

Please submit solutions to columbia2026@bauerschmidt.ca.

1. Day 1 – Mean-field theory

Problem 1.1. Complete the exercises mentioned in the lecture for the mean-field Ising model and extend the analysis to the mean-field $O(3)$ model.

Problem 1.2. Analyze the mean-field Ising model in a more direct fashion by counting the possible configurations of the magnetization $\frac{1}{N} \sum_{i=1}^N \sigma_i$.

2. Day 2 – Correlation inequalities

Problem 2.1. Adapt high temperature expansion and the Peierls argument from percolation to the Ising model. For example, show that the Ising model on $\mathbb{Z}^2$ with + boundary conditions has positive magnetization.

Use correlation inequalities to extend the conclusion of positive magnetization from $\mathbb{Z}^2$ to $\mathbb{Z}^d$ when $d \geq 2$.

Problem 2.2. Let A be a positive definite matrix and $z > 0$. Use Ginibre's method to derive inequalities for the measure proportional to the following density on $\mathbb{R}^N$:

$$e^{-\frac{1}{2}(\varphi, A\varphi) + z \sum_{i=1}^N \cos(\varphi_i)} \prod_i d\varphi_i,$$

where $d\varphi$ is the Lebesgue measure. In particular, show that, for any test vector $f \in \mathbb{R}^N$:

$$\langle e^{i(\varphi, f)} \rangle = \langle \cos((\varphi, f)) \rangle \text{ is monotone increasing in } z,$$

where $(\varphi, f) = \sum_{i=1}^N \varphi_i f_i$. Deduce that

$$\langle (\varphi, f)^2 \rangle \text{ is monotone decreasing in } z.$$

What happens as $z \rightarrow \infty$?

Problem 2.3. Show that the XY model has an infinite volume with either free or Dirichlet boundary conditions.

3. Day 3 – Ward identities

Problem 3.1. Prove $\langle \sigma_0^1 \sigma_x^1 \rangle \leq \frac{1}{n} \sum_y J_{0y} \langle \sigma_y^1 \sigma_x^1 \rangle$ if $h = 0$. For a transitive graph (such as the torus $\mathbb{Z}^d / L\mathbb{Z}^d$), deduce that $\sum_y J_{0y} < (1 - \varepsilon)n$ implies that $\langle \sigma_0 \cdot \sigma_x \rangle_{J,0}$ decays exponentially.

Problem 3.2. On a finite graph, consider the $O(n)$ model with external field in direction $h \in \mathbb{R}^n$, $h \neq 0$. Show that the magnetization points in the same direction as h , i.e., there is $m(J, h) > 0$ such that

$$\langle \sigma_i \rangle_{J,h} = m_i(J, h) \frac{h}{|h|}.$$

Problem 3.3. Show that the infinitesimal rotations corresponding to the group $SO(3)$ acting on the components of a vector in $\mathbb{R}^3$ are exactly antisymmetric matrices.

4. Day 4 – Reflection positivity and infrared bound

Problem 4.1. Let X be a rotationally invariant vector on $\mathbb{R}^n$ with $X \neq 0$ almost surely. Show that $X/|X|$ is uniform on S^{n-1} and independent of $|X|$. Here $|X|$ is the Euclidean norm of X .

Problem 4.2. Use the infrared bound to show the $O(n)$ model has spontaneous magnetization in $d \geq 3$:

$$\langle e \cdot \sigma_0 \rangle_{\beta, h} \geq 1 - O\left(\frac{1}{\beta}\right) + O_{\beta}\left(\frac{1}{h|\Lambda|}\right),$$

where e is a unit vector in the direction of the magnetic field.

5. Day 5 – Potts and random cluster models

Problem 5.1. Show that $P_J(i \leftrightarrow j) = \frac{1}{q-1} \langle \sigma_i \cdot \sigma_j \rangle_J = \langle \sigma_i^1 \sigma_j^1 \rangle_J$ where the probability measure on the left-hand side is the q -RCM and the expectation on the right-hand side is that of the q -state Potts model.

Generalize this to the Edwards–Sokal coupling: Define a probability measure on $\{1, \dots, q\}^{\Lambda} \times \{0, 1\}^E$ by

$$P(\sigma, \omega) \propto p^{\omega} (1-p)^{-\omega} 1(\sigma \text{ is constant on each cluster of } \omega). \quad (5.1)$$

For $q \in \{2, 3, \dots\}$, show the ω -marginal is the random cluster model with parameters (p, q) and the σ -marginal is the q -state Potts model with $p = 1 - e^{-q\beta}$.

Problem 5.2. Let K be the cluster containing the origin. For $q \in \{2, 3, \dots\}$ and $d \geq 3$, use reflection positivity to show that on the torus $\mathbb{E}|K| \geq |\Lambda|(1 + O(1/\beta))$.

Problem 5.3. Use the Harris inequality to show that the q -state random cluster model stochastically dominates Bernoulli bond percolation when the corresponding p parameters are suitably related. For $q > 1$ and $d \geq 2$, derive a better lower bound on $\mathbb{E}|K|$ as $\beta \rightarrow \infty$ using results for Bernoulli bond percolation.

6. Day 6 – Grassmann variables

Problem 6.1. Show that every even element of a Grassmann algebra commutes with all other elements of the Grassmann algebra (even or odd).

Problem 6.2. Let $C = (C_{ij})$ be a symmetric invertible matrix and define

$$\Delta_C = \sum_{i,j} C_{ij} \frac{\partial}{\partial \psi_j} \frac{\partial}{\partial \bar{\psi}_i}.$$

Show that if $A = C^{-1}$ then

$$\langle F \rangle_C \stackrel{\text{def}}{=} \frac{\int \partial_{\psi} \partial_{\bar{\psi}} e^{\bar{\psi} A \psi} F}{\int \partial_{\psi} \partial_{\bar{\psi}} e^{\bar{\psi} A \psi}} = e^{\Delta_C} F|_0,$$

where $F|_0$ denotes the coefficient of the constant term in F . Derive Wick's formula

$$\left\langle \prod_{k=1}^n \bar{\psi}_{i_k} \psi_{j_k} \right\rangle_C = \det (C_{i_k j_{\ell}})_{k, \ell=1}^n.$$

Problem 6.3. In Lecture 1 we used the formula

$$\exp \left[-\frac{1}{4} \sum_{i,j=1}^N |\sigma_i - \sigma_j|^2 \right] \propto \int_{\mathbb{R}^n} d\varphi \exp \left[-\frac{1}{2} \sum_{i=1}^N |\varphi - \sigma_i|^2 \right].$$

Derive the following analogue for Grassmann variables. Let $\psi_i, \bar{\psi}_i$, $i = 1, \dots, N$ and $\theta, \bar{\theta}$ (no indices) be Grassmann variables. Then

$$\exp \left[\frac{1}{4} \sum_{i,j=1}^N (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \right] \propto \int \partial_{\theta} \partial_{\bar{\theta}} \exp \left[\frac{1}{2} \sum_{i=1}^N (\bar{\theta} - \bar{\psi}_i)(\theta - \psi_i) \right].$$

7. Day 7 – The 0-state Potts model

Problem 7.1. Show the following Ward identity for the 0-state Potts model (analogous to the one for the $O(n)$ model):

$$\langle z_0 \rangle_{\beta, h} = h \sum_{i \in \Lambda} \langle \bar{\psi}_0 \psi_i \rangle_{\beta, h}.$$

Interpret it probabilistically in terms of random forests.

Problem 7.2. Show that

$$\sigma_i \cdot \sigma_j = -z_i z_j + \psi_i \bar{\psi}_j + \psi_j \bar{\psi}_i$$

is invariant not only under the global symmetries $T = \sum_i T_i$ and $\bar{T} = \sum_i \bar{T}_i$, but also under $S = \sum_i S_i$ where

$$S_i = \bar{\psi}_i \partial_{\psi_i} + \psi_i \partial_{\bar{\psi}_i}.$$

Find two further infinitesimal symmetries $U = \sum_i U_i$ and $V = \sum_i V_i$. Show that the generators do not commute. These five infinitesimal symmetries generate an action of the super-Lie algebra $\mathfrak{osp}(1|2)$. Compare with the familiar case of $\mathfrak{so}(3)$ represented by three infinitesimal rotations preserving $\sigma_i \cdot \sigma_j$ where $\sigma_i, \sigma_j \in S^2$.

8. Day 8 – Fermionic localization

Problem 8.1. Show by hand that, for any positive definite matrix A ,

$$\int \prod_i d\Phi_i e^{-\sum_{ij} A_{ij} \Phi_i \cdot \Phi_j} = 1. \quad (8.1)$$

Problem 8.2. Let $\mathbb{H}^2 = \{\sigma = (z, u, v) : -z^2 + u^2 + v^2 = -1, z > 0\}$ be the hyperboloid model of the hyperbolic plane and consider the spin model with spins $\sigma_i \in \mathbb{H}^2$. Show the Ward identity

$$\langle z_0 \rangle_{\beta, h} = \frac{1}{h} \sum_i \langle u_i u_j \rangle_{\beta, h}.$$

9. Day 9 – No forest percolation

Problem 9.1. Show that, on a finite graph, the arboreal gas converges as $\beta \rightarrow 0$ to a uniformly chosen spanning tree. Use the $\mathbb{H}^{0|2}$ representation (0-state Potts model) to derive Kirchhoff's matrix tree theorem by taking the limit $\beta \rightarrow \infty$ after appropriate rescaling. The matrix tree theorem states that

$$\sum_T \prod_{ij} J_{ij} = \det(-\Delta_J^0)$$

where the sum on the left-hand side runs over all spanning trees and Δ_J^0 on the right-hand side is the Laplacian matrix with an arbitrary column and row removed.

Problem 9.2. Let Z_J be the partition function of the arboreal with coupling constants $J = (J_{ij})$. In this problem, we show that $J \in [0, \infty)^E \mapsto Z_J$ is log-concave. This is a special case of the results of Brändén–Huh on Lorentzian Polynomials and Anari et al. (with very different perspective and proof).

Let $0 \in \Lambda$ be an arbitrary vertex and set $t_0 = 0$. Then a variation of what was shown in class gives the following formula for the partition function of the arboreal gas:

$$Z_J = \int_{\mathbb{R}^{\Lambda \setminus \{0\}}} \prod_{i \in \Lambda \setminus \{0\}} \frac{dt_i}{\sqrt{2\pi}} e^{-3 \sum_{i \in \Lambda} t_i} \det(-\Delta_{J(t)}^0)^{3/2} \exp \left[- \sum_{ij} J_{ij} (\cosh(t_i - t_j) - 1) \right]$$

where $\Delta_{J(t)}^0$ denotes the matrix $\Delta_{J(t)}$ with the 0-row and column removed, and

$$(\Delta_{J(t)} f)_i = \sum_{j \in \Lambda} J_{ij} e^{t_i + t_j} (f_j - f_i).$$

Let e^{-t} be the diagonal matrix $e^{-t} = \text{diag}(e^{-t_i})$. Show that

$$e^{-t} \circ \Delta_{J(t)} \circ e^{-t} = \Delta_J - \text{diag } V_J(t)$$

with $V_J(t) = (V_J(t)_i)_{i \in \Lambda}$ a suitable potential you should find. For $V \in \mathbb{R}^\Lambda$, define

$$H_J(V) = -\Delta_J + \text{diag}(V), \quad H_J^0(V) = H_J(V)^0, \quad C_J = \{V \in \mathbb{R}^\Lambda : H_J(V) \geq 0\},$$

where the superscript 0 again means that the 0-row and column are removed. Deduce that

$$\begin{aligned} Z_J &\propto \int_{\mathbb{R}^{\Lambda \setminus \{0\}}} \prod_{i \in \Lambda \setminus \{0\}} dt_i e^{-\frac{1}{2} \sum_{i \in \Lambda} V_J(t)_i} \det H_J^0(V_J(t))^{3/2} \\ &= \frac{1}{2} \int_{C_J} dV e^{-\frac{1}{2} \sum_{i \in \Lambda} V_i} \det H_J^0(V)^{1/2}. \end{aligned}$$

The second line follows by changing variables from $(t, s) \in \mathbb{R}^{\Lambda \setminus \{0\}} \times [0, \infty)$ to $V = V_J(t) + s\delta_0$, where δ_0 is the unit vector at the root, with Jacobian

$$\left| \det \frac{\partial V}{\partial (t, s)} \right| = \det H_J^0(V_J(t)).$$

Use the last formula for Z_J and the Prekopa–Leindler Theorem to show that $J \in [0, \infty)^E \mapsto Z_J$ is log-concave.