right-)continuous martingale and A a continuous process that is locally of bounded variation

Rk. The decomposition is unique.

Defn. For a continuous semimartingale $X = X_0 + M + A$,
 $\langle X \rangle = \langle M \rangle$.

and similarly for the covariation. For a locally bounded process, set

$$\int \alpha dX = \int \alpha dM + \int \alpha dA$$

$\uparrow$ $\uparrow$
 Itô integral Lebesgue-Stieltjes integral

3.7. Approximation of Itô integrals

Prop. (Stochastic DCT). Let M be a continuous local martingale and α, α^n locally bounded processes, and β a progressively measurable process. Assume a.s.

$$(i) \alpha_s^n \rightarrow \alpha_s \quad \forall s < t$$

$$(ii) |\alpha_s^n| \leq \beta_s \quad \forall s < t, n \in \mathbb{N}$$

$$(iii) \int_0^t \beta_s^2 d\langle M \rangle_s + \int_0^t \beta_s dA_s < \infty$$

Then: $\int_0^t \alpha_s^n dX_s \rightarrow \int_0^t \alpha_s dX_s$ in probability.

Proof. The usual DCT implies

$$\int_0^t \alpha_s^n dA_s \rightarrow \int_0^t \alpha_s dA_s.$$

Set $\tau_m = \inf \{ t \geq 0 : \int_0^t \beta_s^2 d\langle M \rangle_s > m \}$. Then

$$\mathbb{E} \left[\left(\int_0^{t \wedge \tau_m} \alpha_s^n dM_s - \int_0^{t \wedge \tau_m} \alpha_s dM_s \right)^2 \right] = \mathbb{E} \left[\int_0^{t \wedge \tau_m} (\alpha_s - \alpha_s^n)^2 d\langle M \rangle_s \right] \xrightarrow{\text{a.s.}} 0.$$

Since $t \wedge \tau_m = t$ eventually, a.s., the convergence for fixed t follows in probability.

Cor. Let X be a continuous semimartingale and let α be a locally bounded left-continuous process. Then for a sequence of partitions (t_i^m) s.t. $\Delta(t^m) \rightarrow 0$:

$$\int_0^t \alpha dX = \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} \alpha_{t_{i-1}^m} (X_{t_i^m} - X_{t_{i-1}^m})$$

Proof. Similar to bounded variation case (using now stochastic DCT).

Example.

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} X_{t_{i-1}^m} (X_{t_i^m} - X_{t_{i-1}^m}) = \int_0^t X dX$$

but

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} X_{t_i^m} (X_{t_i^m} - X_{t_{i-1}^m}) &= \int_0^t X dX + \lim_{m \rightarrow \infty} \sum (X_{t_i} - X_{t_{i-1}})^2 \\ &= \int_0^t X dX + \langle X, X \rangle_t \end{aligned}$$

Defn. For X, Y continuous semimartingales, the Stratonovich integral is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2} \langle X, Y \rangle_t$$

It corresponds to

$$\sum_{i=1}^{n_m} \frac{1}{2} (X_{t_i^m} + X_{t_{i-1}^m}) (X_{t_i^m} - X_{t_{i-1}^m})$$

3.8. Itô formula

Thm. Let $X^1, \dots, X^p$ be continuous semimartingales. Then for every $f \in C^2(\mathbb{R}^p)$, for every $t \geq 0$,

$$F(X_t^1, \dots, X_t^p) = F(X_0^1, \dots, X_0^p) + \sum_{i=1}^p \frac{\partial F}{\partial x^i}(X_s^1, \dots, X_s^p) dX_s^i + \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^j}(X_s^1, \dots, X_s^p) d\langle X^i, X^j \rangle_s.$$

Proof for $p=1$. Write $X=X^1$. Let $(t_i^m)_{i=1}^{n_m}$ be nested partitions of $[0, t]$ with step size $\rightarrow 0$. Then

$$F(X_t) = F(X_0) + \sum_{i=1}^{n_m} \underbrace{(F(X_{t_i^m}) - F(X_{t_{i-1}^m}))}_{\begin{aligned} &F'(X_{t_{i-1}^m})(X_{t_i^m} - X_{t_{i-1}^m}) \\ &+ \frac{1}{2} F''(X_{t_{i-1}^m} + \theta(X_{t_i^m} - X_{t_{i-1}^m}))(X_{t_i^m} - X_{t_{i-1}^m})^2 \end{aligned}} \\ \text{for some } \theta \in [0, 1].$$

First term:

$$\sum F'(X_{t_{i-1}^m})(X_{t_i^m} - X_{t_{i-1}^m}) \rightarrow \int_0^t F'(X_s) dX_s$$

Second term:

$$\sup_i |F''(X_{t_{i-1}^m} + \theta(X_{t_i^m} - X_{t_{i-1}^m})) - F''(X_{t_{i-1}^m})| \rightarrow 0 \text{ a.s.}$$

by continuity of X and uniform cont. of F'' on compact intervals.

Since $\sum (X_{t_i^m} - X_{t_{i-1}^m})^2 \rightarrow \langle X \rangle_t$ in probability, hence suffices to show that (in probability)

$$\sum F''(X_{t_{i-1}^m}) (X_{t_i^m} - X_{t_{i-1}^m})^2 \rightarrow \int_0^t F''(X_s) d\langle X \rangle_s.$$

Let $\mu_n(ds) = \sum_{i=1}^m (X_{t_i^m} - X_{t_{i-1}^m})^2 \delta_{t_{i-1}^m}(ds)$

Let $D = \bigcup_{i=1}^n \{t_i^n\}$ and note $D \subset [0, t]$ is dense.

$$\forall s \in D: \mu_n([0, s]) \rightarrow \langle X \rangle_s \text{ in prob.}$$

Thus along a subsequence of n

$$\forall s \in D: \mu_n([0, s]) \rightarrow \langle X \rangle_s \text{ a.s.}$$

$$\Rightarrow \mu_n \xrightarrow{w} 1_{[0, t]}(s) d\langle X \rangle_s \text{ weakly a.s.}$$

$$\Rightarrow \int F''(X_s) \mu_n(ds) \longrightarrow \int_0^t F''(X_s) d\langle X \rangle_s \text{ a.s.}$$

Proof for $p > 1$. Let $X = (X^1, \dots, X^p)$. Then

$$\begin{aligned} F(X_{t_i^m}) - F(X_{t_{i-1}^m}) &= \sum_k \frac{\partial F}{\partial X^k}(X_{t_{i-1}^m}) (X_{t_i^m}^k - X_{t_{i-1}^m}^k) \\ &\quad + \sum \frac{\partial^2 F}{\partial X^k \partial X^l} (\dots) \end{aligned}$$

The rest is similar.

Cor. (Integration by parts).

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

In particular,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X \rangle_t$$

Proof. $F(x, y) = xy$.

Rk. If M is a cont. local martingale, then $M^2 - \langle M \rangle$ is a cont. local martingale. By above,

$$M^2 - \langle M \rangle = M_0^2 + 2 \int_0^t M_s dM_s$$

Example. If $X = B$ is a std. BM,

$$F(B_t) = F(B_0) + \int_0^t F'(B_s) dB_s + \frac{1}{2} \int_0^t F''(B_s) ds.$$

If $X^1 = t$ and $X^2 = B$ then

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(B_s) dB_s + \int_0^t \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) F(s, B_s) ds$$

Thus (as seen before)

$$F(t, B_t) - F(0, B_0) + \int_0^t \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) F(s, B_s) ds \text{ is a mart.}$$

Example Let $M \in \text{Mart}_c^{\text{loc}}$ and $\lambda \in \mathbb{C}$. Then

$$E(\lambda M)_t = \exp\left(\lambda M_t - \frac{1}{2} \lambda^2 \langle M \rangle_t\right)$$

is in $\text{Mart}_c^{\text{loc}}$ and

$$E(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t E(\lambda M)_s dM_s.$$

Proof. Let $F(r, x) = e^{\lambda x - \frac{1}{2} \lambda^2 r}$. Then $\frac{\partial F}{\partial r} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0$.
Thus

$$\begin{aligned} F(\langle M \rangle_t, M_t) &= F(0, M_0) + \int_0^t \underbrace{\frac{\partial F}{\partial x}(\langle M \rangle_s, M_s)}_{\lambda E(\lambda M)_s} dM_s \\ &\quad + \underbrace{\int_0^t \left(\frac{\partial F}{\partial r} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right)(\langle M \rangle_s, M_s) ds}_0 \end{aligned}$$

Rk. In terms of the Stratonovich integral, the Itô formula reads

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) \circ dX_s.$$

Indeed,

$$\int_0^t F'(X_s) \circ dX_s = \int_0^t F'(X_s) dX_s + \frac{1}{2} \langle F'(X), X \rangle_t$$

$$\langle \int_0^t F'(X) dX, X \rangle_t = \int_0^t F''(X_s) d\langle X \rangle_s.$$

Rk. (Computation rules).

$$Z_t - Z_0 = \int_0^t \alpha_s dX_s \Leftrightarrow dZ = \alpha dX$$

$$Z_t - Z_0 = \langle X, Y \rangle_t = \int_0^t d\langle X, Y \rangle_s \Leftrightarrow dZ = dX dY$$

Itô's formula:

$$dF(X_t) = \sum \frac{\partial F}{\partial x^i} dX^i + \frac{1}{2} \sum \frac{\partial^2 F}{\partial x^i \partial x^j} dX^i dX^j$$

Brownian motion: $(dB)^2 = d\langle B \rangle = dt$

$$dF(B_t) = F'(B_t) dB + \frac{1}{2} F''(B_t) dt$$

Associativity:

$$\alpha_t (\beta_t dX_t) = (\alpha_t \beta_t) dX_t \Leftrightarrow \int_0^t \alpha_s \int_0^s \beta_u dX_u = \int_0^t \alpha_s \beta_s dX_s$$

Kunita-Watanabe identity:

$$\alpha_t dX_t dY_t = (\alpha_t dX_t) dY_t \Leftrightarrow \int_0^t \alpha_s d\langle X, Y \rangle_s = \langle \int_0^t \alpha_s dX_s, Y \rangle_t$$

Integration by parts:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

4. Applications to Brownian motion & martingales

4.1. Levy's characterization

Thm. Let $X = (X^1, \dots, X^d)$ be in $(\text{Mart}_c^{\text{loc}})^d$, $X_0 = 0$, and $\langle X^i, X^j \rangle_t = \delta_{ij} t$.

Then X is a standard d -dim. Brownian motion.

Proof. X is a BM iff, for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} E[f(X_t) | \mathcal{F}_s] &= T_{t-s} f(X_s) = f(X_s) + \int_s^t T_u f(X_u) du \\ &= E\left[f(X_s) + \int_s^t \frac{1}{2} \Delta f(X_u) du \mid \mathcal{F}_s\right] \end{aligned}$$

But by Itô's formula and $\langle X^i, X^j \rangle_t = \delta_{ij} t$,

$$f(X_t) - f(X_s) = \int_s^t \nabla f(X_u) dX_u + \frac{1}{2} \int_s^t \Delta f(X_u) du$$

Thus $f(X_t) - f(X_s) - \frac{1}{2} \int_s^t \Delta f(X_u) du$ is a local martingale and bounded, so a martingale.

4.2. Girsanov's Theorem

Example. Let X be an n -dim. Gaussian vector with covariance matrix C

$$E[f(X)] \propto \int e^{-\frac{1}{2}(x, Ax)} f(x) dx, \quad A = C^{-1}.$$

$$\begin{aligned} \Rightarrow E[f(X+y)] &\propto \int e^{-\frac{1}{2}(x-y, A(x-y))} f(x) dx \\ &= E[Z f(X)], \end{aligned}$$

where

$$Z = e^{(y, AX) - \frac{1}{2}(y, Ay)} = e^{Y - \frac{1}{2} \text{Var}(Y)}$$

$$\begin{aligned} \text{since } \text{Var}(Y) &= \text{Var}((Ay, X)) \\ &= (Ay, C Ay) = (y, Ay). \end{aligned}$$

Let $M \in \text{Mart}_c^{\text{loc}}$. The stochastic exponential is

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2} \langle M \rangle_t}.$$

By Itô's formula, $Z = \mathcal{E}(M) \in \text{Mart}_c^{\text{loc}}$ and (if $M_0 = 0$)

$$dZ_t = Z_t dM_t, \quad \text{i.e. } Z_t = 1 + \int_0^t Z_s dM_s.$$

(This is an example of an SDE.)

Thm. Let $L \in \text{Mart}_c^{\text{loc}}$ with $L_0 = 0$. Assume $E(L)$ is uniformly integrable. Define a probability measure Q by

$$\frac{dQ}{dP} = E(L)_{\infty}.$$

Then if $M \in \text{Mart}_c^{\text{loc}}$ with respect to P , then $\tilde{M} = M - \langle M, L \rangle \in \text{Mart}_c^{\text{loc}}$ with respect to Q .

Proof. Let $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then T_n is a stopping time and $T_n \rightarrow \infty$ P -a.s. by continuity of M . Since $Q \ll P$, also $T_n \rightarrow \infty$ Q -a.s. It thus suffices to show M^{T_n} is in $\text{Mart}_c^{\text{loc}}(Q)$.

Let $Y = M^{T_n} - \langle M^{T_n}, L \rangle$ and $Z = E(L)$. Then

$$\begin{aligned} d(ZY)_+ &= Y_t dZ_t + Z_t dY_t + d\langle Z, Y \rangle_t \\ &= Y_t Z_t dL_t + Z_t (dM_t^{T_n} - d\langle M^{T_n}, L \rangle_t) \\ &\quad + Z_t d\langle M^{T_n}, L \rangle_t \end{aligned}$$

since $d\langle Z, Y \rangle = d\langle Z, M^{T_n} \rangle = Z d\langle L, M^{T_n} \rangle$.

$$\Rightarrow d(ZY) = (ZY)dL + ZdM^{T_n}$$

Since L and M^{T_n} are in $\text{Mart}_c^{\text{loc}}(P)$, so is ZY .

Since Z is UI by assumption and Y is bounded, it follows that ZY is UI, so a true martingale.

Finally, we show Y is a Q -martingale:

$$\begin{aligned} E^Q[Y_t - Y_s | \mathcal{F}_s] &= E^P[Z_\infty(Y_t - Y_s) | \mathcal{F}_s] \\ &= E^P[Z_t(Y_t - Y_s) | \mathcal{F}_s] = 0. \end{aligned}$$

Thus $Y = (M - \langle M, L \rangle)^{\hat{T}_n}$ is a Q -martingale and $T_n \rightarrow \infty$ Q -a.s. Thus $M - \langle M, L \rangle \in \text{Mart}_c^{\text{loc}}(Q)$.

Cor. Let B be a std. BM with respect to P . Let $L \in \text{Mart}_c^{\text{loc}}$ with $L_0 = 0$ be s.t. $E(L)$ is UI. Then $\tilde{B} = B - \langle B, L \rangle$ is a std. BM with respect to Q .

Proof. By Girsanov's Theorem, $\tilde{B} \in \text{Mart}_c^{\text{loc}}(Q)$. But $\langle \tilde{B} \rangle = \langle B \rangle = t$, so Levy's characterization implies that $\tilde{B}$ is std. BM with respect to Q .

When is $E(L)$ uniformly integrable?

Prop. Suppose $\langle L \rangle_\infty \leq C$. Then $P[\sup_{t \geq 0} L_t \geq a] \leq e^{-\frac{a^2}{2C}}$. In particular, $E(L)$ is U.I.

Proof. Let $T = \inf \{t \geq 0 : L_t \geq a\}$ and $\theta \in \mathbb{R}$. Then

$$Z_t = e^{\theta L_t^T - \frac{1}{2}\theta^2 \langle L \rangle_t}$$

is a bounded martingale. Thus $E[Z_\infty] = E[Z_0] = 1$.

$$\begin{aligned} \Rightarrow P[\sup L_t \geq a] &= P[L_\infty^T \geq a] \\ &= P[Z_\infty \geq e^{\theta a - \frac{1}{2}\theta^2 C}] \\ &\leq e^{-\theta a + \frac{1}{2}\theta^2 C} \quad \forall \theta \end{aligned}$$

$$\Rightarrow P[\sup L_t \geq a] \leq e^{-\frac{a^2}{2C}}.$$

Consequently,

$$\begin{aligned} E[\sup_t \varepsilon(L)_t] &\leq E[\exp(\sup_t L_t)] \\ &= \int_0^\infty P[\exp(\sup_t L_t) > \lambda] d\lambda \\ &= 1 + \int_1^\infty e^{-\frac{(\log \lambda)^2}{2C}} d\lambda < \infty. \end{aligned}$$

Thus $\varepsilon(L)_t \leq \sup_t \varepsilon(L)_t \in L^1$.

Novikov's condition. Let $M \in \text{Mart}_c^{\text{loc}}$ and $M_0 = 0$. Then $E[e^{\frac{1}{2}\langle M \rangle_\infty}] < \infty$ implies M is UI.

Example. (Weak solns. of SDEs). Consider the SDE
$$dX_t = b(X_t) dt + dB_t, \quad t \leq T.$$

A weak soln. can be constructed as follows.

Let X be a std. BM under the measure P . Set

$$L_t = \int_0^{t \wedge T} b(X_s) dX_s.$$

Assume that $E(L)$ is U.I., for example b bounded

$$\begin{aligned} \Rightarrow B_t &= X_t - \langle X, L \rangle_t = X_t - \int_0^{t \wedge T} b(X_s) d\langle X \rangle_s \\ &= X_t - \int_0^{t \wedge T} b(X_s) ds \end{aligned}$$

is a std. BM under Q , where

$$\frac{dQ}{dP} = E(L)_\infty$$

and

$$X_t = \int_0^{t \wedge T} b(X_s) ds + B_t$$

Specializing to $M = \text{BM}$ and deterministic shifts one obtains the Cameron-Martin formula.

Defn. The Cameron-Martin space (associated with Brownian motion) is

$$\mathcal{H} = \left\{ h: [0, \infty) \rightarrow \mathbb{R} : h(t) = \int_0^t \dot{h}(s) ds, \dot{h} \in L^2([0, \infty)) \right\}.$$

Rk. $\mathcal{H}$ is a Hilbert space with inner product

$$(h, f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s) \dot{f}(s) ds$$

and dual space the completion of

$$\left\{ \mu \in M([0, \infty)) : \int_0^\infty (s \wedge t) \mu(ds) \mu(dt), \mu(\{0\}) = 0 \right\}.$$

See Homework Sheet 2.

Thm (Cameron-Martin formula). Let $h \in \mathcal{H}$ and define P^h by $P^h(A) = P(A-h)$ where P is the Wiener measure on $C[0, \infty)$. Then

$$\frac{dP^h}{dP} = \exp\left(\int_0^\infty \dot{h}(s) dX_s - \frac{1}{2} \int_0^\infty \dot{h}(s)^2 ds\right).$$

Proof. Girsanov with $L_t = \int_0^t \dot{h}(s) dX_s$. Note $\langle L \rangle_t = \int_0^t \dot{h}(s)^2 ds \leq \|h\|_{\mathcal{H}}^2$, so $E(L)$ is UI.

4.3. Burkholder-Davis-Gundy inequality

Thm. Let $M \in \text{Mart}_c^{\text{loc}}$, $M_0 = 0$, and $M_t^* = \sup_{s \leq t} |M_s|$.
 For any $p > 0$, there are C_p, C_p
 (independent of M) s.t. for every stopping time τ ,

$$C_p E \langle M \rangle_\tau^{p/2} \leq E |M_\tau^*|^p \leq C_p E \langle M \rangle_\tau^{p/2}.$$

Proof. (upper bound, $p \geq 2$). It suffices to prove the inequality if M is bounded and $T = \infty$.

Let $f(x) = |x|^p$. Then $f \in C^2$ and

$$f'(x) = p \operatorname{sgn}(x) |x|^{p-1}, \quad f''(x) = p(p-1) |x|^{p-2}.$$

By Itô:

$$|M_t|^p = \underbrace{\int_0^t p \operatorname{sgn}(M_s) |M_s|^{p-1} dM_s}_{\text{martingale}} + \frac{1}{2} \int_0^t p(p-1) |M_s|^{p-2} d\langle M \rangle_s$$

$$\begin{aligned} \Rightarrow E[|M_t|^p] &= \frac{1}{2} p(p-1) E \left[\int_0^t |M_s|^{p-2} d\langle M \rangle_s \right] \\ &\leq |M_t^*|^{p-2} \langle M \rangle_t \\ &\stackrel{\text{Hölder}}{\leq} E[(M_t^*)^p]^{p/2} E[\langle M \rangle_t^{p/2}]^{2/p} \end{aligned}$$

Doob's L^p inequality:

$$E[|M_t^*|^p] \leq \left(\frac{p}{p-1}\right)^p E[|M_t|^p] \leq C_p E[(M_t^*)^p]^{\frac{p-2}{p}} E[\langle M \rangle_t^{p/2}]^{\frac{2}{p}}$$

$$\Rightarrow E[|M_t^*|^p] \leq C_p E[\langle M \rangle_t^{p/2}]$$

Take $t \rightarrow \infty$.

4.4. Dubins-Schwarz Theorem

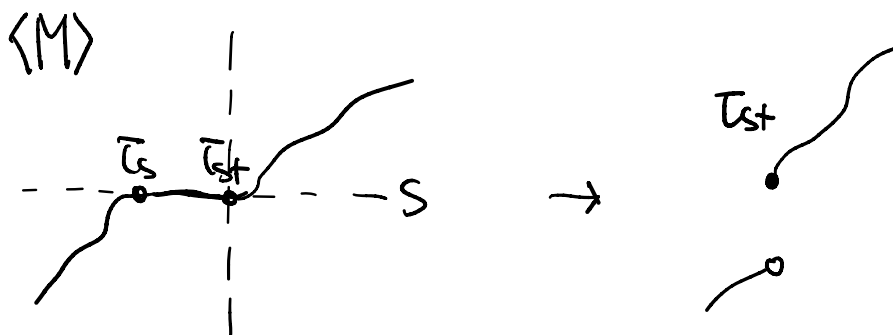
Thm. Let $M \in \text{Mart}_c^{\text{loc}}$ with $M_0 = 0$, $\langle M \rangle_\infty = \infty$.
Then there is a std. Brownian motion B s.t. a.s.

$$\forall t \geq 0 : M_t = B_{\langle M \rangle_t}$$

More precisely, let

$$\tau_s = \inf\{t \geq 0 : \langle M \rangle_t \geq s\}$$

$$\tau_{s+} = \inf\{t \geq 0 : \langle M \rangle_t > s\}$$



Then $B_s = M_{\tau_s} = M_{\tau_{s+}}$ and $B_s \in \mathcal{G}_s = \mathcal{F}_{\tau_s}$.

Lemma. A.s., for all $u < v$

M is const. on $[u, v] \Leftrightarrow \langle M \rangle$ is const. on $[u, v]$

Proof. Since $\langle M \rangle$ is continuous and increasing, it suffices to show that for all $u < v$ fixed,

$$\{M_t = M_u \ \forall t \in [u, v]\} = \{\langle M \rangle_u = \langle M \rangle_v\} \quad \text{a.s.}$$

Let

$$N_t = M_t - M_{t \wedge u} = \int_{t \wedge u}^t dM_s$$

$$\Rightarrow \langle N \rangle_t = \int_{t \wedge u}^t d\langle M \rangle_s = \langle M \rangle_t - \langle M \rangle_{t \wedge u}.$$

Let $T_\varepsilon = \inf\{t \geq 0 : \langle N \rangle_t > \varepsilon\}$. Then $N^{T_\varepsilon} \in \text{Mart}_c^2$ since $\langle N^{T_\varepsilon} \rangle \leq \varepsilon$ and $E[(N^{T_\varepsilon})^2] \leq E[\langle N^{T_\varepsilon} \rangle_t] \leq \varepsilon$.

$$\begin{aligned} \Rightarrow E[1_{\langle M \rangle_u = \langle M \rangle_v} N_t^2] &= E[1_{\langle N \rangle_v = 0} N_t^2] \\ &= E[1_{\langle N \rangle_v = 0} N_{t \wedge T_\varepsilon}^2] \leq \varepsilon \\ &\quad \text{for } t \in [u, v]. \end{aligned}$$

$\Rightarrow N_t = 0$ a.s. on $\{\langle M \rangle_u = \langle M \rangle_v\}$ for $t \in [u, v]$.

Thus $\langle M \rangle_v = \langle M \rangle_u$ implies M is const. on $[u, v]$.
Other direction: exercise.

Proof (theorem). We show that B_s and $B_s^2 - s$ are $(\mathcal{G}_s)$ martingales. The claim then follows from Levy's characterization of Brownian mot.

Since $\langle M^{T_s} \rangle_\infty = \langle M \rangle_{T_s} = s < \infty$, $M^{T_s} \in \text{Mart}_c^2$ and $(M^2 - \langle M \rangle)^{T_s}$ is a uniformly integrable martingale.

Hence, by OST, if $r < s$,

$$E[B_s | \mathcal{G}_r] = E[M_{\alpha}^{\tau_s} | \mathcal{F}_{\tau_r}] = M_{\tau_r} = B_r$$

$$\begin{aligned} E[B_s^2 - s | \mathcal{G}_r] &= E[(M^2 - \langle M \rangle)^{\tau_s} | \mathcal{F}_{\tau_r}] \\ &= M_{\tau_r}^2 - \langle M \rangle_{\tau_r} = B_r^2 - r \end{aligned}$$

as claimed.

5. Stochastic Differential Equations

5.1. Weak and strong solutions

Defn. Let $d, m \in \mathbb{N}$, $b: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$, $\sigma: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be locally bounded. A weak solution to the SDE

$$dX = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

consists of

- a filtered probability space $(\Omega, (\mathcal{F}_t), P)$ (assuming completeness)
- an m -dimensional Brownian motion B ;
- an adapted continuous process $\mathbb{R}^d$ -valued process X s.t.

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds.$$

Defn. For a strong solution, one is given a probability space $(\Omega, \mathcal{F}, P)$ and a Brownian motion B , and takes the completed filtration of B . Then a strong solution is an adapted process X satisfying the same equation.

Defn. For the SDE, there is

- uniqueness in law (or weak uniqueness) if all weak solutions with $X_0 = x \in \mathbb{R}^d$ have the same law.
- pathwise uniqueness (or strong uniqueness) if for fixed $(\Omega, (\mathcal{F}_t), P)$ and B , all solutions are indistinguishable (a.s. equal for all t).

Example (Tanaka). Let X be a std. BM, $X_0 = x$.

Set

$$B_t = \int_0^t \text{sign}(X_s) dX_s$$

$$\Rightarrow x + \int_0^t \text{sign}(X_s) dB_s = x + \int_0^t \text{sign}(X_s)^2 dX_s = X_t.$$

Thus $dX_t = \text{sign}(X_t) dB_t$, $X_0 = x$.

Moreover, X is a std. BM since $\langle B \rangle_t = \int_0^t d\langle X_s \rangle = t$.

In fact, any soln is a std BM, so uniqueness in law holds.

To show that pathwise uniqueness fails, we will show that if X is a soln. with $X_0 = 0$, $-X$ is also one.

Indeed,

$$\begin{aligned} -X_t &= -\int_0^t \text{sign}(X_s) dB_s \\ &= \int_0^t \text{sign}(-X_s) dB_s + 2 \int_0^t 1_{X_s=0} dB_s \end{aligned}$$

Exercise: $\int 1_{X_s=0} dB_s = 0$.

Rk. X is not a strong solution.

Thm (Yamada-Watanabe). Assume the SDE (*) has a weak solution with $X_0 = x$ and that pathwise uniqueness holds. Then uniqueness in law holds and for any $(\Omega, (\mathcal{F}_t), P)$ and B there is a unique strong solution.

Proof omitted.

Thm. Suppose b and σ are locally Lipschitz: there are $K_n > 0$ s.t. for all $t \geq 0$, $|x|, |y| \leq n$,

$$|b(t, x) - b(t, y)| \leq K_n |x - y|$$

$$|\sigma(t, x) - \sigma(t, y)| \leq K_n |x - y|.$$

Then pathwise uniqueness holds.

Proof. Let X and $\tilde{X}$ be two solutions to $E(\sigma, b)$ defined on the same probability space such that $X_0 = \tilde{X}_0$ a.s. Let $T_n = \inf\{t \geq 0: |X_t| \geq n \text{ or } |\tilde{X}_t| \geq n\}$,

$$f_n(t) = E(|X_{t \wedge T_n} - \tilde{X}_{t \wedge T_n}|^2).$$

By continuity of X and $\tilde{X}$ it suffices to show that for all n and t , one has $f_n(t) = 0$.

By Itô's formula,

$$\begin{aligned} |X_{t \wedge T_n} - \tilde{X}_{t \wedge T_n}|^2 &= \int_0^{t \wedge T_n} \underbrace{2(X_s - \tilde{X}_s) \cdot (b(s, X_s) - b(s, \tilde{X}_s))}_{\leq K_n |X_s - \tilde{X}_s|^2} ds \\ &\quad + \int_0^{t \wedge T_n} 2(X_s - \tilde{X}_s) \cdot (\sigma(s, X_s) - \sigma(s, \tilde{X}_s)) dB_s \\ &\quad + \int_0^{t \wedge T_n} \underbrace{2|\sigma(s, X_s) - \sigma(s, \tilde{X}_s)|^2}_{\leq K_n^2 |X_s - \tilde{X}_s|^2} ds \end{aligned}$$

Since $(X_s - \tilde{X}_s) \cdot (\sigma(s, X_s) - \sigma(s, \tilde{X}_s))$ is bounded on $s \leq T_n$, the middle term is a mean 0 martingale. Thus

$$f_n(t) \leq 2(K_n + K_n^2) \int_0^t f_n(s) ds$$

Since $f_n(0) = 0$ and f_n is bounded, Gronwall's inequality implies $f_n(t) = 0$ for all t and n .

Gronwall's inequality. ($\rightarrow$ Homework). Let $T > 0$ and let $f: [0, T] \rightarrow [0, \infty)$ be a bounded Borel function. Then

$$f(t) \leq a + b \int_0^t f(s) ds \quad \text{for all } t \leq T$$

implies

$$f(t) \leq a e^{bt} \quad \text{for all } t \leq T.$$

Rk. The proof of the theorem also shows that solutions defined up to time T must agree up to time T .

5.2. Strong existence for Lipschitz coefficients

Thm. Assume b and σ are globally Lipschitz, i.e., there is $K > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \geq 0$,

$$|b(t, x) - b(t, y)| \leq K |x - y|$$

$$|\sigma(t, x) - \sigma(t, y)| \leq K |x - y|.$$

For any $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, any $(\mathcal{F}_t)$ -Brownian motion B , any $x \in \mathbb{R}^d$, there is a strong solution with $X_0 = x$.

Proof. To simplify notation, assume $d=m=1$. Define

$$F(X)_t = x + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds.$$

Given any $(\mathcal{F}_t)$ we will find an $(\mathcal{F}_t)$ -adapted X such that $F(X) = X$. Such a fixed point is a strong solution since we can take $(\mathcal{F}_t)$ to be the filtration induced by B .

The proof is by Picard iteration. Let $T > 0$. For X continuous adapted, set

$$\|X\|_T = E \left(\sup_{t \leq T} |X_t|^2 \right)^{1/2}$$

Then $B = \{ X : \Omega \times [0, T] \rightarrow \mathbb{R} \text{ continuous adapted} : \|X\|_T < \infty \}$ is a Banach space.

Claim: $\|F(X) - F(Y)\|_T^2 \leq (2T+8)K^2 \int_0^T \|X-Y\|_t^2 dt.$

Indeed, $\|F(X) - F(Y)\|_T^2 \leq 2 \mathbb{E} \left(\sup_{t \leq T} \left| \int_0^t (b(s, X_s) - b(s, Y_s)) ds \right|^2 \right)$
 $(a+b)^2 \leq 2a^2 + 2b^2 \rightarrow$
 $+ 2 \mathbb{E} \left(\sup_{t \leq T} \left| \int_0^t (\sigma(s, X_s) - \sigma(s, Y_s)) dB_s \right|^2 \right)$
 $= \text{(I)} + \text{(II)}$

$$\text{and } (I) \stackrel{(CS)}{\leq} 2T \mathbb{E} \left(\int_0^T |b(s, X_s) - b(s, Y_s)|^2 ds \right) \leq 2TK^2 \int_0^T \|X - Y\|_t^2 dt$$

$$(II) \leq 8 \mathbb{E} \left(\int_0^T |\sigma(s, X_s) - \sigma(s, Y_s)|^2 ds \right) \leq 8 K^2 \int_0^T \|X - Y\|_t^2 dt$$

$$\mathbb{E} \sup_{t \leq T} |M_t|^2 \leq 4 \mathbb{E} M_T^2 = 4 \mathbb{E} \langle M \rangle_T.$$

Claim: $\|F(0)\|_T < \infty$

$$F(0)_t = x + \int_0^t b(s, 0) \, ds + \int_0^t \sigma(s, 0) \, dB_s$$

$$\Rightarrow \|F\|_T^2 \leq 3 \left(|x|^2 + \underbrace{\left\| \int_0^t b(s,0) ds \right\|_T^2}_{\leq T \int_0^T |b(s,0)|^2 ds} + \underbrace{\left\| \int_0^t \sigma(s,0) dB_s \right\|_T^2}_{\leq 4 \mathbb{E} \left(\int_0^T |\sigma(s,0)|^2 ds \right)} \right) < \infty$$

Let $X_t^0 = 0$ and $X^{i+1} = F(X^i)$. Then

$$\begin{aligned} \|X^{i+1} - X^i\|_T^2 &\leq C \int_0^T \|X^i - X^{i-1}\|_t^2 dt \\ &\leq C^2 \int_0^T \int_0^t \|X^{i-1} - X^{i-2}\|_s^2 ds dt \\ &\leq C^n \underbrace{\int_0^T \int_0^{t_{n-1}} \dots \left(\int_0^{t_1} dt_0 \right) dt_1 \dots dt_{n-1}}_{\frac{T^n}{n!}} \underbrace{\|X^1 - X^0\|_T^2}_{\|F(0)\|_T^2} \end{aligned}$$

$$\Rightarrow \sum_{i=1}^{\infty} \|X^{i+1} - X^i\|_T < \infty$$

$\Rightarrow X^i$ converges uniformly on $[0, T]$, a.s.

$\Rightarrow F(X) = X$ on $[0, T]$

By uniqueness, solutions defined using different T must agree when both are defined. Hence we can find a solution defined on all of $[0, \infty)$.

Rk When $\sigma = 1$, the integral equation $X_t = X_0 + \int_0^t b(s, X_s) ds + B$ can be solved for any continuous function by the same argument replacing $\|\cdot\|_T$ by $\|\cdot\|_T = \sup_{t \leq T} |X_t|$.

5.3. The solution map

The following proposition provides a rough estimate on the dependence of the solution on the initial condition.

Prop. Under the same assumptions as in the theorem, let X^x be the solution with initial condition $X_0^x = x$. For $p \geq 2$,

$$\mathbb{E} \left(\sup_{s \leq t} |X_s^x - X_s^y|^p \right) \leq C_{p,t} |x-y|^p$$

Proof. Fix $x, y \in \mathbb{R}^d$ and let $T = \inf\{t \geq 0 : |X_t^x| \geq n \text{ or } |X_t^y| \geq n\}$. Since $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$,

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \leq t} |X_{s \wedge T}^x - X_{s \wedge T}^y|^p \right) \\ & \leq 3^{p-1} \left[|x-y|^p + \underbrace{\mathbb{E} \left(\sup_{s \leq t} \left| \int_0^{s \wedge T} (\sigma(r, X_r^x) - \sigma(r, X_r^y)) dB_r \right|^p \right)}_{(A)} \right. \\ & \quad \left. + \underbrace{\mathbb{E} \left(\sup_{s \leq t} \left| \int_0^{s \wedge T} (b(r, X_r^x) - b(r, X_r^y)) dr \right|^p \right)}_{(B)} \right] \end{aligned}$$

$$\begin{aligned} \text{where } (A) & \stackrel{(BDG)}{\leq} C_p \mathbb{E} \left(\left| \int_0^{t \wedge T} |\sigma(r, X_r^x) - \sigma(r, X_r^y)|^2 dr \right|^{p/2} \right) \\ & \stackrel{(\text{H\"older})}{\leq} C_p t^{\frac{p}{2}-1} \mathbb{E} \left(\int_0^{t \wedge T} |\sigma(r, X_r^x) - \sigma(r, X_r^y)|^p dr \right) \\ (B) & \stackrel{(\text{H\"older})}{\leq} C_p t^{p-1} \mathbb{E} \left(\int_0^{t \wedge T} |b(r, X_r^x) - b(r, X_r^y)|^p dr \right) \end{aligned}$$

Using the Lipschitz condition for σ and b , therefore there are $C_{p,t}$ such that

$$\underbrace{\mathbb{E}\left(\sup_{s \leq t} |X_{s \wedge T}^x - X_{s \wedge T}^y|^p\right)}_{f(t)} \leq 3^{p-1} |x-y|^p + C_{p,t} \int_0^t \underbrace{\mathbb{E} |X_{s \wedge T}^x - X_{s \wedge T}^y|^p}_{\leq f(s)} ds$$

Note that f is bounded because of the stopping time T . By Gronwall, thus

$$f(t) \leq 3^{p-1} |x-y|^p \exp(C_{p,t} t).$$

By Fatou, we can take $n \rightarrow \infty$ and get the claim.

Strong solutions are functions of Brownian motion in the following sense. Recall the d -dimensional Wiener space $(W^d, \mathcal{W}^d, P)$ where $W^d = C(\mathbb{R}_+, \mathbb{R}^d)$. The space W^d can be given the topology of uniform convergence on compact intervals, which is induced by the metric

$$d(w, \tilde{w}) = \sum_{k=1}^{\infty} \alpha_k \left(\sup_{t \leq k} |w(t) - \tilde{w}(t)| \wedge 1 \right)$$

for any sequence $(\alpha_k) \subset (0, \infty)$ with $\sum \alpha_k < \infty$. This metric makes W^d a complete separable metric space (a 'Polish space').

Thm. Under the same assumptions as the previous theorem, for $x \in \mathbb{R}^d$ there exist maps

$$F_x: C(\mathbb{R}_+, \mathbb{R}^m) \longrightarrow C(\mathbb{R}_+, \mathbb{R}^d)$$

measurable w.r.t. the completion of $\mathcal{W}^m$ and w.r.t. $\mathcal{W}^d$ such that

(i) $\forall t \geq 0, x \in \mathbb{R}^d: F_x(w)_t$ is measurable w.r.t. $\sigma(w(s): s \leq t)$, for P -a.e. $w \in \mathcal{W}^d$

(ii) $\forall w \in C(\mathbb{R}_+, \mathbb{R}^m): x \mapsto F_x(w) \in C(\mathbb{R}_+, \mathbb{R}^d)$ is continuous.

(iii) $\forall x \in \mathbb{R}^d, \forall (\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions, every $(\mathcal{F}_t)$ -Brownian motion B with $B_0 = 0$, the unique solution to $E(\sigma, b)$ with $X_0 = x$ is $X_t = F_x(B)_t$.

(iv) In the set up of (iii), if $U \in \mathcal{F}_0$ then $F_U(B)_t$ is the unique solution with $X_0 = U$.

Cor. The solutions to $E(\sigma, b)$ with $X_0 = x \in \mathbb{R}^d$ can be constructed for all $x \in \mathbb{R}^d$ simultaneously such that a.s. X^x is continuous in the initial condition x .

Proof. For simpler notation, $d=m=1$. Let

$$\mathcal{G}_t = \sigma(w(s) : 0 \leq s \leq t) \vee \mathcal{N}, \quad \mathcal{G} = \mathcal{G}_\infty$$

where $\mathcal{N}$ are the P -nullsets. By the last theorem, there is a unique strong solution X^x to $E(\sigma, b)$ with $X_0^x = x$, with respect to $(W, \mathcal{G}, (\mathcal{G}_t), P)$ and $B_t(w) = w(t)$.

By the last proposition, choosing (α_k) appropriately,

$$\begin{aligned} E d(X^x, X^y)^p &\leq E \left(\sum_k \alpha_k \sup_{t \leq k} |X_t^x - X_t^y| \right)^p \\ &\stackrel{(\text{Jensen})}{\leq} \sum_k \alpha_k E \left(\sup_{t \leq k} |X_t^x - X_t^y|^p \right) \\ &\leq |x-y|^p \sum_k \alpha_k C_{p,k} \leq C |x-y|^p. \end{aligned} \quad (†)$$

A version of Kolmogorov's continuity criterion applies to processes X^x indexed by $\mathbb{R}^d$ and values in a complete metric space, provided $(†)$ holds with $p > d$. Hence there is a modification $(\tilde{X}^x)_{x \in \mathbb{R}^d}$ of $(X^x)_{x \in \mathbb{R}^d}$ that is continuous in $x \in \mathbb{R}^d$. Set

$$F_x(w) = \tilde{X}^x(w) = (\tilde{X}_t^x(w))_{t \geq 0}.$$

Then (ii) is immediate. To see (i), note $w \mapsto X_t^x(w)$ is $\mathcal{G}_t$ -measurable. Since $X^x = \tilde{X}^x$ a.s. this gives (i).

To show (iii), assume $(\Omega, \mathcal{F}, P)$ and $\hat{B}$ are given, and set

$$X_t = F_x(\hat{B})_t.$$

Since F_x maps into $C(\mathbb{R}_+, \mathbb{R}^d)$, X is continuous in t . Since $F(\hat{B})$ coincides a.s. with a random variable measurable with respect to the completed filtration generated by $\hat{B}$ and $\mathcal{F}$ contains this filtration, X is adapted. By definition,

$$\begin{aligned}\tilde{X}_t &= x + \int_0^t \sigma(s, \tilde{X}_s) dB_s + \int_0^t b(s, \tilde{X}_s) ds \\ &= x + \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} \sigma(t_{i-1}, \tilde{X}_{t_{i-1}}^m) (B_{t_i} - B_{t_{i-1}}) + \int_0^t b(s, \tilde{X}_s) ds\end{aligned}$$

in P -probability and thus P -a.s. along a subsequence which we now fix

Since $\tilde{X}(w) = F(w)$ thus, for P -a.e. $w \in W^m$,

$$F_x(w)_t = x + \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} \sigma(t_{i-1}, F_x(w)_{t_{i-1}}) (F_x(w)_{t_i} - F_x(w)_{t_{i-1}}) + \int_0^t b(s, X_s) ds.$$

Since $\hat{B}$ has law P on W^m , substituting $w = \hat{B}$ and then undoing the approximation of the stochastic integral gives

$$X_t = x + \int_0^t \sigma(s, X_s) d\hat{B}_s + \int_0^t b(s, X_s) ds$$

as claimed.

(iv) omitted (proof is similar).

Example: Geometric Brownian motion

For $w \in C(\mathbb{R}_+, \mathbb{R})$ define $F_x(w)$ by

$$[F_x(w)](t) = x \exp(\sigma w(t) + (\mu - \frac{\sigma^2}{2})t).$$

If B is a standard Brownian motion with $B_0 = 0$ then $X_t = F_x(B)_t$ satisfies

$$dX_t = \sigma X_t dB_t + \mu X_t dt.$$

On the other hand, if we choose w to be any smooth path, then $y_t = F_x(w)_t$ satisfies the ODE

$$dy_t = \sigma y_t dw_t + (\mu - \frac{\sigma^2}{2}) dt.$$

Thus the Itô map F satisfies the 'wrong' equation on smooth paths.

5.4. Examples of SDEs

The Ornstein-Uhlenbeck process. Let $\lambda > 0$. The Ornstein-Uhlenbeck process is the unique solution to

$$dX_t = -\lambda X_t dt + dB_t.$$

It is an example that can be solved explicitly. Apply Itô's formula to $e^{\lambda t} X_t$:

$$d(e^{\lambda t} X_t) = \lambda e^{\lambda t} X_t dt + e^{\lambda t} dX_t = e^{\lambda t} dB_t$$

$$\Leftrightarrow e^{\lambda t} X_t - X_0 = \int_0^t e^{\lambda s} dB_s$$

$$\Leftrightarrow X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dB_s.$$

Wiener integral

Exercise. Fix $X_0 = x \in \mathbb{R}$ (or Gaussian). Then (X_t) is a Gaussian process, i.e., $(X_{t_i})_{i=1}^n$ is jointly Gaussian for all $t_0 = 0 < t_1 < \dots < t_n$.

Fact. $\mathbb{E} X_t = e^{-\lambda t} x$, $\text{Cov}(X_t, X_s) = \frac{1}{2\lambda} (e^{-\lambda|t-s|} - e^{-\lambda(t+s)})$

Proof. $\mathbb{E} X_t = e^{-\lambda t} \mathbb{E} X_0 + \underbrace{\mathbb{E} \int_0^t e^{-\lambda(t-s)} dB_s}_0 = e^{-\lambda t} x.$

By the Kunita-Watanabe identity,

$$\begin{aligned}
 \text{Cov}(X_t, X_s) &= \mathbb{E}((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)) \\
 &= \mathbb{E}\left(\int_0^t e^{-\lambda(t-u)} dB_u \int_0^s e^{-\lambda(s-u)} dB_u\right) \\
 &= \mathbb{E}((H \cdot B)(K \cdot B)) = \mathbb{E}\langle H \cdot B, K \cdot B \rangle_\infty = \mathbb{E}(HK \cdot \langle B \rangle)_\infty \\
 &= \int_0^\infty (1_{u \leq t} e^{-\lambda(t-u)})(1_{u \leq s} e^{-\lambda(s-u)}) du \\
 &= e^{-\lambda(t+s)} \int_0^{s \wedge t} e^{2\lambda u} du = \frac{1}{2\lambda} e^{-\lambda(t+s)} (e^{2\lambda(s \wedge t)} - 1)
 \end{aligned}$$

Cor. $X_t \sim \mathcal{N}(\underbrace{e^{-\lambda t} x}_{\rightarrow 0}, \underbrace{\frac{1-e^{-2\lambda t}}{2\lambda}}_{\rightarrow \frac{1}{2\lambda}})$ for every $t \geq 0$, $X_0 = x \in \mathbb{R}$.
as $t \rightarrow \infty$

This suggests the distribution $\mathcal{N}(0, \frac{1}{2\lambda})$ is invariant. This is indeed easy to check.

Fact. If $X_0 \sim \mathcal{N}(0, \frac{1}{2\lambda})$ then $X_t \sim \mathcal{N}(0, \frac{1}{2\lambda})$ for all $t \geq 0$, and X_t is a stationary Gaussian process with

$$\text{Cov}(X_t, X_s) = \text{Cov}(X_0, X_{|t-s|}) = \frac{1}{2\lambda} e^{-\lambda|t-s|}.$$

5.5. Local Solutions

As for ODEs, solutions to SDEs do not always exist for all times. For SDEs the explosion time is random.

Prop. (Local Itô formula). Let $X = (X^1, \dots, X^d)$ be continuous semimartingales. Let $U \subset \mathbb{R}^d$ be open and let $f: U \rightarrow \mathbb{R}$ be C^2 . Set $T = \inf\{t \geq 0: X_t \notin U\}$. Then for all $t < T$,

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x^i}(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j \rangle_s.$$

Sketch. Apply Itô's formula to X^{T_n} where

$$T_n = \inf\{t \geq 0: \text{dist}(X_t, U^c) \leq \frac{1}{n}\}.$$

Example. Let B be a standard one-dimensional Brownian motion with $B_0 = 1$. Taking $U = (0, \infty)$, $f(x) = \sqrt{x}$ gives

$$\sqrt{B_t} = 1 + \frac{1}{2} \int_0^t B_s^{-1/2} ds - \frac{1}{8} \int_0^t B_s^{-3/2} ds$$

for $t < T = \inf\{t \geq 0: B_t = 0\}$.

Thm. Let $U \subset \mathbb{R}^d$ be open and $b: \mathbb{R}_+ \times U \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}_+ \times U \rightarrow \mathbb{R}^{d \times m}$ be locally Lipschitz continuous. Then for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B , every $x \in U$, there exists a stopping time T such that, for $t < T$,

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s,$$

and for all compact $K \subset U$, $\sup\{t < T: X_t \in K\} < T$. This stopping time T is called the explosion time.

Proof. Fix $K_n \subset U$ compact such that $K_{n+1} \supseteq K_n$ and $\bigcup K_n = U$. One can find b_n and σ_n defined on all of $\mathbb{R}^d$ suchⁿ that $b_n|_{\mathbb{R}_+ \times K_n} = b|_{\mathbb{R}_+ \times K_n}$ and $\sigma_n|_{\mathbb{R}_+ \times K_n} = \sigma|_{\mathbb{R}_+ \times K_n}$ and such that b_n and σ_n are globally Lipschitz continuous. Hence, for any $x \in U$ and n large enough that $x \in K_n$, there are unique solutions X^n to $E(\sigma_n, b_n)$ with $X_0 = x$. Let

$$T_n = \inf\{t \geq 0: X_t^n \notin K_n\}$$

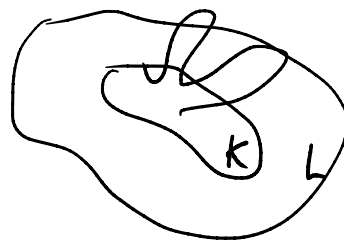
By uniqueness, X^{n+1} also solves $E(\sigma_n, b_n)$ up to time T_n . Thus $X_t^{n+1} = X_t^n$ for $t < T_n$ and we can define X_t for $t < T := \sup_n T_n$ by requiring that $X_t = X_t^n$ for $t < T_n$.

Claim: Let $K \cup U$ be compact. Then on $\{T < \infty\}$ a.s.

$$\sup\{t < T : X_t \in K\} < T.$$

Let L be another compact set such that $K \subset L^\circ \subset L \subset U$.
Let $\varphi: U \rightarrow \mathbb{R}$ be C^∞ such that $\varphi|_K = 1$ and $\varphi|_{L^c} = 0$.

$$\begin{aligned} \text{Let } R_1 &= \inf\{t \geq 0 : X_t \notin L\} \\ S_n &= \inf\{t \geq R_n : X_t \in K\} \wedge T \\ R_n &= \inf\{t \geq S_{n-1} : X_t \notin L\} \end{aligned}$$



Let N be the number of crossings of X from L^c to K .

Then on $\{T \leq t, N \geq n\}$,

$$\begin{aligned} n &= \sum_{k=1}^n (\varphi(X_{S_k}) - \varphi(X_{R_k})) \\ &= \sum_{k=1}^n \int_{R_k}^{S_k} \left(\nabla \varphi(X_s) \cdot dX_s + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j \rangle_s \right) \\ &= \int_0^t (H_s^n dB_s + \tilde{H}_s^n ds) = Z_t^n \end{aligned}$$

with H^n and $\tilde{H}^n$ and bounded uniformly in n .

$$\Rightarrow n^2 \mathbb{1}_{\{T \leq t, N \geq n\}} \leq (Z_t^n)^2 \Rightarrow P(T \leq t, N \geq n) \leq n^{-2} E((Z_t^n)^2) \leq \frac{C(t)}{n^2}.$$

$$\Rightarrow P(T \leq t, N = \infty) = 0 \Rightarrow P(T < \infty, N = \infty) = 0 \text{ as claimed.}$$

Example. Consider the SDEs

$$dX_t^i = b^i(X_t) dt + dB_t^i, \quad X_0 = x.$$

Assume there are $a > 0, c > 0$ such that

$$x \cdot b(x) \leq a|x|^2 + c.$$

Then the SDE has a global solution, i.e., $T = \infty$ a.s.

Proof. Let $T_n = \inf \{t \geq 0 : |X_t|^2 \geq n\}$. Then by Itô's formula

$$\begin{aligned} \mathbb{E}|X_{t \wedge T_n}|^2 &= \mathbb{E}|X_0|^2 + 2 \mathbb{E} \int_0^{t \wedge T_n} X_s \cdot b(X_s) ds + (t \wedge T_n) d \\ &\leq \mathbb{E}|X_0|^2 + 2a \mathbb{E} \int_0^{t \wedge T_n} |X_s|^2 ds + (d+c)t \\ &\leq \mathbb{E}|X_0|^2 + (d+c)t + 2a \int_0^t \mathbb{E}|X_{s \wedge T_n}|^2 ds. \end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}|X_{t \wedge T_n}|^2 \leq \underbrace{(\mathbb{E}|X_0|^2 + (d+c)t)}_{C(t)} e^{2at}$$

$$\Rightarrow \mathbb{P}(T_n \leq t) \leq \frac{C(t)}{n}$$

The explosion time T satisfies $T = \lim_{n \rightarrow \infty} T_n$ by the theorem.
Hence by Fatou,

$$P(T \leq t) = 0 \text{ for every } t > 0.$$

$$\Rightarrow P(T = \infty) = 1.$$

5.6. Markov property

Let $B(\mathbb{R}^d)$ be the Banach space of bounded Borel functions with $\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|$ for $f \in B(\mathbb{R}^d)$.

Defn. (i) A collection of bounded linear operators Q_t on $B(\mathbb{R}^d)$ is a transition semigroup if $Q_t f \geq 0$ a.e. if $f \geq 0$ a.e., $Q_t 1 = 1$ where $1(x) = 1$ for all $x \in \mathbb{R}^d$, $\|Q_t\| \leq 1$, and

$$Q_{t+s} = Q_t Q_s \quad \text{for all } t, s \geq 0.$$

(ii) An $(\mathcal{F}_t)$ -adapted process X is a Markov process with transition semigroup (Q_t) if

$$\mathbb{E}(f(X_{s+t}) | \mathcal{F}_s) = Q_t f(X_s) \quad \text{for all } s, t \geq 0, f \in B(\mathbb{R}^d).$$

Thm. Let $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be Lipschitz.

Assume X is a solution to $E(\sigma, b)$ on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Then X is a Markov process with semigroup

$$Q_t f(x) = \mathbb{E}_x f(X_t) = \int f(F_x(w)_t) P(dw)$$

where F_x is the solution map on Wiener space and P is the Wiener measure.

Proof. Let X be a solution to $E(\sigma, b)$.

Claim: $E(f(X_{t+s}) | \mathcal{F}_s) = Q_t f(X_s)$

Indeed, $X_t = X_0 + \int_0^t \sigma(X_u) dB_u + \int_0^t b(X_u) du$

$$X_{t+s} = X_0 + \int_0^{t+s} \sigma(X_u) dB_u + \int_0^{t+s} b(X_u) du$$

Setting $\tilde{X}_t = X_{t+s}$, $\tilde{\mathcal{F}}_t = \mathcal{F}_{t+s}$, and $\tilde{B}_t = B_{t+s} - B_s$, note that $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), P)$ is another filtered probability space obeying the usual conditions and $\tilde{B}$ an $(\tilde{\mathcal{F}}_t)$ -Brownian motion.

$$\Rightarrow \tilde{X}_t = \tilde{X}_0 + \int_0^t \sigma(\tilde{X}_u) d\tilde{B}_u + \int_0^t b(\tilde{X}_u) du$$

Indeed, this follows from $\int_s^{s+t} \sigma(X_u) dB_u = \int_0^t \sigma(\tilde{X}_u) d\tilde{B}_u$ which can be seen by approximating both sides by sums.

Thus $\tilde{X}$ solves $E(\sigma, b)$ with $\tilde{X}_0 = X_s$ and therefore in terms of the solution map $\tilde{X} = F_{X_s}(\tilde{B})$ a.s.

$$\Rightarrow E(f(X_{s+t}) | \mathcal{F}_s) = E(f(\tilde{X}_t) | \tilde{\mathcal{F}}_s) \stackrel{\uparrow}{=} E(f(F_{X_s}(\tilde{B}))_t | \tilde{\mathcal{F}}_s) = Q_t f(X_s)$$

and $\tilde{B}$ is indep. of X_s

$$Q_{t+s} f(x) = E_x f(X_{t+s}) = E_x \left(\underbrace{E_x(f(X_{t+s}) | \mathcal{F}_s)}_{Q_t f(X_s)} \right) = Q_s Q_t f(x).$$

Rk. Using the strong Markov property for Brownian motion, one can prove this property for X similarly.

Let $C_0(\mathbb{R}^d)$ be the Banach space of continuous functions tending to 0 at ∞ with norm $\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|$.

Defn. A transition semigroup has the Feller property if

- (i) $\forall f \in C_0, t > 0: Q_t f \in C_0$
- (ii) $\forall f \in C_0: \|Q_t f - f\| \rightarrow 0$ as $t \rightarrow 0$.

Thm. The semigroup of the last theorem is Feller.

Proof (assuming σ and b are bounded). Let $f \in C_0$.

Claim: $Q_t f$ is continuous

Indeed, since $x \mapsto F_x(w)$ is continuous, by DCT,

$$|Q_t f(x) - Q_t f(y)| \leq \int |f(F_x(w)_t) - f(F_y(w)_t)| P(dw) \rightarrow 0.$$

Claim: $Q_t f(x) \rightarrow 0$ as $|x| \rightarrow \infty$

Since $X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dB_s$ and σ, b are bounded,

$$\mathbb{E}|X_t^x - x|^2 \leq C(t^2 + t)$$

Markov $\Rightarrow P(|X_t^x - x| > A) \leq A^{-1} C_t \rightarrow 0$ as $A \rightarrow \infty$.

$$\Rightarrow |Q_t f(x)| \leq \mathbb{E}|f(X_t^x)| \mathbf{1}_{|X_t^x - x| \leq A} + \|f\| P(|X_t^x - x| > A)$$

$$\Rightarrow \limsup_{|x| \rightarrow \infty} |Q_t f(x)| \leq 0 + \|f\| P(|X_t^x - x| > A) \quad \forall A$$

$$\Rightarrow \lim_{|x| \rightarrow \infty} |Q_t f(x)| = 0. \quad \text{Since } f(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Claim: $\|Q_t f - f\| \rightarrow 0$

$$\sup_x |E(f(X_t^x) - f(x))| \leq \sup_{|x-y| \leq \varepsilon} |f(x) - f(y)| + 2\|f\| \underbrace{\sup_x P(|X_t^x - x| > \varepsilon)}_{\leq C\varepsilon^{-1}(t^2 + t)}$$

$$\Rightarrow \lim_{t \rightarrow 0} \|Q_t f - f\| = 0.$$

$\rightarrow 0$ as $t \rightarrow 0$

Defn. Let

$$D(L) = \{f \in C_0 : \frac{1}{t}(Q_t f - f) \text{ converges in } C_0 \text{ as } t \downarrow 0\}$$

$$Lf = \lim_{t \downarrow 0} \frac{1}{t}(Q_t f - f) \text{ for } f \in D(L)$$

The linear operator $L: D(L) \rightarrow C_0$ is called the generator of the semigroup (Q_t) .

Prop. Let $f \in D(L)$. Then $Q_t f \in D(L)$, $L(Q_t f) = Q_t(Lf)$, and

$$Q_t f = f + \int_0^t Q_s Lf \, ds = f + \int_0^t LQ_s f \, ds$$

Proof. To see $Q_t f \in D(L)$ and $L(Q_t f) = Q_t(Lf)$, note

$$\frac{1}{s}(Q_s(Q_t f) - Q_t f) = Q_t \left(\underbrace{\frac{1}{s}(Q_s f - f)}_{\rightarrow Lf} \right) \rightarrow Q_t(Lf) \text{ as } s \downarrow 0. \quad \text{since } \|Q_t\| \leq 1.$$

Since the left-hand side is $\frac{1}{s}(Q_{t+s} f - Q_t f)$, $Q_t f$ is also right-differentiable with right-derivative $Q_t Lf$. Since this right-derivative is continuous in t , in fact $Q_t f$ is differentiable in t .

Thm Let $f, g \in C_0$. Then the following are equivalent:

(i) $f \in D(L)$ and $Lf = g$.

(ii) For every $x \in \mathbb{R}^d$, $f(X_t^x) - \int_0^t g(X_s^x) ds$ is a martingale.

Proof. (i) $\Rightarrow$ (ii).

$$\mathbb{E}(f(X_{t+s}^x) - \int_0^{t+s} g(X_r^x) dr \mid \mathcal{F}_t)$$

$$= \mathbb{E}(\underbrace{Q_s f(X_t^x)} - \int_0^{t+s} g(X_r^x) dr \mid \mathcal{F}_t)$$

$$Q_s f = f + \int_0^s Q_r g dr \text{ by (i)}$$

$$= f(X_t^x) + \int_0^t g(X_r^x) dr + \underbrace{\mathbb{E}(\int_0^s Q_r g(X_t^x) dr - \int_0^s g(X_{r+t}^x) dr \mid \mathcal{F}_t)}_{=0}$$

(ii) $\Rightarrow$ (i). By (ii),

$$f(x) = \mathbb{E}(f(X_t^x) - \int_0^t g(X_r^x) dr)$$

$$= Q_t f(x) - \int_0^t Q_r g(x) dr$$

$$\Rightarrow \frac{1}{t}(Q_t f - f) = \frac{1}{t} \int_0^t Q_r g(x) dr \rightarrow g(x) \text{ as } t \downarrow 0.$$

Cor. The generator of the semigroup of the SDE as in the last theorems satisfies $C_c^2(\mathbb{R}^d) \subset D(L)$, and for $f \in C_c^2(\mathbb{R}^d)$,

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x^i}(x), \quad a = \sigma \sigma^T.$$

5.7. Convergence to equilibrium

Defn. Let (Q_t) be a transition semigroup and let μ be a probability measure on $\mathbb{R}^d$. Then μ is

(i) invariant under (Q_t) if

$$\int Q_t f(x) \mu(dx) = \int f(x) \mu(dx) \quad \forall f \in B(\mathbb{R}^d)$$

(ii) reversible under (Q_t) if

$$\int g(x) Q_t f(x) \mu(dx) = \int Q_t g(x) f(x) \mu(dx) \quad \forall f, g \in B(\mathbb{R}^d)$$

Fact. reversible $\Rightarrow$ invariant

Proof. Take $g=1$ and use $Q_t 1 = 1$.

Fact. The measure μ is

(i) invariant iff $\int Lf \, d\mu = 0$ for all $f \in D(L)$

(ii) reversible iff $\int g Lf \, d\mu = \int (Lg) f \, d\mu$ for all $f, g \in D(L)$.

Rk. In practice, $D(L)$ is rarely known explicitly. However, if $D \subset D(L)$ is a dense subspace such that $Q_t f \in D$ for $f \in D$ and $t > 0$ then it suffices to check the conditions for $f \in D$. For example, for $f \in D$,

$$\frac{d}{dt} \int Q_t g \, d\mu = \int L \overbrace{Q_t g}^f \, d\mu = \int Lf \, d\mu = 0.$$

Prop. Let Q_t be the transition semigroup of the SDE

$$dX_t = -\nabla H(X_t) dt + \sqrt{2} dB_t.$$

Assume that $H \in C_b^2(\mathbb{R}^d)$ with $\int e^{-H}(1+|\nabla H|) < \infty$. Then the probability measure $\mu(dx) = \frac{1}{Z} e^{-H(x)} dx$ is reversible.

Proof. For $f, g \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \int g(Lf) e^{-H} dx &= \int g \underbrace{(\Delta f - \nabla H \cdot \nabla f)}_{\nabla \cdot (\nabla f) e^{-H}} e^{-H} dx \\ &= -\int \nabla f \cdot \nabla g e^{-H} dx \stackrel{\text{same argument}}{=} \int (Lf) g e^{-H} dx. \end{aligned}$$

Thm. Assume that H is uniformly convex, i.e., there is $\lambda > 0$ such that

$$\xi^T \text{Hess } H(x) \xi \geq \lambda |\xi|^2 \quad \text{for all } x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Let X and Y be two solutions to the SDE w.r.t. the same Brownian motion. Then, a.s.,

$$|X_t - Y_t| \leq |X_0 - Y_0| e^{-\lambda t} \quad \swarrow \text{indep. of dimension}$$

Proof. By Itô's formula,

$$\begin{aligned} e^{2\lambda t} |X_t - Y_t|^2 &= |X_0 - Y_0|^2 + \int_0^t 2\lambda |X_s - Y_s|^2 e^{2\lambda s} ds \\ &\quad - 2 \int_0^t \underbrace{(X_s - Y_s) \cdot (\nabla H(X_s) - \nabla H(Y_s))}_{\geq \lambda |X_s - Y_s|^2} e^{2\lambda s} ds \\ &\leq |X_0 - Y_0|^2. \end{aligned}$$

Cor. For every $f \in C_b(\mathbb{R}^d)$, $\mathbb{E}_x f(X_t) \rightarrow \mathbb{E}_\mu f$ where μ is the invariant probability measure $\mu(dx) \propto e^{-H(x)} dx$.

Proof. It suffices to assume that $f \in C_b^1$. Then if $Y_0 \sim \mu$,

$$\mathbb{E} f(X_t) = \mathbb{E} f(Y_t) + O(\|\nabla f\|_\infty \underbrace{\mathbb{E}|X_t - Y_t|}_{\leq e^{-\lambda t} \mathbb{E}|X_0 - Y_0|}) \rightarrow \mathbb{E} f(Y_0) = \mathbb{E}_\mu f.$$

Defn. For two probability measures, define

$$d_{TV}(\mu, \nu) = \frac{1}{2} \|\mu - \nu\|_{TV} = \frac{1}{2} |\mu - \nu|(\Omega).$$

If $\nu \ll \mu$ with $\frac{d\nu}{d\mu} = g$ then

$$d_{TV}(\nu, \mu) = \frac{1}{2} \mathbb{E}_\mu[|g - 1|]$$

Fact. $d_{TV}(\mu, \nu) = \inf_{\substack{X \sim \mu \\ Y \sim \nu}} P[X \neq Y]$

Sketch. If (X, Y) is a coupling of (μ, ν) ,

$$\int f d\mu - \int f d\nu = \mathbb{E}[f(X) - f(Y)] \leq 2 \|f\| P[X \neq Y]$$

$$\Rightarrow \|\mu - \nu\|_{TV} \leq \sup_{\|f\| \leq 1} \|\int f d\mu - \int f d\nu\| \leq 2 P[X \neq Y].$$

Other direction: Exercise.

Cor. $d_{TV}(\mu_t, \mu) \leq P[X_t \neq Y_t] \leq P[|X_t - Y_t| > \delta] \leq \frac{\mathbb{E}[|X_0 - Y_0|]}{\delta} e^{-\lambda t}$
where $Y_0 \sim \mu$

Total variation is a natural distance (metric) between measures, but in practice it is often useful to work with other distances ("divergences") which are not metrics.

Fact. $2d_{TV}(\mu_t, \mu)^2 \stackrel{CS}{\leq} \frac{1}{2} E_{\mu}[|g_t - 1|^2]$ χ^2 divergence
 $= \frac{1}{2} E_{\mu}[|g_t - E_{\mu}[g_t]|^2] = \text{Var}_{\mu}(g_t).$

Pinsker's inequality. $2d_{TV}(\mu_t, \mu)^2 \leq \text{Ent}_{\mu}(g_t)$

Proof. The following elementary ineq. holds:

$$|x-1| \leq \frac{1}{\sqrt{3}} \sqrt{4+2x} \sqrt{x \log x - x + 1}$$

$$\begin{aligned} \Rightarrow E_{\mu}[|g_t - 1|^2] &\leq \underbrace{\frac{1}{3} E_{\mu}[4+2g_t]}_6 E_{\mu}[g_t \log g_t - g_t + 1] \\ &= 2 \text{Ent}_{\mu}(g_t) \quad \text{KL divergence} \end{aligned}$$

Prop. Let μ_t be the law of a Markov process (X_t) with reversible measure μ . Suppose

$$d\mu_t = g_t d\mu$$

for some smooth $g_t \in L^2(\mu)$. Then $\dot{g}_t = Lg_t$.

Proof. $Q_t f(x) = E_x[f(X_t)]$

$$= \int f(x) d\mu_t$$

$$= \int f(x) g_t(x) \mu(dx)$$

$$\Rightarrow \frac{\partial}{\partial t} Q_t f(x) = Q_t Lf(x)$$

$$= \int Lf(x) g_t(x) \mu(dx)$$

$$= \int f(x) Lg_t(x) \mu(dx)$$

$$\Rightarrow \int f(x) \left(\frac{\partial}{\partial t} g_t(x) - Lg_t(x) \right) \mu(dx) = 0$$

Exercise. Poincaré ineq. $\Leftrightarrow \text{Var}_\mu(Q_t f) \leq e^{-2\lambda t} \text{Var}_\mu f$
 Log-Sob. ineq. $\Leftrightarrow \text{Ent}_\mu(Q_t f) \leq e^{-2\lambda t} \text{Ent}_\mu(f)$.
 Log-Sob. ineq. $\Rightarrow$ Poincaré ineq.

Sketch. For $f \in D(L)$,

$$\frac{\partial}{\partial t} \text{Var}_\mu(Q_t f) = \frac{1}{2} \frac{\partial}{\partial t} \int |Q_t f - \bar{f}|^2 d\mu$$

$$= \int (Q_t f - \bar{f}) LQ_t f d\mu$$

$$= - \int (\nabla Q_t f)^2 d\mu \leq -\lambda \text{Var}(Q_t f)$$

Apply Gronwall.

Back to the example $\mu(dx) \propto e^{-H(x)} dx$ with
 $\text{Hess } H(x) \geq \lambda \text{ id} \quad \forall x$.

Cor. The measure μ satisfies the Poincaré inequality
 $\text{Var}_\mu(f) \leq \frac{1}{\lambda} \mathbb{E}_\mu |\nabla f|^2 \quad \forall f \in C_b^1(\mathbb{R}^d)$.

Proof. By the mean-value theorem,

$$|Q_t f(x) - Q_t f(y)| = \mathbb{E} |f(X_t^x) - f(X_t^y)| \leq e^{-\lambda t} \mathbb{E} |\nabla f(Z_t)| \cdot |x - y|$$

$\Rightarrow \frac{|Q_t f(x) - Q_t f(y)|}{|x - y|} \leq e^{-\lambda t} \mathbb{E} |\nabla f(Z_t)|$ where Z_t lies on the line from X_t^x to X_t^y .

By continuity of the solution map,

$$\Rightarrow |\nabla Q_t f(x)| \leq e^{-\lambda t} \mathbb{E} |\nabla f(X_t)| = e^{-\lambda t} Q_t |\nabla f|(x).$$

$$\begin{aligned} \text{Thus } \text{Var}_\mu f &= \mathbb{E}_\mu f^2 - (\mathbb{E}_\mu f)^2 = \lim_{t \rightarrow \infty} (\mathbb{E}_\mu (Q_0 f)^2 - \mathbb{E}_\mu (Q_t f)^2) \\ &= - \int_0^\infty \mathbb{E}_\mu 2(Q_s f) L(Q_s f) ds \\ &= 2 \int_0^\infty \mathbb{E}_\mu |\nabla Q_t f|^2 \\ &\leq 2 \int_0^\infty e^{-2\lambda t} \mathbb{E}_\mu |\nabla f|^2 = \frac{1}{\lambda} \mathbb{E}_\mu |\nabla f|^2. \end{aligned}$$

In fact, the following stronger inequality holds.

Cor. The measure μ satisfies the Log-Sobolev inequality:
For all $f \in C_b'(\mathbb{R}^d)$ with $f > 0$,

$$\underbrace{\mathbb{E}_\mu(f \log f) - (\mathbb{E}_\mu f)(\log \mathbb{E}_\mu f)}_{=: \text{Ent}_\mu(f)} \leq \frac{1}{2\lambda} \mathbb{E}_\mu \frac{|\nabla f|^2}{f}$$

Proof. As in the last proof,

$$|\nabla Q_t f| \leq e^{-\lambda t} Q_t |\nabla f|.$$

Since $Q_t f(x) = \mathbb{E} f(X_t^x)$, for all strictly positive f ,

$$\frac{(Q_t |\nabla f|)^2}{Q_t f} = \frac{(\mathbb{E} |\nabla f|)^2}{\mathbb{E} f} \quad \text{if } \mathbb{E} \text{ denotes the expectation w.r.t. the law of } X_t^x$$

$$= \left(\mathbb{E}_f \frac{|\nabla f|}{f} \right) (\mathbb{E} f) \quad \text{if } \mathbb{E}_f X := \mathbb{E}(fX) / \mathbb{E} f$$

$$\stackrel{(CS)}{\leq} \left(\mathbb{E}_f \frac{|\nabla f|^2}{f^2} \right) (\mathbb{E} f) = \mathbb{E} \frac{|\nabla f|^2}{f} = Q_t \left(\frac{|\nabla f|^2}{f} \right).$$

$$\Rightarrow \frac{|\nabla Q_t f|^2}{Q_t f} \leq e^{-2\lambda t} Q_t \left(\frac{|\nabla f|^2}{f} \right).$$

$$\Rightarrow \mathbb{E}_\mu f \log f - \mathbb{E}_\mu f (\log \mathbb{E}_\mu f) = \int_0^\infty \mathbb{E}_\mu \frac{|\nabla Q_t f|^2}{Q_t f} \leq \frac{1}{2\lambda} \mathbb{E}_\mu \frac{|\nabla f|^2}{f}.$$

6. Jump processes

6.1. Poisson and compound Poisson processes

Defn. $(N_t)_{t \geq 0}$ is a Poisson process if

- (i) Almost surely, $N \in D[0, \infty)$ with $N_0 = 0$ and $N_t \in \mathbb{N}$ for all $t \geq 0$.
- (ii) For each $h > 0$, the distribution of $N_{t+h} - N_t$ is independent of t .
- (iii) If $[s_j, t_j]$ are nonoverlapping intervals, $N_{t_j} - N_{s_j}$ are mutually independent.
- (iv) $P[N_{t+h} - N_t \geq 2] = P[N_h \geq 2] = o(h)$ as $h \rightarrow 0$.

Thm. Let (N_t) be a PP. Then:

- (i) Almost surely, N_t is a step function increasing by steps of 1.
- (ii) There is $\lambda \geq 0$ s.t. $N_{t+s} - N_s \sim \text{Poisson}(\lambda t)$.
- (iii) The gaps $\tau_1, \tau_2, \dots$ between jumps are i.i.d. $\text{Exp}(\lambda)$.

Cor. The Poisson process of rate $\lambda > 0$ exists.

Proof. (i) Let

Ξ_n = number of intervals $[\frac{kT}{n}, \frac{(k+1)T}{n}]$, $0 \leq k < n$,
such that $N(\frac{(k+1)T}{n}) - N(\frac{kT}{n}) \geq 2$.

$$\Rightarrow P[\Xi_n \geq 1] \leq E[\Xi_n] = n P[N(\frac{T}{n}) \geq 2] \\ = n \cdot o(\frac{1}{n}) = o(1).$$

(ii) By right-continuity,

$$P[N_t \geq 1] \rightarrow 0 \text{ as } t \rightarrow 0.$$

$$\Rightarrow E[e^{-\sigma N_t}] \stackrel{(*)}{=} \lim_{n \rightarrow \infty} E[e^{-\sigma N_{t/n}}]^n = e^{-tg(\sigma)}$$

where $g(\sigma) = \lim_{h \rightarrow 0} \frac{1}{h} (1 - E[e^{-\sigma N_h}])$ and the limit exists since $(*)$ exists (is indep. of n).

$$g(\sigma) = \lim_{h \rightarrow 0} \frac{1}{h} E[1 - e^{-\sigma N_h}] \\ = (1 - e^{-\sigma}) P[N_h = 1] + o(h)$$

$$= (1 - e^{-\sigma}) \lim_{h \rightarrow 0} \frac{1}{h} P[N_h = 1] \\ \lambda \quad (\text{the limit must exist})$$

$$\Rightarrow E[e^{-\sigma N_t}] = e^{-\lambda t(1-e^{-\sigma})} \Rightarrow N_t \sim \text{Poisson}(\lambda t).$$

$$(iii) \quad P[\tau_1 \geq t] = P[N_t = 0] = e^{-\lambda t} \text{ since } N_t \sim \text{Poi}(\lambda t).$$

The claim follows by induction from the strong Markov property:

$$N_{\tau+t} - N_\tau = N_{\tau+t} - 1$$

is a Poisson process independent of τ .

This is proved in the usual way. Given a stopping time, consider $\tau^{(k)} = 2^{-k} \lceil 2^k \tau \rceil$. Assuming the strong Markov property holds for $\tau^{(k)}$ one can take $k \rightarrow \infty$ using $\tau < \tau^{(k)}$ and right-continuity.

To prove the strong Markov property for $\tau^{(k)}$, let $A \in \mathcal{F}_{\tau^{(k)}}$.

Then $A \cap \{\tau^{(k)} = 2^{-k} n\} \in \mathcal{F}_{2^{-k} n}$ and $(N_{t+2^{-k} n} - N_{2^{-k} n})_t$ is independent of $A \cap \{\tau^{(k)} = 2^{-k} n\}$.

Sum over n ,

Exercise.

$N_t - \lambda t$ and $(N_t - \lambda t)^2 - \lambda t$ are martingales. Thus $N_t = (N_t - \lambda t) + \lambda t$ is the semimartingale decomposition of N_t .

Define $T_t f(x) = E[f(x + N_t)]$ (the semigroup of PP).

Fact. The generator of the Poisson process is
$$Lf(x) = \lambda(f(x+1) - f(x))$$

Proof.

$$\begin{aligned} \frac{T_t f(x) - f(x)}{t} &= \frac{E[f(x + N_t) - f(x)]}{t} \\ &= (f(x+1) - f(x)) \left(\frac{P[N_t=1]}{t} + o(1) \right) \end{aligned}$$

Cor.

$$M_t^f = f(N_t) - f(0) - \lambda \int_0^t (f(N_s + 1) - f(N_s)) ds$$
 is a martingale

Exercise: Check by hand.

Defn. Let $X_1, X_2, \dots$ be i.i.d. with distribution α , and let N_t be a PP with rate λ . Define the compound Poisson process

$$X_t = S_{N_t}, \quad S_n = X_1 + \dots + X_n.$$

Then X_t has independent increments since each time that N_t jumps X_t changes by an indep. α -distributed random variable.

The distribution of $X_{t+s} - X_s$ is that of X_t and

$$\begin{aligned} E[e^{iyX_t}] &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \hat{\alpha}(y)^n \\ &= e^{\lambda t(\hat{\alpha}(y) - 1)} = \exp \left[\frac{\lambda t \int (e^{iyx} - 1) d\alpha(x)}{t \int (e^{iyx} - 1) dM(x)} \right]. \end{aligned}$$

X_t thus has an infinitely divisible distribution and M is called its Levy-Khintine measure.

Exercise: The following are martingales.

$$X_t - At, \quad A = \int x dM(x)$$

$$(X_t - At)^2 - Bt, \quad B = \int x^2 dM(x)$$

Exercise: X_t is a Markov process with generator
$$L f(x) = \int (f(x+z) - f(x)) dM(z)$$

Sketch. One can proceed as for Poisson process.
Suppose $f(x) = e^{iyx}$. Then

$$\begin{aligned} T_t f(x) &= f(x) E[e^{iyX_t}] \\ &= f(x) \exp \left[t \int (e^{iyz} - 1) dM(z) \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{T_t f(x) - f(x)}{t} &\rightarrow f(x) \int (e^{iyz} - 1) dM(z) \\ &= \int (f(x+z) - f(x)) dM(z). \end{aligned}$$

For a nice functions $f(x) = \int \hat{f}(y) e^{iyx} dy$.

The above can be generalized to stationary processes with independent increments which are characterized by their Levy measure.

6.2. Markov jump processes

Let $(E, \mathcal{E})$ be a measurable space (state space).

Let $\Pi(x, A)$ be a transition probability:

$\forall A \in \mathcal{E}: x \mapsto \Pi(x, A)$ is $\mathcal{E}$ -measurable

$\forall x \in E: \Pi(x, \cdot)$ is a probability measure

Then there is a probability measure on $E^{\mathbb{N}}$ under which $(Y_n)_{n \in \mathbb{N}}$ is a Markov process with transition probability Π .

The n -step transition probability is defined by

$$\Pi^{(n)}(x, A) = \int_E \Pi^n(y, A) \Pi(x, dy)$$

Let (N_t) be a Poisson process with rate λ .

Define $X_t = Y_{N_t}$. Then

- $P_x[X_t \in A] = p(t, x, A) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Pi^{(k)}(x, A)$
- $P[X_t \in A | \mathcal{F}_s] = p(t-s, X_s, A)$
- $p(t+s, x, A) = \int_E p(s, y, A) p(t, x, dy)$.

Proof.

$$\begin{aligned}
 P[X_t \in A \mid X_s = x] &= P[Y_{N_t} \in A \mid Y_{N_s} = x] \\
 &= P[Y_{N_t - N_s} \in A \mid Y_0 = x] \\
 &= P_x[X_{t-s} \in A] = p(t-s, x, A)
 \end{aligned}$$

$$p(t+s, x, A) = e^{-\lambda(t+s)} \sum_k \frac{(t+s)^k}{k!} \pi^{(k)}(x, A)$$

$$\begin{aligned}
 &\underbrace{\frac{1}{k!} \sum_{m=0}^k \binom{k}{m} t^{k-m} s^m}_{=} \\
 &= \sum_{m=0}^k \frac{t^{k-m}}{(k-m)!} \frac{s^m}{m!}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n,m} e^{-\lambda t} \frac{t^n}{n!} e^{-\lambda s} \frac{s^m}{m!} \underbrace{\pi^{(n+m)}(x, A)}_{\int \pi^{(n)}(x, dy) \pi^{(m)}(y, A)} \\
 &= \int p(t, x, dy) p(s, y, A)
 \end{aligned}$$

Let B denote the Banach space of bounded functions $f: E \rightarrow \mathbb{R}$ with

$$\|f\| = \sup_{x \in E} |f(x)|.$$

Define the transition operator $\Pi: B \rightarrow B$:

$$\Pi f(x) = \int \Pi(x, dy) f(y)$$

for any transition kernel $\Pi(x, dy)$. Then:

- $\|\Pi f\| \leq \|f\|$ and $\Pi 1 = 1$
- $\Pi f \geq 0$ if $f \geq 0$

If $T_t f(x) = \int p(t, x, dy) f(y) dy$ then also

- $T_0 = \text{id}$, $T_{t+s} = T_t T_s = T_s T_t$ for any $s, t \geq 0$.

If $p(t, x, dy)$ is defined in terms of Π as above:

$$\begin{aligned} T_t &= \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Pi^k = e^{-\lambda t} \exp(\lambda t \Pi) \\ &= \exp(tL), \quad L = \lambda(\Pi - \text{id}). \end{aligned}$$

Thus

$$Lf(x) = \lambda \int (f(y) - f(x)) \Pi(x, dy).$$

is the generator.

Example. Let $a: E \rightarrow (0, \infty)$ be bounded and

$$Lf(x) = a(x) \int (f(y) - f(x)) \Pi(x, dy)$$

Then if $\lambda \geq a(x)$ for all x , can set

$$\hat{\Pi}(x, A) = \frac{a(x)}{\lambda} \Pi(x, A) + \left(1 - \frac{a(x)}{\lambda}\right) 1_{x \in A}$$

so that

$$Lf(x) = \lambda \int (f(y) - f(x)) \hat{\Pi}(x, dy).$$

Thm. Let (X_t) be a Markov jump process with generator L as above where a is bounded.

Then the strong Markov property holds: for any stopping time τ ,

$$P_x[X_{t+\tau} \in A \mid \mathcal{F}_\tau] = p(t, X_\tau, A).$$

Sketch. Similar to similar statements before:

Prove first for τ taking finitely many values (which is straightforward from standard Markov property). Then take limit.

Thm. Under the same conditions,

$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds$
is a martingale.

Proof. Similar to indep. incr. case.

Cor. For any λ ,

$$\begin{aligned} M_t^f e^{-\lambda t} &= M_0^f + \lambda \int_0^t M_s^f e^{-\lambda s} ds \\ &= f(X_t) e^{-\lambda t} - f(X_0) - \int_0^t e^{-\lambda s} [Lf(X_s) - \lambda f(X_s)] ds \end{aligned}$$

and this is a martingale.

Proof.

$$\begin{aligned} M_t^f e^{-\lambda t} &= M_0^f + \lambda \int_0^t M_s^f e^{-\lambda s} ds \\ &= e^{-\lambda t} \left[f(X_t) - f(X_0) - \underbrace{\int_0^t Lf(X_s) ds}_{g(t)} \right] + \lambda \int_0^t M_s^f e^{-\lambda s} ds \end{aligned}$$

Since $e^{-\lambda t} g(t) = g(0) - \int_0^t \underbrace{(\lambda e^{-\lambda s} g(s) - e^{-\lambda s} g'(s))}_{-(e^{-\lambda s} g(s))'} ds$
 this is

$$\begin{aligned}
 &= e^{-\lambda t} f(X_t) - f(X_0) \\
 &\quad + \int_0^t \left[\lambda e^{-\lambda s} f(X_0) + \lambda e^{-\lambda s} \int_0^s Lf(X_u) du - e^{-\lambda s} Lf(X_s) \right. \\
 &\quad \left. + \lambda e^{-\lambda s} (f(X_s) - f(X_0) - \int_0^s Lf(X_u) du) \right] ds \\
 &= e^{-\lambda t} f(X_t) - f(X_0) + \int_0^t (\lambda f(X_s) - Lf(X_s)) e^{-\lambda s} ds
 \end{aligned}$$

Since M^f is a martingale:

$$\begin{aligned}
 &E \left[M_t^f e^{-\lambda t} - M_0^f - \int_0^t M_r^f \lambda e^{-\lambda r} dr \mid \mathcal{F}_s \right] \\
 &= M_s^f e^{-\lambda t} - M_0^f - \int_0^s M_r^f \lambda e^{-\lambda r} dr + M_s^f \underbrace{\int_s^t \lambda e^{-\lambda r} dr}_{e^{-\lambda s} - e^{-\lambda t}} \\
 &= M_s^f e^{-\lambda s} - M_0^f - \int_0^s M_r^f \lambda e^{-\lambda r} dr
 \end{aligned}$$

Thm. Let $\tau_A = \inf\{t \geq 0 : X_t \notin A\}$. For $\lambda > 0$,

$$u(x) = E_x[e^{-\lambda \tau_A} f(X_{\tau_A})]$$

is the unique solution to

$$\begin{aligned} Lu(x) &= \lambda u(x) & (x \in A) \\ u(x) &= f(x) & (x \notin A). \end{aligned}$$

Proof. Let $x \in A$. Let $\tau = \tau_1$ be the time of first jump: an $\text{Exp}(a(x))$ random variable. Since τ and X_τ are independent (by construction):

$$\begin{aligned} u(x) &= E_x[e^{-\lambda \tau_A} f(X_{\tau_A})] \\ &= E_x[e^{-\lambda \tau} \underbrace{E_{x_\tau}[e^{-\lambda \tau_A} f(X_{\tau_A})]}_{u(x_\tau)}] \end{aligned}$$

$$= E_x[e^{-\lambda \tau}] E_x[u(X_\tau)]$$

$$= \frac{a(x)}{\lambda + a(x)} \int u(y) \Pi(x, dy)$$

$$\Leftrightarrow (\lambda + a(x)) u(x) = a(x) \int u(y) \Pi(x, dy)$$

$$\Leftrightarrow Lu(x) = a(x) \int \Pi(x, dy) (u(y) - u(x)) = \lambda u(x).$$

Conversely, let $Lu = \lambda u$ on A , $u = f$ on A^c .

$$e^{-\lambda t} u(X_t) - u(X_0) - \int_0^t e^{-\lambda s} [Lu - \lambda u](X_s) ds$$

is a martingale, so if OST can be applied,

$$(L - \lambda)u(X_s) = 0 \quad \text{for } 0 \leq s < \tau_A$$

$$u(X_{\tau_A}) = f(X_{\tau_A}),$$

we get

$$E_x[e^{-\lambda \tau_A} f(X_{\tau_A})] = u(x),$$

proving uniqueness.

To make sure, OST can be applied, replace τ_A by $\tau_A \wedge t$. Then

$$E_x[e^{-\lambda(\tau_A \wedge t)} u(X_{\tau_A \wedge t})] = u(x).$$

Take $t \rightarrow \infty$.

6.3. Explosion

Example. Let $E = \mathbb{Z}_+ = \{n: n \geq 0\}$. Given birth rates $\lambda_n > 0$ and death rates $\mu_n > 0$, the birth-death process has generator

$$(Lf)_n = \sum_m L_{nm} f_m$$

with

$$L_{n,n+1} = \lambda_n$$

$$L_{n,n-1} = \mu_n$$

$$L_{n,n} = -(\lambda_n + \mu_n)$$

and $L_{n,m} = 0$ else.

Now consider $\lambda_n = (n+1)^2$, $\mu_n = 0$. The waiting time at step n has expectation $(n+1)^{-2}$. Thus

$$\sum_{n=0}^{\infty} E[\tau_n] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty.$$

Thus

$$\tilde{Z} = \lim_{n \rightarrow \infty} T_n < \infty \quad \text{a.s.}$$

Thus the process will reach ∞ (exit from all finite sets) by time $\tilde{Z}$. It is called the explosion time.

This can happen whenever $a(x)$ in the generator is unbounded.

Thm. The Markov process corresponding to

$$Lf(x) = a(x) \int (f(y) - f(x)) \Pi(x, dy)$$

starting from x_0 explodes in finite time with positive probability iff there is a $0 \leq u \leq C$ with $u(x_0) > 0$ s.t.

$$Lu(x) \geq \sigma u(x) \quad \text{for all } x \in E$$

for some $\sigma > 0$.

Proof. For $\sigma > 0$ set

$$u_\sigma(x) = E_x[e^{-\sigma \zeta}]$$

Then there is explosion iff $u_\sigma(x_0) > 0$.

Let τ be the time of first jump. Then (as before)

$$\begin{aligned} u_\sigma(x) &= E_x[E_x[e^{-\sigma \zeta} | \mathcal{F}_\tau]] \\ &= E_x[e^{-\sigma \tau} u_\sigma(X_\tau)] \\ &= E[e^{-\sigma \tau}] E_x[u_\sigma(X_\tau)] \end{aligned}$$

$$= \frac{a(x)}{a(x) + \sigma} \int U_0(y) \Pi(x, dy)$$

$$\Rightarrow L U_0(x) = \sigma U_0(x).$$

Conversely, given U , let

$$\begin{aligned} M_t &= e^{-\sigma t} U(X_t) - \int_0^t e^{-\sigma s} (\underbrace{L U - \sigma U}_{\geq 0})(X_s) ds \\ &= e^{-\sigma t} U(X_t) - A_t \end{aligned}$$

with A increasing.

$\Rightarrow e^{-\sigma t} U(X_t)$ is a submartingale

Since it is bounded, OST with $T_n = \tau_1 + \dots + \tau_n$ the time of the n -th jump gives

$$U(x_0) \leq E_{x_0}[e^{-\sigma T_n} U(X_{T_n})] \leq C E_{x_0}[e^{-\sigma T_n}]$$

$$\Rightarrow U(x_0) \leq C E_{x_0}[e^{-\sigma \zeta}]$$

$$\Rightarrow P_{x_0}[\zeta < \infty] \geq \frac{U(x_0)}{C} > 0.$$

Thm. Let $\sigma > 0$ and assume there is $U \geq 0$ s.t.

$$LU(x) \leq \sigma U(x) \quad \text{for all } x$$

and $U(x) \rightarrow \infty$ whenever $a(x) \rightarrow \infty$. Then there cannot be explosion.

Proof. Now $e^{-\sigma t} U(X_t)$ is a supermartingale.

$$\stackrel{\text{OST}}{\Rightarrow} U(x) \geq E_x[e^{-\sigma(T_n \wedge T)} U(X_{T_n \wedge T})]$$

$$\stackrel{\text{Fatou}}{\Rightarrow} U(x) \geq E_x[e^{-\sigma T_n} U(X_{T_n})]$$

Let $\Omega_0 = \{\lim T_n < \infty\}$. Since the waiting time at T_n is $\text{Exp}(a(X_{T_n}))$,

$$\sum_n \frac{1}{a(X_{T_n})} < \infty$$

on Ω_0 . Thus $a(X_{T_n}) \rightarrow \infty$ and $U(X_{T_n}) \rightarrow \infty$ by assumption. This contradicts

$$U(x) \geq E_x[\underbrace{e^{-\sigma T_n}}_{>0} \underbrace{U(X_{T_n})}_{\rightarrow \infty}]$$

For $a(x)$ unbounded, L is unbounded.

Example Consider again the birth-death process with rates λ_n and μ_n .

For $i \geq 1$,

$$LU(i) = \sigma U(i)$$

$$\Leftrightarrow \lambda_i U(i+1) + \mu_i U(i-1) - (\lambda_i + \mu_i) U(i) = \sigma U(i)$$

$$\Leftrightarrow U(i+1) = \frac{(\sigma + \lambda_i + \mu_i) U(i) - \mu_i U(i-1)}{\lambda_i}$$

and for $i=0$,

$$\lambda_0 U(1) = (\lambda_0 + \sigma) U(0).$$

$$\Leftrightarrow U(1) = \frac{\lambda_0 + \sigma}{\lambda_0} U(0).$$

We know that if U is the soln with $U(n) = 1$ and $\tau_n = \inf\{t : X_t = n\}$ then

$$U(i) = E_i[e^{-\sigma \tau_n}].$$

Thus for any soln,

$$\frac{U(i)}{U(n)} = E_i[e^{-\sigma \tau_n}] \quad \text{for } 0 \leq i \leq n-1$$

Note from the recursion that $U(n)$ is increasing.

Nonexplosion is equivalent to $U(n) \rightarrow \infty$.

To determine when $U(n) \rightarrow \infty$ take $U(0) = 1$. Then

$$U(1) = 1 + \frac{\sigma}{\lambda_0}$$

and

$$U(i+1) - U(i) = \frac{\mu_i}{\lambda_i} (U(i) - U(i-1)) + \frac{\sigma}{\lambda_i} U(i)$$

Exercise Necessary and sufficient conditions for boundedness are

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} \sum_{j=1}^n \frac{\lambda_1 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_{j-1}} \frac{1}{\mu_j} < \infty$$

$$\& \sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$$

Example If $\lambda_n - \mu_n \leq C(n+1)$ there is no explosion.

Proof. Let $U(n) = (n+1)^\alpha$, $\alpha < 1$. Then

$$\begin{aligned} LU(n) &= \lambda_n (U(n+1) - U(n)) + \mu_n (U(n) - U(n-1)) \\ &\leq (\lambda_n - \mu_n) (U(n) - U(n-1)) \\ &\leq C(n+1) \alpha (n+1)^{\alpha-1} = C \alpha U(n). \end{aligned}$$

