

Lattice Models of Statistical Physics (SLMath Summer School)

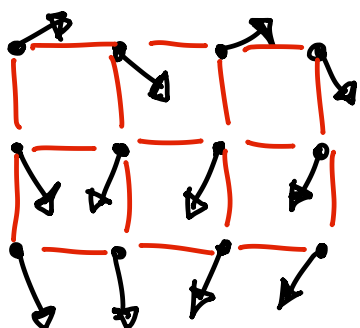
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1. Spin models and mean-field theory

- $\Lambda \subset \mathbb{Z}^d$ finite set of points
- $J_{xy} = J_{yx} \geq 0$ ferromagnetic coupling constants
- $\sigma: \Lambda \rightarrow S^{n-1} \subset \mathbb{R}^n$ spin configurations



$n=1$: Ising model, $\sigma_x \in \{\pm 1\}$

$n=2$: XY model, $\sigma_x = \begin{pmatrix} \cos \theta_x \\ \sin \theta_x \end{pmatrix}$

$n=3$: classical Heisenberg model

Defn. For $\beta \geq 0$, the $O(n)$ model on Λ with coupling constants J is given by prob. meas.

$$\nu_\beta(d\sigma) \propto e^{\beta \sum_{x,y} J_{xy} \sigma_x \cdot \sigma_y} \prod_{x \in \Lambda} d\sigma_x$$

$$\propto e^{-\frac{\beta}{4} \sum_{x,y} J_{xy} (\sigma_x - \sigma_y)^2} \prod_{x \in \Lambda} d\sigma_x$$

Expectation:

$$\langle F \rangle_\beta = \int F(\sigma) \nu_\beta(d\sigma)$$

Two-point function:

$$G_\beta(x,y) = \langle \sigma_x \cdot \sigma_y \rangle_\beta.$$

Main question: Do spins align?

- LRO: $G_{\beta}(x,y) \gtrsim c > 0$
- Exp. decay: $G_{\beta}(x,y) \lesssim e^{-c|x-y|}$
- Poly. decay: $G_{\beta}(x,y) \approx |x-y|^{-\alpha}$

General results:

- Exp. decay for β small, any d : high temp. expansion.
- Exp. decay for $d=1$, any β : transfer matrix
- LRO for $n=1$, $d \geq 2$ if β large: Peierls argument.
- LRO for $n \geq 2$, $d \geq 3$ if β large, $\wedge$ torus: RP
- no LRO for $n \geq 2$, $d=2$, any $\beta > 0$: Mermin-Wagner
- Poly. decay for $n=2$, $d=2$, β large: KT phase

Polyakov conjecture: Exp. decay for $n \geq 3$, $d=2$, $\beta > 0$.

Goldstone picture: LRO for $n \geq 3$, $\beta \gg 1$.

Critical behavior: Near $\beta \sim \beta_c$.

Exercise: Similar to percolation arguments:

- High temperature expansion ($\beta \ll 1$).
- Peierls argument ($\beta \gg 1$, $n=1$, $d=2$).

Mean-field model: $\Lambda = [N] = \{1, \dots, N\}$, $J_{ij} = \frac{1}{N}$.

Exercise: Enumerate the configurations of the Ising model on the complete graph and compute $m(\beta) = \langle (\frac{1}{N} \sum \sigma_i)^2 \rangle^{1/2}$.

Lemma. For any $\sigma = (\sigma_i)_{i=1}^N \in (\mathbb{R}^n)^N$,

$$\exp\left[\frac{\beta}{2N} \sum_{i,j=1}^N \sigma_i \cdot \sigma_j\right] \propto \int_{\mathbb{R}^n} d\varphi \exp\left[-\frac{1}{2} \beta N |\varphi|^2 + \beta \varphi \cdot \sum_{i=1}^N \sigma_i\right]$$

Equivalently,

$$\exp\left[\frac{\beta}{4N} \sum_{i,j=1}^N |\sigma_i - \sigma_j|^2\right] \propto \int d\varphi \exp\left[-\frac{\beta}{2} \sum_{i=1}^N |\varphi - \sigma_i|^2\right]$$

↑ N spins ↑ single 'block' spin

Proof. By rescaling, can take $\beta=1$.

$$\begin{aligned} C_N &= \int d\varphi \exp\left[-\frac{N}{2} |\varphi|^2\right] \\ &= \int d\varphi \exp\left[-\frac{N}{2} \left|\varphi - \frac{1}{N} \sum \sigma_i\right|^2\right] \\ &= \int d\varphi \exp\left[-\frac{N}{2} |\varphi|^2 + \varphi \cdot \sum \sigma_i - \frac{1}{2N} \left|\sum \sigma_i\right|^2\right]. \end{aligned}$$

Rearrange.

Partition function:

$$Z_\beta = \int_{\mathbb{R}^{nN}} e^{-\frac{\beta}{4N} \sum (\sigma_i - \sigma_j)^2 + h \sum_i \mathbf{e} \cdot \sigma_i} \prod_{i=1}^N d\sigma_i$$

unit vector

$$= \int_{\mathbb{R}^n} d\varphi \left(\underbrace{\int e^{-\frac{\beta}{2} |\varphi - \sigma_1|^2 + h \mathbf{e} \cdot \sigma_1} d\sigma_1}_{e^{-NV_{\beta,h}(\varphi)}} \right)^N$$

For $n=1$: $V_{\beta,h}(\varphi) = -\log \int e^{-\frac{\beta}{2} (\varphi - \sigma)^2 + h\sigma} d\sigma$
 (sing)

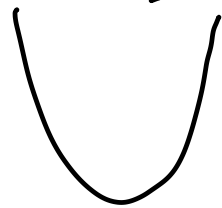
$$= -\log (e^{-\frac{\beta}{2} (\varphi-1)^2 + h} + e^{-\frac{\beta}{2} (\varphi+1)^2 + h})$$

$$= \frac{\beta}{2} \varphi^2 - \log \cosh(\beta\varphi + h) + \text{const.}$$

Exercise: Plot the function $V_{\beta,h}$. For $h=0$,

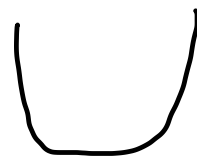
- For $\beta < 1$, V_β is strictly convex:

$$V_\beta''(\varphi) \geq \beta - \beta^2$$



- For $\beta = 1$, V_β is convex but

$$V_\beta''(0) = 0, \quad V_\beta(\varphi) \sim |\varphi|^4$$



- For $\beta > 1$, V_β has two minima



Laplace Principle. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be

- continuous, bounded below
- $\{\varphi: V(\varphi) \leq \min V + 1\}$ is compact
- $\int e^{-V(\varphi)} d\varphi < \infty$.

Then

$$-\frac{1}{N} \log \int e^{-NV(\varphi)} d\varphi \xrightarrow{N \rightarrow \infty} \min V.$$

What are the minima of $V_{\beta,h}$?

$$V_{\beta,h}(\varphi) = \frac{\beta}{2} \varphi^2 - \log \cosh(\beta\varphi + h)$$

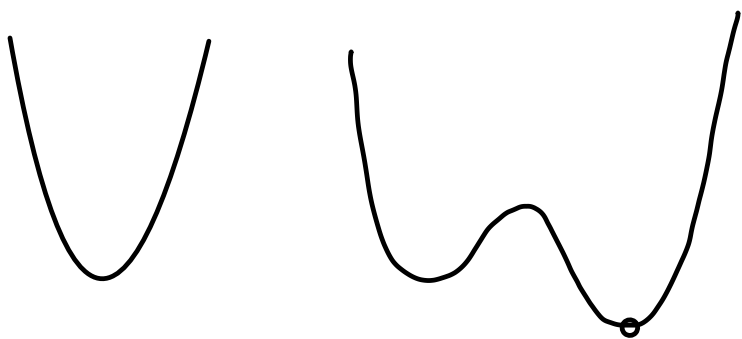
$$\Rightarrow V'_{\beta,h}(\varphi) = \beta\varphi - \beta \tanh(\beta\varphi + h)$$

$$V''_{\beta,h}(\varphi) = \beta - \frac{\beta^2}{\cosh^2(\beta\varphi + h)}$$

$$\text{So } V'_{\beta,h}(\varphi_0) = 0 \Leftrightarrow \varphi_0 = \tanh(\beta\varphi_0 + h) \\ \text{self-consistent eqn.}$$

Exercise: for $h > 0$, there is a unique min. $\varphi_0(\beta, h)$.

- If $\beta \leq 1$ then $\varphi_0(\beta, h) \rightarrow 0$ as $h \downarrow 0$.
- If $\beta > 1$ then $\varphi_0(\beta, h) \rightarrow \varphi_0(\beta, 0+) > 0$ as $h \downarrow 0$.



Cor. If $\beta > 1$ then

$$\lim_{h \downarrow 0} \lim_{N \rightarrow \infty} \langle \sigma_i \rangle_{\beta, h} = \lim_{h \downarrow 0} \lim_{N \rightarrow \infty} \frac{\partial}{\partial h} \underbrace{\frac{1}{N} \log Z_{\beta, h}}_{-F^N(\beta, h)} > 0.$$

Proof. $h \mapsto -F^N(\beta, h)$ is convex (check!).

If F^N are differentiable convex functions, $F^N(h) \rightarrow F(h)$ for all h , then F is convex and at every h for which F is differentiable, $\nabla F^N(h) \rightarrow \nabla F(h)$.

By Laplace Principle, $F^N_{\beta}(h) \rightarrow F_{\beta}(h) = \min V_{\beta, h}$.

$$\Rightarrow \lim_{h \downarrow 0} \lim_{N \rightarrow \infty} \langle \sigma_i \rangle_{\beta, h} \stackrel{\text{(above)}}{=} - \lim_{h \downarrow 0} \frac{\partial}{\partial h} (V_{\beta, h}(\varphi_0(\beta, h))) = - \lim_{h \downarrow 0} V'_{\beta, h}(\varphi_0)$$

where φ_0 is ! soln to $\varphi = \tanh(\beta\varphi + h)$.

$$\Rightarrow \langle \sigma_i \rangle_{\beta, \text{ot}} = \lim_{h \downarrow 0} \beta \tanh(\beta\varphi_0 + h) = \lim_{h \downarrow 0} \beta \varphi_0(\beta, h).$$

This is called spontaneous symmetry breaking.
 Symmetry implies, for any $\beta \neq 0$,

$$\lim_{h \downarrow 0} \langle \sigma_1 \rangle_{\beta, h} = \langle \sigma_1 \rangle_{\beta, 0} = 0$$

Thus the limits do not commute.

Exercise: Compute other observables. For example,

$$\langle \sigma_1 \rangle_{\beta, 0+} \sim (3(\beta - \beta_c))^{1/2} \quad (\beta \downarrow \beta_c = 1)$$

$$\langle \sigma_1 \rangle_{\beta_c, h} \sim (3h)^{1/3} \quad (h \downarrow 0)$$

The susceptibility $\chi(\beta, h) = \frac{\partial \langle \sigma_1 \rangle_{\beta, h}}{\partial h}$ satisf.

$$\chi(\beta, 0) = \frac{1}{\beta_c - \beta}$$

$$(\beta < \beta_c)$$

$$\chi(\beta, 0+) \sim \frac{1}{2(\beta - \beta_c)}$$

The above are examples of critical exponents.

Exercise: Generalize this to the $O(3)$ model.

2. Correlation inequalities

uniform on S^{n-1}

O(n) model:

$$\langle F \rangle_{\beta, h} \propto \int F(\sigma) e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i e \cdot \sigma_i} \prod d\sigma_i$$

unit vector $\nearrow$

1st Griffith inequality. For $n \geq 1$, $J_{ij} \geq 0$, $h_i \geq 0$,

$$\langle \sigma_{x_1}^{m_1} \dots \sigma_{x_k}^{m_k} \rangle \geq 0, \quad m_\ell \in [n]. \quad (\text{GI})$$

2nd Griffith inequality. For $n = 1, 2$, $J_{ij} \geq 0$, $h_i \geq 0$,

$$\langle \underbrace{\sigma_i \cdot \sigma_j}_{\text{unit vector}}; \sigma_k \cdot \sigma_\ell \rangle \geq 0. \quad (\text{GII})$$

$$\langle (\sigma_i \cdot \sigma_j)(\sigma_k \cdot \sigma_\ell) \rangle - \langle \sigma_i \cdot \sigma_j \rangle \langle \sigma_k \cdot \sigma_\ell \rangle$$

Open question: $n \geq 3$?

Cor. $\langle \sigma_0 \cdot \sigma_x \rangle_{\beta, h}$ is increasing in $\beta \geq 0$, $h \geq 0$.

Proof.

$$\begin{aligned} \frac{\partial}{\partial \beta} \langle F \rangle_{\beta, h} &= \frac{\partial}{\partial \beta} \frac{\int e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i \sigma_i \cdot e} F}{\int e^{+\beta \sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i \sigma_i \cdot e}} \\ &= \langle F; \sum J_{ij} \sigma_i \cdot \sigma_j \rangle_{\beta, h} \geq 0. \end{aligned}$$

Proof of (GI). By expanding

$$e^{\sum J_{ij} \sigma_i \cdot \sigma_j + \sum h_i \sigma_i}$$

it suffices to consider $J=0$, $h=0$. Thus the measure is product. For any $O(n)$ -invariant measure of a single spin one can check that

$$\int s^{m_1} \dots s^{m_k} \mu(ds) \geq 0.$$

Proof of (GII), $n=1$.

$$\langle \sigma_i \cdot \sigma_j ; \sigma_k \cdot \sigma_\ell \rangle = \frac{1}{2} \langle (\sigma_i \cdot \sigma_j - \sigma'_i \cdot \sigma'_j) (\sigma_k \cdot \sigma_\ell - \sigma'_k \cdot \sigma'_\ell) \rangle$$

where σ' is an independent copy.

$$H(\sigma) + H(\sigma') = \sum J_{ij} (\sigma_i \cdot \sigma_j + \sigma'_i \cdot \sigma'_j)$$

Write

$$\begin{aligned} t_i &= \frac{1}{2}(\sigma_i + \sigma'_i) \\ q_i &= \frac{1}{2}(\sigma_i - \sigma'_i) \end{aligned} \left\{ \Rightarrow \begin{aligned} \sigma_i \sigma_j + \sigma'_i \sigma'_j &= 2(t_i t_j + q_i q_j) \\ \sigma_i \sigma_j - \sigma'_i \sigma'_j &= 2(t_i q_j + t_j q_i) \end{aligned} \right.$$

$$\Rightarrow H(\sigma) + H(\sigma') = -\sum 2J_{ij} (t_i t_j + q_i q_j)$$

$$(\sigma_i \sigma_j - \sigma'_i \sigma'_j) (\sigma_k \sigma_\ell - \sigma'_k \sigma'_\ell) = 4(t_i q_j + t_j q_i)(t_k q_\ell + t_\ell q_k)$$

$$\Rightarrow \langle \sigma_i \sigma_j ; \sigma_k \sigma_e \rangle = \sum (\text{pos.}) \int e^{+2 \sum J_{ij} (t_i t_j + q_i q_j)} \dots$$

Expand.

Exercise. Show $\langle \sigma_{x_1} \dots \sigma_{x_k} ; \sigma_{y_1} \dots \sigma_{y_e} \rangle \geq 0$.

Proof of (GI), $n=2$. Use polar coordinates.

$$\sigma_i = \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix}.$$

For the duplicated system, set

$$\varphi = \frac{1}{2}(\theta + \theta'), \quad \varphi' = \frac{1}{2}(\theta' - \theta)$$

Then (with $m\theta = \sum m_i \theta_i$)

$$\cos(m\theta) + \cos(m\theta') = 2 \cos(m\varphi) \cos(m\varphi')$$

$$\cos(m\theta) - \cos(m\theta') = 2 \sin(m\varphi) \sin(m\varphi')$$

Thus

$$\begin{aligned} -H(\theta) - H(\theta') &= \sum_{ij} J_{ij} (\cos(\theta_i - \theta_j) + \cos(\theta'_i - \theta'_j)) \\ &= \sum_{ij} 2J_{ij} \cos(\varphi_i - \varphi_j) \cos(\varphi'_i - \varphi'_j) \end{aligned}$$

$$\Rightarrow e^{-H} = \sum (\text{positive}) \prod \cos(m\varphi) \cos(m\varphi')$$

Thus

$$\begin{aligned}
 & \langle \sigma_i \cdot \sigma_j ; \sigma_k \cdot \sigma_e \rangle \\
 &= \langle \cos(\theta_i - \theta_j) ; \cos(\theta_k - \theta_e) \rangle \\
 &= \frac{1}{2} \langle \underbrace{(\cos(\theta_i - \theta_j) - \cos(\theta'_i - \theta'_j))}_{2\sin(\theta_i - \theta_j)\sin(\theta'_i - \theta'_j)} \underbrace{(\cos(\theta_k - \theta_e) - \dots)}_{\dots} \rangle \\
 &\propto \sum (\text{positive}) \left(\int \prod \sin(m\theta) \prod \cos(n\theta) d\theta \right)^2 \geq 0.
 \end{aligned}$$

Rk. The proof immediately implies the more general inequality

$$\langle \cos(\sum_i m_i \theta_i) ; \cos(\sum_i n_i \theta_i) \rangle \geq 0$$

for every $m_i \in \mathbb{Z}, n_i \in \mathbb{Z}$.

Example. $\langle \sigma_i \cdot \sigma_j \rangle_J^{XY} \leq \langle \sigma_i \cdot \sigma_j \rangle_J^{\text{Ising}}$.

Proof. Let

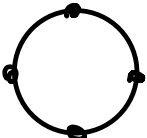
$$\langle F \rangle_J^\lambda \propto \int F(\sigma) e^{\sum_{i,j} J_{ij} \cos(\theta_i - \theta_j) + \lambda \sum_i \cos(4\theta_i)} \prod_i d\theta_i$$

$\uparrow$
 $\sigma = (\cos \theta, \sin \theta)$.

Then $\langle \cdot \rangle_J^{XY} = \langle \cdot \rangle_J^0$ and

$$\lim_{\lambda \rightarrow \infty} \langle \cdot \rangle_J^\lambda \propto \sum_{\sigma \in C_4} F(\sigma) e^{\sum_{i,j} J_{ij} \sigma_i \cdot \sigma_j}$$

where (since $\cos 4\theta_i = 1 \Leftrightarrow \theta_i \in \frac{\pi}{2}\mathbb{Z}$)

$$C_4 = \{(\pm 1, 0), (0, \pm 1)\}.$$


$$\text{Ginibre: } \frac{\partial}{\partial \lambda} \langle \sigma_i \cdot \sigma_j \rangle^\lambda \geq 0$$

$$\Rightarrow \langle \sigma_i \cdot \sigma_j \rangle_J^{XY} \leq \langle \sigma_i \cdot \sigma_j \rangle_J^\infty$$

Let $t_i = \sigma_i^1 + \sigma_i^2$, $q_i = \sigma_i^1 - \sigma_i^2$. Then

$$2 \sum_{i,j} J_{ij} \sigma_i \cdot \sigma_j = \sum_{i,j} J_{ij} (t_i t_j + q_i q_j)$$

and if $\sigma_i \sim C_4$ then $(t_i, q_i) \sim \{(\pm 1), (\pm 1)\}$. Thus

$$\langle \sigma_i \cdot \sigma_j \rangle_J^\infty = \langle t_i t_j \rangle_{\frac{1}{2}J}^{\text{Ising}}$$

Example. Consider the XY model on $\Lambda \subset \mathbb{Z}^d$ with

- free boundary conditions:

$$\langle F \rangle_{\Lambda, f} \propto \int_{(S^1)^\Lambda} F e^{\sum_{i,j \in \Lambda} J_{ij} \cos(\theta_i - \theta_j)} \prod_i d\theta_i$$

- Dirichlet boundary conditions:

$$\langle F \rangle_{\Lambda, +} \propto \int_{(S^1)^\Lambda} e^{\sum_{i,j \in \Lambda} J_{ij} \cos(\theta_i - \theta_j) + \sum_{\substack{i \in \Lambda \\ j \notin \Lambda}} J_{ij} \cos(\theta_i)} \prod_i d\theta_i.$$

Then if $F = \cos(\sum m_i \theta_i)$, $\Lambda_1 \subset \Lambda_2$,

$$\langle F \rangle_{\Lambda_1, f} \leq \langle F \rangle_{\Lambda_2, f} \quad (F)$$

$$\langle F \rangle_{\Lambda_1, +} \geq \langle F \rangle_{\Lambda_2, +}. \quad (D)$$

As a consequence, if $\Lambda_i \nearrow \mathbb{Z}^d$, the limits

$$\langle F \rangle^\# = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle F \rangle_{\Lambda, \#}$$

exist for any trigonometric polynomial F (and by density for all local continuous observables).

3. Word identities & the Mermin-Wagner thm

Principle: Symmetries imply identities.

Consider the $O(n)$ model with ext. field $h \in \mathbb{R}^n$:

$$\nu_{\beta, h}(d\sigma) \propto e^{\beta \sum_{ij} J_{ij} \sigma_i \cdot \sigma_j + \sum_i h \cdot \sigma_i} \prod_i d\sigma_i$$

$$= e^{-H(\sigma)} \prod_i d\sigma_i$$

invariant under global
rotations if $h=0$.

$\sigma_i \rightarrow R\sigma_i$, same $R \in SO(n)$.

invariant under local
rotations: $\sigma_i \rightarrow R_i \sigma_i$

Fact. Let $R=(R_i)$ be a local rotation. Then

$$\langle F(R\sigma) \rangle = \langle e^{-H(R^{-1}\sigma) + H(\sigma)} F(\sigma) \rangle.$$

In particular, if $R=R(t)$ and $TF(\sigma) = \left. \frac{d}{dt} F(R(t)\sigma) \right|_{t=0}$ is the action of an infinitesimal rotation,

$$\langle TF \rangle = \langle F(TH) \rangle$$

Example. Let $R(t)$ be a rotation by angle t .
Then in polar coordinates ($n=2$),

$$\sigma = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad R(t)\sigma = \begin{pmatrix} \cos \theta + t \\ \sin \theta + t \end{pmatrix}$$

$$\Rightarrow T\sigma^1 = -\sin\theta = -\sigma^2$$

$$T\sigma^2 = \cos\theta = \sigma^1$$

$$\Rightarrow T_i(\sigma_i \cdot \sigma_j) = T_i(\sigma_i^1 \sigma_j^1 + \sigma_i^2 \sigma_j^2) = -\sigma_i^2 \sigma_j^1 + \sigma_i^1 \sigma_j^2$$

↑ rotation acting only on σ_i

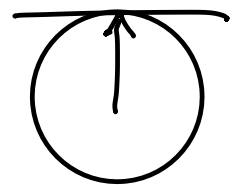
$$\Rightarrow (T_i + T_j)(\sigma_i \cdot \sigma_j) = 0$$

Example. Let $T = \sum_i T_i$ be a global rotation.

$$\Rightarrow TH = T\left(-\sum_{i,j} J_{ij} \sigma_i \cdot \sigma_j - \sum_i h_i \sigma_i^1\right) = \sum_i h_i \sigma_i^2$$

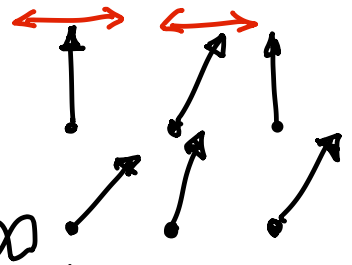
Thus if $h_i = h$ is constant, $F = \sigma_0^2$,

$$\langle \sigma_0^1 \rangle_{J,h} = h \sum_i \langle \sigma_0^2 \sigma_i^2 \rangle_{J,h}.$$



Cor. If $\lim_{h \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma_0^1 \rangle_{\beta,h} > 0$

then $\lim_{h \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_x \langle \sigma_0^2 \sigma_x^2 \rangle_{\beta,h} \rightarrow \infty$



The transversal two-point function cannot decay exponentially if there is spontaneous magn.

Example. Let $R_j(t)$ be the local rotation ($n=2$)

$$R_j(t)\sigma = \begin{cases} \sigma_j & (j \neq 0) \\ e^{Mt}\sigma_0 & (j = 0) \end{cases}$$

$$\Rightarrow TH = \frac{2}{\partial t} \left(- \sum_j J_{0j} \cos(\theta_0 - \theta_j + t) - h \cos(\theta_0 + t) \right) \Big|_{t=0}$$

$$\Rightarrow TH = \sum_j J_{0j} \sin(\theta_0 - \theta_j) + h \sin \theta_0.$$

$$\Rightarrow \underbrace{\langle \cos \theta_0 \rangle}_{T \sin \theta_0} = \left\langle \sin \theta_0 \left(\sum_j J_{0j} \underbrace{\sin(\theta_0 - \theta_j)}_{\sin \theta_0 \cos \theta_j - \cos \theta_0 \sin \theta_j} + h \sin \theta_0 \right) \right\rangle$$

$$\leq \sum_j J_{0j} \langle (\sin \theta_0)^2 \cos \theta_j \rangle + h \langle (\sin \theta_0)^2 \rangle$$

Now

$$\begin{aligned} \langle \cos \theta_j \rangle &= \langle (\cos^2 \theta_0 + \sin^2 \theta_0) \cos \theta_j \rangle \\ &= 2 \langle \sin^2 \theta_0 \cos \theta_j \rangle \\ &\quad - \underbrace{\langle (\cos^2 \theta_0 - \sin^2 \theta_0) \cos \theta_j \rangle}_{\cos(2\theta_0)} \end{aligned}$$

$$\leq 2 \langle \sin^2 \theta_0 \cos \theta_j \rangle$$

$$1 \leq 2 \langle \sin^2 \theta_0 \rangle$$

$$\Rightarrow \langle \cos \theta_0 \rangle \leq \frac{1}{n} \sum_j J_{0j} \langle \cos \theta_j \rangle + h$$

Exercise: Generalize proof to $n \geq 2$.

Exercise. If $h=0$ then also show

$$\langle \cos \theta_0 \cos \theta_j \rangle \leq \frac{1}{n} \sum_i J_{0i} \langle \cos \theta_i \cos \theta_j \rangle$$

Cor. On any vertex-transitive graph,

$$m(\beta, h) = \langle \cos \theta_0 \rangle_{\beta, h}$$

$$G_\beta(x) = \langle \cos \theta_0 \cos \theta_x \rangle_{\beta, 0}$$

satisfy ($|J| = \sum J_{0j}$)

$$m(\beta, h) \leq \frac{|J|\beta}{n} m(\beta, h) + h$$

$$G_\beta(x) \leq \frac{|J|\beta}{n} \sum_{e \sim 0} G_\beta(x-e).$$

Exercise: If $\beta < 1/|J|$ then

$$\lim_{h \rightarrow 0} \limsup_{\beta \uparrow 1/|J|} m^*(\beta, h) = 0 \quad (\text{no spont. mag.})$$

$$G_\beta(x) \leq C e^{-c|x|} \quad (\text{exp. decay})$$

Compare these results with the results for the complete graph.

Aside: Fourier series on the lattice

Let $\Lambda \simeq \{1, \dots, L\}^d$ be a discr. torus of side length L

Let $\Lambda^* = \left\{ \frac{2\pi}{L} n : n \in \Lambda \right\}$.

For $f: \Lambda \rightarrow \mathbb{R}$, define

$$\hat{f}(p) = \sum_{x \in \Lambda} f_x e^{-ip \cdot x} = (f, e_p) \sqrt{|\Lambda|}$$

$$\Rightarrow f_x = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \hat{f}(p) e^{ip \cdot x} = \sum_{p \in \Lambda^*} (f, e_p) e_p(x)$$

$$\stackrel{L \gg 1}{\approx} \int_{[-\pi, \pi]^d} \frac{dp}{(2\pi)^d} \hat{f}(p) e^{ip \cdot x}$$

Parseval identity:

$$\sum_{x \in \Lambda} f_x^2 = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} |\hat{f}(p)|^2$$

Laplace operator on Λ :

$$(\Delta f)_x = \sum_y J_{xy} (f_y - f_x)$$

$$\Rightarrow \widehat{\Delta f}(p) = \hat{\Delta}(p) \hat{f}(p),$$

where if $J_{xy} = 1_{x \sim y}$

$$\hat{\Delta}(p) = \sum_{e \sim 0} (e^{ip \cdot e} - 1) \approx -|p|^2$$

Mermin-Wagner Theorem. Let Λ be a torus, $J_{ij} = J(i-j) = J(j-i)$. Then for any $p \in \Lambda^*$,

$$\sum_j \langle \sigma_0^2 \sigma_j^2 \rangle_{\beta, h} e^{ip \cdot j} \geq \frac{\langle \sigma_0^1 \rangle^2}{-\hat{\Delta}(p) + h \langle \sigma_0^1 \rangle}$$

$$\Rightarrow 1 \geq \langle \sigma_0^2 \sigma_0^2 \rangle_{\beta, h} \geq \langle \sigma_0^1 \rangle_{\beta, h}^2 \int \frac{1}{-\hat{\Delta}(p) + h \langle \sigma_0^1 \rangle} \frac{dp}{(2\pi)^d}$$

In $d=2$,

$\langle \sigma_0^1 \rangle_{\beta, h} \downarrow 0$ as $h \downarrow 0$ uniformly in Λ .

Proof. Let

$$\hat{T}(p) = \sum_j e^{-ip \cdot j} T_j$$

$$\hat{S}(p) = \sum_j e^{-ip \cdot j} \sigma_j^2$$

$$\Rightarrow \frac{|\langle \hat{S}(p) \hat{T}(p) H \rangle|^2}{\langle \hat{T}(p) \hat{S}(p) \rangle} \leq \langle |\hat{S}(p)|^2 \rangle \frac{\langle |\hat{T}(p) H|^2 \rangle}{\langle \hat{T}(p) H \overline{\hat{T}(p) H} \rangle}$$

$$= \langle \hat{T}(p) \hat{T}(p) H \rangle$$

$$\Rightarrow \langle |\hat{S}(p)|^2 \rangle \geq \frac{|\langle \hat{T}(p) \hat{S}(p) \rangle|^2}{\langle \hat{T}(p) \hat{T}(p) H \rangle}$$

Now:

$$\begin{aligned} \frac{1}{N} \langle |\hat{S}(p)|^2 \rangle &= \frac{1}{N} \sum_{j,k} e^{i(j-k) \cdot p} \langle \sigma_j^2 \sigma_k^2 \rangle \\ &= \sum_k e^{ik \cdot p} \langle \sigma_0^2 \sigma_k^2 \rangle \end{aligned}$$

$$\begin{aligned} \frac{1}{N} \langle \overline{\hat{T}(p)} \hat{S}(p) \rangle &= \frac{1}{N} \sum_{j,k} e^{i(j-k) \cdot p} \underbrace{\langle T_j \sigma_k^2 \rangle}_{\delta_{jk} \sigma_k^1} \\ &= \langle \sigma_0^1 \rangle_{\beta, h} \end{aligned}$$

$$\frac{1}{N} \langle \hat{T}(p) \overline{\hat{T}(p)} H \rangle = \frac{1}{N} \sum_{j,k} e^{i(j-k) \cdot p} \langle T_j T_k H \rangle$$

For $j \neq k$:

$$\begin{aligned} &T_j T_k \left(-\sum_{u,v} J_{u-v} \sigma_u \cdot \sigma_v - h \sum \sigma_u^1 \right) \\ &= -T_j \left(\sum_v J_{k-v} (-\sigma_k^2 \sigma_v^1 + \sigma_k^1 \sigma_v^2) + h \sigma_k^2 \right) \\ &= -J_{k-j} (\sigma_k^2 \sigma_j^2 + \sigma_k^1 \sigma_j^1) \end{aligned}$$

For $j=k$:

$$\begin{aligned}
 & T_j^2 (-\sum_u J_{u-v} \sigma_u \cdot \sigma_v - h \sum \sigma_u^1) \\
 &= -\sum_v J_{u-v} \underbrace{T_j^2 (\sigma_j \cdot \sigma_v)}_{T_j (\sigma_j^2 \sigma_v^1 + \sigma_j^1 \sigma_v^2)} + h \sigma_j^1 \\
 &= (-\sigma_j^1 \sigma_v^1 - \sigma_j^2 \sigma_v^2)
 \end{aligned}$$

$$\Rightarrow \langle \hat{T}(p) \overline{\hat{T}(p)} H \rangle = \underbrace{\sum_k J_k (e^{-ip \cdot k} - 1)}_{\hat{J}(p) - \hat{J}(0) = \hat{\Delta}(p)} \langle \sigma_0^1 \sigma_e^1 + \sigma_0^2 \sigma_e^2 \rangle$$

$$\Rightarrow |\langle \hat{T}(p) \overline{\hat{T}(p)} H \rangle| \leq -\hat{\Delta}(p) + h \langle \sigma_0^1 \rangle$$

Thus

$$\begin{aligned}
 \sum_k \langle \sigma_0^2 \sigma_k^2 \rangle e^{ip \cdot k} &\geq \frac{\langle \sigma_0^1 \rangle^2}{-\hat{\Delta}(p) \langle \sigma_0^1 \sigma_e^1 + \sigma_0^2 \sigma_e^2 \rangle + h \langle \sigma_0^1 \rangle} \\
 &\geq \frac{\langle \sigma_0^1 \rangle^2}{-\hat{\Delta}(p) + h \langle \sigma_0^1 \rangle}
 \end{aligned}$$

4. Reflection positivity

Let Λ be a discrete torus, Δ_Λ the n.n. Laplacian:

$$(\Delta_\Lambda f)_x = \sum_{y \sim x} f_y - f_x$$

Let μ be any measure on $\mathbb{R}^n$ with (say)

$$\int e^{\epsilon |s|^2} \mu(ds) < \infty. \quad \leftarrow \text{no } O(n) \text{ invariance req.}$$

Define

$$\langle F \rangle \propto \int e^{\frac{\beta}{2}(\sigma, \Delta \sigma)} \prod_{x \in \Lambda} \mu(d\sigma_x)$$

Thm. For any $f: \Lambda \rightarrow \mathbb{R}^n$, $\sum f_x = 0$,

$$\langle e^{(f, \sigma)} \rangle \leq e^{\frac{1}{2\beta} (f, \Delta f)}.$$

In particular,

$$\frac{1}{|\Lambda|} \langle |\hat{\sigma}(p)|^2 \rangle \leq (-\beta \hat{\Delta}(p))^{-1} \approx (\beta |p|^2)^{-1}$$

$$\sum_x \langle \sigma_0 \cdot \sigma_x \rangle e^{ip \cdot x} = \hat{G}_\beta(p)$$

Cor. Let $d \geq 3$. Then the $O(n)$ model satisfies

$$\frac{1}{|\Lambda|} \sum_x \langle \sigma_0 \cdot \sigma_x \rangle_{\beta,0} \geq 1 - O\left(\frac{1}{\beta}\right) \quad \text{LRO}$$

$$\langle \sigma_0 \cdot e \rangle_{\beta,h} \geq 1 - O\left(\frac{1}{\beta}\right) + O_\beta\left(\frac{1}{h|\Lambda|}\right). \quad \text{spont. mag.}$$

Proof. Parseval identity:

$$\sum_{x \in \Lambda} |\sigma_x|^2 = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} |\hat{\sigma}(p)|^2 = \frac{1}{|\Lambda|} |\hat{\sigma}(0)|^2 + \frac{1}{|\Lambda|} \sum_{p \neq 0} |\hat{\sigma}(p)|^2$$

On the other hand,

$$\sum_x \langle \sigma_0 \cdot \sigma_x \rangle = \frac{1}{|\Lambda|} \sum_{x,y} \langle \sigma_x \cdot \sigma_y \rangle = \left\langle \frac{1}{|\Lambda|} \left| \sum \sigma_x \right|^2 \right\rangle = \left\langle \frac{|\hat{\sigma}(0)|^2}{|\Lambda|} \right\rangle$$

$$\frac{1}{|\Lambda|} \left\langle \sum_{x \in \Lambda} |\sigma_x|^2 \right\rangle = 1.$$

$$\begin{aligned} \Rightarrow \frac{1}{|\Lambda|} \sum_x \langle \sigma_0 \cdot \sigma_x \rangle &= 1 - \frac{1}{|\Lambda|^2} \sum_{p \neq 0} |\hat{\sigma}(p)|^2 \\ &\geq 1 - \underbrace{\frac{1}{|\Lambda|} \sum_{p \neq 0} (-\beta \hat{\Delta}(p))^{-1}}_{\approx \int \frac{d^d p}{(2\pi)^d} \frac{1}{\beta |p|^2} \leq \frac{C}{\beta} \quad (d > 2)}. \end{aligned}$$

Magnetization: Exercise.

Proof of infrared bound.

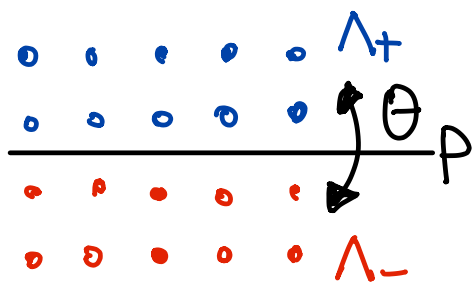
$$\text{Let } H(\sigma) = \frac{1}{2}(\sigma, -\Delta\sigma).$$

$$\Rightarrow H(\sigma) - (f, \sigma) \stackrel{\uparrow \Sigma f = 0}{=} H(\sigma - (-\Delta)^{-1}f) - \frac{1}{2}(f, (-\Delta)^{-1}f)$$

Thus

$$\int e^{-H(\sigma) + (f, \sigma)} \pi d\sigma_x = e^{-\frac{1}{2}(f, (-\Delta)^{-1}f)} \underbrace{\int e^{-H(\sigma - (-\Delta)^{-1}f)} \pi d\sigma_x}_x = Z(-(-\Delta)^{-1}f).$$

Suffices to prove $Z(f) \leq Z(0)$.



Split Λ into $\Lambda = \Lambda_+ \cup \Lambda_-$

Let $\theta: \Lambda_{\pm} \rightarrow \Lambda_{\mp}$ be reflection.

$$(\theta\sigma)_x = \sigma_{\theta x}$$

$$(\theta F)(\sigma) = F(\theta\sigma).$$

Defn. $\langle \cdot \rangle$ is RP if for any $F, G: (\mathbb{R}^n)^{\Lambda_+} \rightarrow \mathbb{R}$,

$$\langle F\theta G \rangle = \langle (\theta F)G \rangle \quad \langle F\theta F \rangle \geq 0.$$

Fact. Any i.i.d. measure is RP since

$$\langle F\theta G \rangle = \langle F \chi(\theta G) \rangle = \langle F \rangle \langle G \rangle$$

RP means $(F, G) \mapsto \langle F \Theta G \rangle$ is pos. semidefinite.

Cauchy-Schwarz:

$$\langle F \Theta G \rangle^2 \leq \langle F \Theta F \rangle \langle G \Theta G \rangle$$

Lemma. For $A, B, C, D: (\mathbb{R}^n)^{\wedge 1} \rightarrow \mathbb{R}$,

$$\langle e^{A+\Theta B+\Sigma C \Theta D} \rangle \leq \langle e^{A+\Theta A+\Sigma C \Theta C} \rangle \langle e^{B+\Theta B+\Sigma D \Theta D} \rangle$$

Proof.

$$e^{A+\Theta B+C \Theta D} = \sum_k \frac{1}{k!} \underbrace{(e^A C^k)}_{X_k} \Theta \underbrace{(e^B D^k)}_{Y_k}$$

$$\Rightarrow \langle e^{A+\Theta B+C \Theta D} \rangle \stackrel{\text{RP}}{\leq} \sum_k \frac{1}{k!} \langle X_k \Theta X_k \rangle^{1/2} \langle Y_k \Theta Y_k \rangle^{1/2}$$

$$\begin{aligned} \Rightarrow \langle e^{A+\Theta B+C \Theta D} \rangle^2 &\stackrel{\text{CS}}{\leq} \sum_k \frac{1}{k!} \langle X_k \Theta X_k \rangle \sum_l \frac{1}{l!} \langle Y_l \Theta Y_l \rangle \\ &= \langle e^{A+\Theta A+C \Theta C} \rangle \langle e^{B+\Theta B+D \Theta D} \rangle \end{aligned}$$

The case of $\Sigma C \Theta D$ is analogous.

Lemma. Let $f_{\pm} = f$ on $\Lambda_{\pm}$, $f_{\pm} = \theta f$ on $\Lambda_{\mp}$. Then

$$Z(f)^2 \leq Z(f_+)Z(f_-).$$

Proof.

$$H(\sigma) = \underbrace{\frac{1}{2} \sum_{xy \in E_+} |\sigma_x - \sigma_y|^2}_{H_+(\sigma)} + \underbrace{\frac{1}{2} \sum_{xy \in E_-} |\sigma_x - \sigma_y|^2}_{H_-(\sigma) = \theta H_+(\sigma)} + \underbrace{\frac{1}{2} \sum_{xy \in E_0} |\sigma_x - \sigma_y|^2}_{H_0(\sigma)}$$

Since $H_0(\sigma) = \frac{1}{2} \sum_{xy \in E_0} |\sigma_x|^2 + \theta |\sigma_x|^2 + 2\sigma_x \cdot \theta \sigma_x.$

$$\Rightarrow H(\sigma) = A + \theta A + \sum C \theta C.$$

$$\Rightarrow H(\sigma + f) = A_{f_+} + \theta A_{f_-} + \sum C_{f_+} \theta C_{f_-}$$

Apply prev. lemma.

Iteratively applying lemma for all hyperplanes gives

$$Z(f) \leq \sup_{g: \nabla g = 0} Z(g) = Z(0).$$

since $\nabla_e f_{\pm} = 0$ for $e \in E_0$.

5. Potts and random cluster models

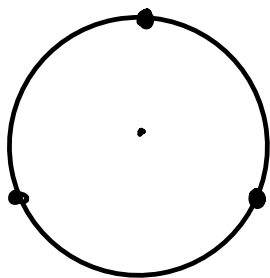
Ising model: σ_x takes two possible values

$$\sigma_x \sigma_y + 1 = 2 \mathbb{1}_{\sigma_x = \sigma_y} \text{ if } \sigma_x \in \{\pm 1\}$$

Potts model: σ_x takes q possible values

$$\sigma_x \cdot \sigma_y + 1 = q \mathbb{1}_{\sigma_x = \sigma_y} \text{ if } \sigma_x \in T_q \subset \mathbb{R}^{q-1}$$

Exercise: $T_3 = \frac{1}{\sqrt{2}} \{(0, 2), (\sqrt{3}, -1), (-\sqrt{3}, -1)\}$



Prop.
$$\begin{aligned} Z_J &= \int e^{\sum J_{ij} (\sigma_i \cdot \sigma_j + 1)} \prod_i d\sigma_i \quad \int d\sigma = \frac{1}{q} \sum_{\sigma \in T_q} \\ &= \int e^{q \sum J_{ij} \mathbb{1}_{\sigma_i = \sigma_j}} \prod_i d\sigma_i \\ &= q^{-n} \sum_{\omega \in CE} (q\alpha)^\omega q^{k(\omega)}, \quad \alpha_{ij} = \frac{1}{q} (e^{J_{ij} q} - 1) \\ &= q^{-n} \sum_{\omega \in CE} p^\omega (1-p)^{-\omega} q^{k(\omega)}, \quad p_{ij} = 1 - e^{-J_{ij} q} \end{aligned}$$

where $\omega \in CE$ denotes all subgraphs and $k(\omega)$ is the number of clusters in ω .

Proof. Since $\sigma_i \cdot \sigma_j + 1 \in \{0, q\}$ if $\sigma_i, \sigma_j \in T_q$,

$$e^{J_{ij}(\sigma_i \cdot \sigma_j + 1)} = 1 + \alpha_{ij}(\sigma_i \cdot \sigma_j + 1), \quad \alpha_{ij} = \frac{1}{q}(e^{J_{ij}q} - 1).$$

Indeed,

$$\sigma_i = \sigma_j \Rightarrow \text{both sides } e^{J_{ij}q}$$

$$\sigma_i \neq \sigma_j \Rightarrow \text{both sides } 1.$$

$$\begin{aligned} \Rightarrow e^{\sum J_{ij}(\sigma_i \cdot \sigma_j + 1)} &= \prod_{ij} (1 + \alpha_{ij}(\sigma_i \cdot \sigma_j + 1)) \\ &= \sum_{\omega} \prod_{ij \in \omega} \alpha_{ij} \prod_{ij \in \omega} (\sigma_i \cdot \sigma_j + 1) \end{aligned}$$

The claim follows from the observation that for any graph ω on Λ :

$$\int \prod_{ij \in \omega} \underbrace{(\sigma_i \cdot \sigma_j + 1)}_{\in \{q, 0\}} d\sigma = q^{|\omega| + k(\omega) - N}$$

In other words, for each connected ω ,

$$\int \prod_{ij \in \omega} (\sigma_i \cdot \sigma_j + 1) d\sigma = q^{1 + |\omega| - N(\omega)}$$

Defn For $q > 0$, the q -random cluster measure is

$$P_J^q(\omega) \propto p^w (1-p)^{-w} q^{k(\omega)}, \quad p_{ij} = 1 - e^{-qJ_{ij}}$$

Exercise.

$$\langle \sigma_i \cdot \sigma_j \rangle_{J,0} = (q-1) P_J^q(i \leftrightarrow j)$$

and equivalently

$$\langle \sigma_i^\alpha \sigma_j^\alpha \rangle_{J,0} = P_J^q(i \leftrightarrow j)$$

More generally, derive the Edwards-Sokal coupling.

Exercise. Introduce an external field $h > 0$ in the Potts model:

$$\langle F \rangle_{J,h} \propto \int F e^{\sum_{ij} J_{ij} \sigma_i \cdot \sigma_j + h \sum_i \sigma_i \cdot e} \prod d\sigma_i$$

where $e \in T_q$ is arbitrary. Show that

$$\langle \sigma_i \cdot e \rangle_{J,h} = P_{J,h}(i \leftrightarrow \mathbb{G})$$

where $P_{J,h}$ is the RCM on $\Lambda \cup \{\mathbb{G}\}$ with couplings $J_{i\mathbb{G}} = h$ for all $i \in \Lambda$.

Exercise For $q \in \{2, 3, \dots\}$ use reflection positivity to show that on the torus, for $\beta > 0$ large,

$$E|K| = \sum_x P(0 \leftrightarrow x) \geq |A| (1 + O(\frac{1}{\beta}))$$

The I-RCM P_J^I is Bernoulli bond percolation:

$$P_J^I(\omega) \propto \prod_{ij \in \omega} \frac{P_{ij}}{1 - P_{ij}} = \prod_{ij \in \omega} \frac{1 - e^{-J_{ij}}}{e^{-J_{ij}}} = \prod_{ij} (e^{J_{ij}} - 1)$$

The O-RCM P_J^O is the random forest model (also called arboreal gas):

$$P_J^O(\omega) \propto \prod_{ij \in \omega} J_{ij} \mathbb{1}(\omega \text{ is a forest}).$$

Indeed, recall that, for each connected ω ,

$$\int \prod_{ij \in \omega} (\sigma_i \cdot \sigma_j + 1) d\sigma = q^{1 + |\omega| - N(\omega)}$$

As $q \rightarrow 0$, want to minimize $|1 + |\omega| - N(\omega)|$. Note that $|1 + |\omega| - N(\omega)| = 0$ if ω is a tree.

Prop

$$P(i \leftrightarrow j) \leq P'(i \leftrightarrow j).$$

Proof: Note that

$\{i \leftrightarrow j\}$ is increasing

$\{\text{forest}\}$ is decreasing

Thus by the Harris-FGK inequality,

$$P'(\{i \leftrightarrow j\} \cap \{\text{forest}\}) \leq P'(i \leftrightarrow j) P'(\text{forest})$$

$$\Leftrightarrow P^0(i \leftrightarrow j) = P'(i \leftrightarrow j \mid \text{forest}) \leq P'(i \leftrightarrow j).$$

Cor. For $J < J_c^{\text{Percolation}}$ on $\mathbb{Z}^d$,

$$P_J^0(i \leftrightarrow j) \leq C e^{-c|i-j|}$$

What happens at low temperature?

6. Grassmann variables and forests

Want to find 'spins' σ_i s.t. for w connected:

$$\int \prod_{i \in w} (1 + \sigma_i \cdot \sigma_j) = \mathbb{1}(w \text{ is a tree}) \\ = \lim_{q \rightarrow 0} q^{|w| - |V(w)|}$$

Let Λ be finite. For $i \in \Lambda$, let $\psi_i, \bar{\psi}_i$ be Grassmann variables satisfying

$$\psi_i \psi_j = -\psi_j \psi_i, \quad \bar{\psi}_i \psi_j = -\psi_j \bar{\psi}_i, \quad \bar{\psi}_i \bar{\psi}_j = -\bar{\psi}_j \bar{\psi}_i$$

An element of $\Omega^{\Lambda \sqcup \bar{\Lambda}}$ is a noncommutative polynomial in the $\psi, \bar{\psi}$:

$$F = \sum_n \frac{1}{n!} \sum_{z_1} \cdots \sum_{z_n} F_{z_1, \dots, z_n} \psi^{z_1} \cdots \psi^{z_n}$$

with $z \in \Lambda \sqcup \bar{\Lambda}$ where $\bar{\Lambda}$ is a copy of Λ and $\psi^{\bar{x}} = \bar{\psi}^x$.

Example. $\bar{\psi}_i \psi_j \in \Omega^{\Lambda \sqcup \bar{\Lambda}}$

$$e^{\bar{\psi}_i \psi_j} = 1 + \bar{\psi}_i \psi_j \in \Omega^{\Lambda \sqcup \bar{\Lambda}}$$

Defn. F is even if only $n \in 2\mathbb{N}$ contribute,
odd if only $n \in 1 + 2\mathbb{N}$ contribute.

Exercise: Even elements commute with all others

Defn. Let $g: \mathbb{R}^N \rightarrow \mathbb{R}$ be smooth near 0 and $F_1, \dots, F_N$ be even. Define

$$g(F_1, \dots, F_N) = \text{Taylor polynomial}(F_1, \dots, F_N).$$

Defn. Define $\partial_{\psi_i}: \Omega \rightarrow \Omega$ linear by

$$\partial_{\psi_i}(\psi_i F) = F$$

whenever F is a monomial s.t. $\psi_i F \neq 0$.

Defn. Define $\int \partial_{\psi_i}: \Omega \rightarrow \Omega$ linear by

$$\int \partial_{\psi_i} F = \partial_{\psi_i} F$$

and $\int \partial_{\psi} \partial_{\bar{\psi}} F = \int \partial_{\psi_{i_1}} \partial_{\bar{\psi}_{i_1}} \dots \partial_{\psi_{i_n}} \partial_{\bar{\psi}_{i_n}} F \in \mathbb{R}$

Example.

$$\int \partial_{\psi} \partial_{\bar{\psi}} e^{a \bar{\psi}_1 \psi_1} = \int \partial_{\psi} \partial_{\bar{\psi}_1} (1 + a \bar{\psi}_1 \psi_1)$$

$$= \int \partial_{\psi_1} \partial_{\bar{\psi}_1} a \bar{\psi}_1 \psi_1 = a$$

$$\int \partial_{\psi} \partial_{\bar{\psi}} \psi_1 = 0$$

Example. For F even or odd,

$$\begin{aligned}\int \partial_\psi \partial_{\bar\psi} (\partial_{\psi_i} F) G &= \int \partial_\psi \partial_{\bar\psi} (\partial_{\psi_i} (FG)) - (-1)^{\deg F} F \partial_{\psi_i} G \\ &= 0 - (-1)^{\deg F} \int \partial_\psi \partial_{\bar\psi} F \partial_{\psi_i} G.\end{aligned}$$

Example Let $\bar\psi A \psi = \sum_{ij} A_{ij} \bar\psi_i \psi_j$. Then

$$\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} = \det(A)$$

Proof. Let $X = \{1, \dots, N\}$.

$$e^{\bar\psi A \psi} = 1 + \sum A_{ij} \bar\psi_i \psi_j + \frac{1}{2} \sum A_{ij} A_{ke} \bar\psi_i \psi_j \bar\psi_k \psi_e + \dots$$

Only the last term of maximal degree contributes

$$\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} = \int \partial_\psi \partial_{\bar\psi} \frac{1}{N!} \sum_{i_1, \dots, i_N} \dots \sum_{j_1, \dots, j_N} A_{i_1 j_1} \dots A_{i_N j_N} \bar\psi_{i_1} \psi_{j_1} \dots \bar\psi_{i_N} \psi_{j_N}.$$

Only terms with $\{i_1, \dots, i_N\} = \{1, \dots, N\}$ contribute:

$$\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} = \int \partial_\psi \partial_{\bar\psi} \sum_{j_1} \dots \sum_{j_N} A_{1j_1} \dots A_{Nj_N} \bar\psi_1 \psi_{j_1} \dots \bar\psi_N \psi_{j_N}$$

Only terms with $\{j_1, \dots, j_N\} = \{1, \dots, N\}$ contribute.

Thus there is $\sigma \in S_N$ s.t. $j_k = \sigma(k)$ and

$$\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} = \int \partial_\psi \partial_{\bar\psi} \sum_{\sigma \in S_N} A_{1\sigma(1)} \cdots A_{N\sigma(N)} \bar\psi_1 \psi_{\sigma(1)} \cdots$$

Since

$$\bar\psi_1 \psi_{j_1} \bar\psi_2 \psi_{j_2} \cdots = - \bar\psi_1 \psi_{j_2} \bar\psi_2 \psi_{j_1} \cdots$$

finally

$$\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} = \sum_{\sigma} (-1)^\sigma A_{1\sigma(1)} \cdots A_{N\sigma(N)} = \det(A).$$

Exercise. Prove the Wick formula:

$$\left\langle \prod_{k=1}^n \bar\psi_{i_k} \psi_{j_k} \right\rangle_C = \det(C_{i_k j_e})_{k,e=1}^n.$$

if

$$\langle F \rangle_C = \frac{\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi} F}{\int \partial_\psi \partial_{\bar\psi} e^{\bar\psi A \psi}}, \quad A = C^{-1}.$$

7. The 0-state Potts model

Let $z_i = \sqrt{1 - 2\bar{\psi}_i \psi_i} = 1 - \bar{\psi}_i \psi_i$

and define $\sigma_i = (z_i, \bar{\psi}_i, \psi_i)$ as a triple of elements of the Grassmann algebra. Set

$$\sigma_i \cdot \sigma_j = -z_i z_j + \psi_i \bar{\psi}_j + \bar{\psi}_i \psi_j.$$

Note that

$$\sigma_i \cdot \sigma_i = -z_i^2 + 2\psi_i \bar{\psi}_i = -1 + 2\bar{\psi}_i \psi_i + 2\psi_i \bar{\psi}_i = -1.$$

This resembles the sphere constraint $\sigma_i \cdot \sigma_i = 1$.

Let $T_j = z_j \frac{\partial}{\partial \psi_j}$, $\bar{T}_j = z_j \frac{\partial}{\partial \bar{\psi}_j}$

$T, \bar{T}$ are analogues of infinitesimal rot.

Prop. $T_j \psi_j = z_j$, $T_j \bar{\psi}_j = 0$, $T_j z_j = \bar{\psi}_j$

$$\bar{T}_j \psi_j = 0, \quad \bar{T}_j \bar{\psi}_j = z_j, \quad \bar{T}_j z_j = -\psi_j$$

$$\& \quad (T_i + T_j)(\sigma_i \cdot \sigma_j) = (\bar{T}_i + \bar{T}_j)(\sigma_i \cdot \sigma_j) = 0.$$

Proof $T_j \psi_j = z_j$, $T_j z_j = T_j (1 - \bar{\psi}_j \psi_j) = z_j \bar{\psi}_j = \bar{\psi}_j$
 $\bar{T}_j \bar{\psi}_j = z_j$, $\bar{T}_j z_j = -\psi_j$

$$\Rightarrow T_i (\sigma_i \cdot \sigma_j) = T_i (-z_i z_j + \psi_i \bar{\psi}_j + \psi_j \bar{\psi}_i)$$

$$= -\bar{\psi}_i z_j + z_i \bar{\psi}_j$$

$$\Rightarrow (T_i + \bar{T}_i) (\sigma_i \cdot \sigma_j) = 0.$$

Rk Together with three further 'symplectic' inf. rotations preserving $\sigma_i \cdot \sigma_j$ the $T, \bar{T}$ generate the super-Lie algebra $\text{osp}(1|2)$, an analog of $\mathfrak{o}(3)$.

The $O(n)$ Haar measure satisfies

$$\int_{S^{n-1}} (TF) d\sigma = 0$$

What is the $H^{0|2}$ analogue of the Haar meas.?

Define

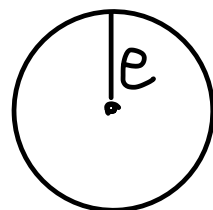
$$\int_{H^{0|2}} d\sigma F = \int \partial_\psi \partial_{\bar{\psi}} \frac{1}{Z} F$$

Fact. For any $F \in \Lambda$,

$$\int d\sigma_i T_i F = \int d\bar{\sigma}_i \bar{T}_i F = \dots = 0$$

$$\int d\sigma_i 1 = 1$$

$$\int d\sigma_i \sigma_i \cdot \sigma_j = \int d\sigma_i \sigma_i \cdot e = 0$$



Proof. (1) Since $\partial_4^2 = 0$,

$$\int \partial_4 \partial_{\bar{4}} \frac{1}{Z} T F = \int \partial_4 \partial_{\bar{4}} \frac{1}{Z} Z (\partial_4 F) = \int \partial_4 \partial_{\bar{4}} \partial_4 F = 0$$

$$(2) \int d\sigma 1 = \int \partial_4 \partial_{\bar{4}} \frac{1}{1 - \bar{4}4} = \int \partial_4 \partial_{\bar{4}} (1 + \bar{4}4) = 1.$$

$$(3) \int d\sigma_2 \sigma_1 \cdot \sigma_2$$

$$= \int \partial_{4_2} \partial_{\bar{4}_2} \frac{1}{Z_2} (-Z_1 Z_2 + 4_1 \bar{4}_2 + 4_2 \bar{4}_1)$$

$$= \int \partial_{4_2} \partial_{\bar{4}_2} \left(-Z_1 + \underbrace{(1 + \bar{4}_2 4_2)}_{\frac{1}{Z_2}} (4_1 \bar{4}_2 + 4_2 \bar{4}_1) \right)$$

$$= \int \partial_{4_2} \partial_{\bar{4}_2} (-Z_1) = 0$$

Prop. For any connected graph ω ,

$$\int \prod_i d\sigma_i \prod_{ij \in \omega} (1 + \sigma_i \cdot \sigma_j) = 1 (\omega \text{ is a tree}).$$

Proof. Using that

$$\int d\sigma_i (1 + \sigma_i \cdot \sigma_j) = 1$$

repeatedly, it follows that for every tree ω ,

$$\int \prod d\sigma_i \prod_{ij \in \omega} (1 + \sigma_i \cdot \sigma_j) = 1.$$

Thus it suffices to show that if ω is a cycle,

$$\prod_{ij \in \omega} (1 + \sigma_i \cdot \sigma_j) = 0.$$

By computation,

$$1 + \sigma_i \cdot \sigma_j = (\bar{\varphi}_i - \bar{\varphi}_j)(\varphi_i - \varphi_j)[1 - \bar{\varphi}_j \varphi_j]$$

Indeed,

$$\begin{aligned} 1 + \sigma_i \cdot \sigma_j &= -\frac{1}{2}(\sigma_i - \sigma_j) \cdot (\sigma_i - \sigma_j) \\ &= (\bar{\varphi}_i - \bar{\varphi}_j)(\varphi_i - \varphi_j) + \frac{1}{2}(z_i - z_j)^2 \end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}(z_i - z_j)^2 &= \frac{1}{2}(-\bar{y}_i y_i + \bar{y}_j y_j)^2 \\ &= -\bar{y}_i y_i \bar{y}_j y_j\end{aligned}$$

$$= -(\bar{y}_i - \bar{y}_j)(y_i - y_j)\bar{y}_j y_j$$

the claim follows. WLOG the cycle is $\{1, \dots, n\}$.

$$\prod_k (1 + \sigma_{i_k} \cdot \sigma_{i_{k+1}}) = \prod_k (\bar{y}_{i_k} - \bar{y}_{i_{k+1}})(y_{i_k} - y_{i_{k+1}}) \dots$$

and it suffices to show

$$\prod_k (y_{i_k} - y_{i_{k+1}}) = 0$$

The last term is $y_{i_k} - y_{i_1} = (y_{i_k} - y_{i_{k-1}}) + \dots + (y_{i_2} - y_{i_1})$.

$$\begin{aligned}\Rightarrow \prod_{k=1}^n (y_{i_k} - y_{i_{k+1}}) &= \\ &= \left[(y_{i_n} - y_{i_{n-1}}) + \dots + (y_{i_2} - y_{i_1}) \right] \prod_{k=1}^{n-1} (y_{i_k} - y_{i_{k+1}})\end{aligned}$$

Cor. $\int \prod d\sigma_i e^{\sum_{i,j} J_{ij}(\sigma_i \cdot \sigma_j + 1)} = \sum_{\omega} \prod_{i,j \in \omega} J_{ij} 1(\omega \text{ forest}).$

Proof.

$$\begin{aligned} & \int \prod d\sigma_i e^{\sum_{i,j} J_{ij}(\sigma_i \cdot \sigma_j + 1)} \quad \leftarrow (\sigma_i \cdot \sigma_j + 1)^2 = 0 \\ &= \int \prod d\sigma_i \prod_{i,j} (1 + J_{ij}(\sigma_i \cdot \sigma_j)) \\ &= \sum_{\omega} \int \prod d\sigma_i \prod_{i,j \in \omega} J_{ij} (1 + \sigma_i \cdot \sigma_j) = \sum_{\omega} \prod_{i,j} J_{ij} 1(\omega \text{ forest}). \end{aligned}$$

Prop.

$$\langle \sigma_i \cdot \sigma_j \rangle = -P(i \leftrightarrow j) = (q-1)P(i \leftrightarrow j) \text{ if } q=0.$$

Proof. Suppose $i, j \in \omega$ connected. Then

$$\begin{aligned} & \int \sigma_i \cdot \sigma_j \prod_{k \in \omega} (1 + \sigma_k \cdot \sigma_l) \\ &= - \underbrace{\int \prod_{k \in \omega} (1 + \sigma_k \cdot \sigma_l)}_{1(\omega \text{ free})} + \underbrace{\int \prod_{k \in \omega} (1 + \sigma_k \cdot \sigma_l) (1 + \sigma_i \cdot \sigma_j)}_{=0 \text{ cycle}} \end{aligned}$$

If $i \in \omega_1$ and $j \in \omega_2$ then

$$\sigma_j \cdot \int \sigma_i \prod_{\omega} (1 + \sigma_k \cdot \sigma_l) = \dots = \sigma_j \int \sigma_i = 0.$$

Exercise: Derive formula combinatorially using

$$J_{ij} \langle 1 + u_i \cdot u_j \rangle = P[ij \in \omega].$$

Exercise. $\int d\sigma_2 (\sigma_1 \cdot \sigma_2)(\sigma_2 \cdot \sigma_3) = \sigma_1 \cdot \sigma_3$

Compare with $O(n)$.

Cor. Let $p_t(x) = 1 + e^{-t}x$. Then

$$\int d\sigma_2 p_t(\sigma_1 \cdot \sigma_2) = 1$$

$$\int d\sigma_2 p_t(\sigma_1 \cdot \sigma_2) p_s(\sigma_2 \cdot \sigma_3) = p_{t+s}(\sigma_1 \cdot \sigma_3)$$

and $f(t, \sigma_1) = \int d\sigma_1 p_t(\sigma_1 \cdot \sigma_2) f(\sigma_2)$ satisfies

$$\frac{\partial f}{\partial t} = \Delta_{H^{0|2}} f$$

where $\Delta_{H^{0|2}} = 2(1 + \bar{\psi}\psi)\partial_{\psi}\partial_{\bar{\psi}} - 2\bar{\psi}\partial_{\bar{\psi}} - 2\psi\partial_{\psi}$.

8. Fermionic localization

Let $\psi_1, \dots, \psi_N$ be Grassmann variables.

The algebra of forms $\Omega^N(\mathbb{R}^M)$ consists of

$$F = \sum_{n=0}^N \frac{1}{n!} \sum_{z_1, \dots, z_n} F_{z_1, \dots, z_n} \psi_{z_1} \dots \psi_{z_n}$$

where each $F_{z_1, \dots, z_n} \in C^\infty(\mathbb{R}^M)$. Write $C^\infty(\mathbb{R}^{M|N})$.

Example As before, replace N by $2N$ and write the generators as

$$\psi_1, \bar{\psi}_1, \dots, \psi_N, \bar{\psi}_N.$$

Replace M by $2M$ and use complex coordinates

$$\phi_1, \bar{\phi}_1, \dots, \phi_M, \bar{\phi}_M.$$

Thus $\phi_j = \frac{1}{\sqrt{2}}(u_j + i v_j)$ with $(u_i, v_i) \in \mathbb{R}^2$

Then

$$F = \phi_1 \bar{\phi}_1 + \psi_1 \bar{\psi}_1$$

$$F = \phi_1 \phi_1 \psi_1 \bar{\psi}_1$$

are examples of forms.

Defn. Let $g: \mathbb{R}^k \rightarrow \mathbb{R}$ be C^∞ and $F_1, \dots, F_k$ even.
Define

$$g(F_1, \dots, F_k) = \sum_a \frac{1}{a!} g^{(a)}(F_1^0, \dots, F_k^0) (F - F^0)^a$$

Example

$$e^{-\Phi_1 \bar{\Phi}_1 - \psi_1 \bar{\psi}_1} = e^{-\Phi_1 \bar{\Phi}_1} (1 - \psi_1 \bar{\psi}_1)$$

$$e^{-\Phi_1 \bar{\Phi}_1 - \psi_1 \bar{\psi}_1 - \psi_2 \bar{\psi}_2} = e^{-\Phi_1 \bar{\Phi}_1} (1 - \psi_1 \bar{\psi}_1 - \psi_2 \bar{\psi}_2 - \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2)$$

Write

$$\underline{\Phi} = (\Phi, \bar{\Phi}, \psi, \bar{\psi})$$

$$\underline{\Phi}_i \cdot \underline{\Phi}_j = \underbrace{\Phi_i \bar{\Phi}_j + \Phi_j \bar{\Phi}_i}_{u_i u_j + v_i v_j} + \psi_i \bar{\psi}_j + \psi_j \bar{\psi}_i.$$

Define

$$Q_i = \psi_i \frac{\partial}{\partial \Phi_i} + \bar{\psi}_i \frac{\partial}{\partial \bar{\Phi}_i} - \Phi_i \frac{\partial}{\partial \psi_i} + \bar{\Phi}_i \frac{\partial}{\partial \bar{\psi}_i}.$$

This is an inf. rotation in superspace $\mathbb{R}^{2N|2N}$:

$$\begin{aligned} Q_j \Phi_j &= \psi_j, & Q_j \bar{\Phi}_j &= \bar{\psi}_j \\ Q_j \psi_j &= -\Phi_j, & Q_j \bar{\psi}_j &= -\bar{\Phi}_j \end{aligned}$$

Exercise: $Q^2 = 0$

$$Q(\underline{\Phi}_i \cdot \underline{\Phi}_j) = 0, \quad \underline{\Phi}_i \cdot \underline{\Phi}_j = Q \lambda_{ij}$$

$$Qf((\Phi_i \cdot \bar{\Phi}_j)_{i,j}) = 0 \quad \forall f \in C^\infty$$

Thm. Suppose $F \in \Omega^{2N}(\mathbb{R}^{2N})$ is integrable and
 $QF = 0$. (global SUSY)

Then

$$\int \frac{d\bar{\Phi} d\Phi}{(2\pi i)^N} \partial_{\bar{\Phi}} \partial_{\Phi} F = \int_{(\mathbb{R}^{2N})} d\bar{\Phi} F = F_0(0).$$

↖ $\phi = \frac{1}{\sqrt{2}}(u + iv) \Rightarrow d\bar{\phi} \wedge d\phi = i du \wedge dv$
 $\Rightarrow \frac{d\bar{\phi} \wedge d\phi}{2\pi i} = \frac{du dv}{2\pi}$

Exercise. Check explicitly when $F = e^{-\sum_{i,j} A_{ij} \bar{\Phi}_i \cdot \Phi_j}$.

This can be used to extend to general F .
 The following proof is more interesting.

Proof (Witten). Suffices to show:

1. Laplace principle:

$$\int e^{-t \sum_i \bar{\Phi}_i \cdot \Phi_i} F \xrightarrow{t \rightarrow \infty} F_0(0)$$

2. Laplace principle is exact:

$$\frac{\partial}{\partial t} \int e^{-t \sum_i \bar{\Phi}_i \cdot \Phi_i} F = 0$$

The more interesting step is 2. Note

$$\Phi_i \cdot \bar{\Phi}_i = Q \lambda_i, \quad \lambda_i = \Phi_i \bar{\Psi}_i + \bar{\Phi}_i \Psi_i$$

$$\begin{aligned} \Rightarrow -\frac{\partial}{\partial t} \int e^{-t \sum_i \Phi_i \cdot \bar{\Phi}_i} F &= \int e^{-t \sum \Phi \bar{\Phi}} \underbrace{(Q \lambda) F}_{Q(\lambda F)}, \quad \lambda = \sum \lambda_i \\ &= \int Q(e^{-t \sum \Phi \bar{\Phi}} \lambda F) = 0. \end{aligned}$$

Exercise: Check $\int QF = 0$ for every nice F .

For Step 1: Change variables:

$$\int e^{-t \sum \Phi_i \cdot \bar{\Phi}_i} F(\Phi) = \int e^{-\sum \Phi_i \cdot \bar{\Phi}_i} F\left(\frac{1}{\sqrt{t}} \Phi\right)$$

Write

$$\begin{aligned} \sum_i \Phi_i \cdot \bar{\Phi}_i &= \sum_i |\Phi_i|^2 - \omega, \quad \omega = \sum_i \bar{\Psi}_i \Psi_i \\ \Rightarrow \int e^{-\sum \Phi_i \cdot \bar{\Phi}_i} F\left(\frac{1}{\sqrt{t}} \Phi\right) &= \sum_{n=0}^N \int e^{-\sum |\Phi_i|^2} \frac{\omega^n}{n!} \underbrace{F\left(\frac{1}{\sqrt{t}} \Phi\right)}_{F_0\left(\frac{1}{\sqrt{t}} \Phi\right) + G} \end{aligned}$$

The F_0 term gives

$$\sum_{n=0}^N \int \dots = \int e^{-\sum |\Phi|^2} \frac{\omega^N}{N!} F_0\left(\frac{1}{\sqrt{t}} \Phi\right)$$

$$+ \underbrace{\left(\int e^{-\sum |\Phi|^2} \frac{\omega^N}{N!} \right)}_1 F_0(0)$$

The R term gives

$$\sum_{n=0}^{N-1} \int e^{-\sum |\Phi|^2} \frac{\omega^n}{n!} \underbrace{R\left(\frac{1}{\sqrt{t}} \Phi\right)} \rightarrow 0$$

$$\sum_{\substack{\alpha, \beta \\ |\alpha|+|\beta| \geq 1}} R_{\alpha} \left(\frac{\Phi}{\sqrt{t}}, \frac{\bar{\Phi}}{\sqrt{t}} \right) \left(\frac{1}{\sqrt{t}} \psi \right)^{\alpha} \left(\frac{1}{\sqrt{t}} \bar{\psi} \right)^{\beta}$$

so there is at least one factor $1/t$.

Cor.

$$\begin{aligned} P^0(i \leftrightarrow j) &= \langle \sigma_i \cdot \sigma_j \rangle_{H^{0|2}} \\ &= \langle \sigma_i \cdot \sigma_j \rangle_{H^{2|4}} \end{aligned}$$

$$\Phi = (\phi, \bar{\phi}, \psi^2, \bar{\psi}^2)$$

where in $H^{2|4}$

$$\sigma_i \cdot \sigma_j = -z_i z_j + \psi_i \bar{\psi}_j + \psi_j \bar{\psi}_i + \underbrace{\bar{\phi}_i \cdot \bar{\phi}_j}_{\downarrow}$$

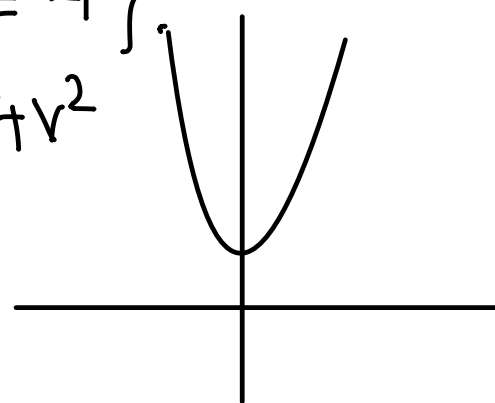
$$z_i = \sqrt{1 - 2 \bar{\psi}_i \psi_i + 2 \bar{\phi}_i \cdot \bar{\phi}_i}$$

$$d\sigma_i = \frac{d\phi d\bar{\phi}}{2\pi i} \partial\psi \partial\bar{\psi} \partial\psi^2 \partial\bar{\psi}^2 \frac{1}{z}$$

9. Hyperbolic space

$$\mathbb{H}^2 = \{ \underbrace{(z, u, v)}_{\sigma} : z > 0, 2|\Phi|^2 - z^2 = -1 \}$$

$$\uparrow 2|\Phi|^2 = u^2 + v^2$$



$$\sigma_i \cdot \sigma_j = \phi_i \bar{\phi}_j + \phi_j \bar{\phi}_i - z_i z_j$$

$$\int_{\mathbb{H}^2} d\sigma = \int_{\mathbb{H}^2} du dv \frac{1}{z(u, v)}$$

Exercise: Find the infinitesimal symmetries of $\mathbb{H}^2$. They generate $\mathfrak{so}(2, 1)$.

Harospherical coordinates: Write $(u, v, z) \in \mathbb{H}^2$ as $\underbrace{(u, v)}_{\in \mathbb{R}^2}$

$$\begin{cases} u = \sinh t - \frac{1}{2} s^2 e^t \\ v = e^t s \\ z = \cosh t + \frac{1}{2} s^2 e^t \end{cases}$$

Fact. Jacobian for the change of variable $(s, t) \rightarrow (u, v)$ is

$$J(s, t) = \det \begin{pmatrix} \cosh t - \frac{1}{2} s^2 e^t & -s e^t \\ +s e^t & e^t \end{pmatrix} = e^t \overbrace{z}^{(n-1)}$$

$$\Rightarrow \int_{\mathbb{H}^2} F = \int du dv \frac{1}{z(u, v)} F(u, v, z(u, v)) = \int ds dt e^t F(\dots)$$

Moreover,

$$\sum J_{ij}(\sigma_i \cdot \sigma_j + 1) = \sum J_{ij}(\cosh(t_i - t_j) - 1 + \frac{1}{2}e^{t_i + t_j}(s_i - s_j)^2)$$

$$\sum_i h_i z_i = \sum h_i (\cosh(t_i) + \frac{1}{2}s^2 e^t)$$

$$z_i = \sigma_i \cdot e, \quad e = (1, 0, 0)$$

Thus

$$\int_{(\mathbb{H}^2)^n} \prod d\sigma_i e^{\sum J_{ij}(\sigma_i \cdot \sigma_j + 1) - h \sum z_i}$$

$$= \int \prod_i dt_i ds_i e^{t_i} e^{-\sum J_{ij}(\cosh(t_i - t_j) - 1) - \frac{1}{2} \sum J_{ij} e^{t_i + t_j} (s_i - s_j)^2}$$

Gaussian!

$$e^{-\frac{1}{2}(S, \Delta_S(t) S)}$$

$$(\Delta_S(t) f)_i = \sum_j J_{ij} e^{t_i + t_j} (f_i - f_j)$$

$$= \int \prod_i dt_i e^{t_i} e^{-\sum J_{ij}(\cosh(t_i - t_j) - 1)} (\det(-\Delta_S(t) + h/t))^{-1/2}$$

For $\mathbb{H}^n$ analogously,

$$\int \prod_i dt_i e^{(n-1)t_i} e^{-\sum J_{ij}(\cosh(t_i - t_j) - 1)} (\det(-\Delta_S(t) + h/t))^{-(n-1)/2}$$

Exercise. Show the Ward identity for H^n :

$$\langle z_0 \rangle_{J,h} = \frac{1}{h} \sum_i \langle \sigma_0 \cdot \sigma_i \rangle_{J,h}.$$

Prop. For the arboreal gas, $n = 2 - 4 = -2$,

$$Z_{J,h} = \int \prod_i dt_i e^{-3t_i} e^{-\sum_j (\cosh(t_i - t_j) - 1) (-\Delta_{J(t)} + h I t)}^{+3/2}$$

Prop. (Hemospherical coordinates for $H^{2|4}$).

$$u = \sinh t - \left(\frac{1}{2} s^2 + \theta^1 \bar{\theta}^1 + \theta^2 \bar{\theta}^2 \right)$$

$$v = s e^t$$

$$z = \cosh t + \left(\frac{1}{2} s^2 + \theta^1 \bar{\theta}^1 + \theta^2 \bar{\theta}^2 \right)$$

$$\psi^i = \theta^i e^t$$

$$\bar{\psi}^i = \bar{\theta}^i e^t$$

Then

$$\int_{H^{2|4}} d\sigma F(\sigma) = \int \frac{dt ds d\theta d\bar{\theta}}{2\pi} e^{-3t} F(u(t,s,\dots), e^t s, \dots)$$

Proof. Can be derived for general change of var.
formula for superintegrals. Here are some examples.

Defn. Collections $(\tilde{\Phi}_i)$ of even elements and odd elements $(\tilde{\Psi}_i)$ are generators of $\Omega^N(\mathbb{R}^N)$ if every $F \in \Omega^N(\mathbb{R}^N)$ can be written

$$F^0(\tilde{\Phi}, \dots, \tilde{\Phi}_N) + F^1(\tilde{\Phi}, \dots, \tilde{\Phi}_N) \tilde{\Psi}_1 + \dots$$

Example The coordinate maps

$$\tilde{\Phi}_i(\phi) = \phi_i, \quad \tilde{\Psi}_i = \psi_i$$

Example. Let $\Phi: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a diffeomorphism. The the following are generators

$$\tilde{\Phi}_i = \Phi_i(\phi_1, \dots, \phi_N), \quad \tilde{\Psi}_i = \psi_i$$

and (change of variable formula)

$$\int d\phi \partial_\psi F(\phi, \psi) = \int d\tilde{\Phi} \partial_{\tilde{\Psi}} (\det D\Phi) F(\tilde{\Phi}, \psi)$$

Example. Let ϕ, ψ_1, ψ_2 be generators for $\Omega^2(\mathbb{R})$. Then

$$\phi + g(\phi) \psi_1 \psi_2, \quad \psi_1, \psi_2$$

are also generators and

$$\int d\phi \partial_{\psi_1} \partial_{\psi_2} F(\phi, \psi_1, \psi_2)$$

$$= \int d\phi \partial_{\psi_1} \partial_{\psi_2} F(\phi + g(\phi) \psi_1 \psi_2, \psi_1, \psi_2) (1 + g'(\phi) \psi_1 \psi_2)$$

Exercise In the $O(n)$ model, show

$$\langle \sigma_0 \cdot \sigma_j \rangle_{J,0} = \langle \sigma_j \cdot e \rangle_{J,0}^0$$

where $\langle \cdot \rangle^0$ refers to measure with 'pinning' $\sigma_0 = 0$:

$$\langle F \rangle^0 = \int \prod_{i \neq 0} d\sigma_i e^{\sum_{i,j} J_{ij} (\sigma_i \cdot \sigma_j + 1)}$$

Observe that this is equivalent to model on $\Lambda \setminus \{0\}$ with external field $h_i = J_{0i}$ and 0 interpreted as the ghost vertex.

Exercise For $h=0$,

$$Z_J = \int \prod_{i \neq 0} dt_i e^{-3t_i} e^{-\sum_{i,j} J_{ij} (\cosh(t_i - t_j) - 1)} (-\Delta_{J(t)}^0)^{+3/2}$$

and

$$\begin{aligned} P_j(0 \leftrightarrow j) &= -\langle \sigma_0 \cdot \sigma_j \rangle_{J,0}^0 \leftarrow \text{w.r.t. } H^{0|2} \text{ or } H^{2|4} \\ &= \langle z_j \rangle_{J,0}^0 \quad \vdots \\ &= \langle z_j + x_j \rangle_{J,0}^0 \end{aligned}$$

$$x_i + z_j = e^{t_j} \rightarrow \int \prod_{i \neq 0} dt_i e^{-3t_i} e^{t_j} e^{-\sum_{i,j} J_{ij} (\cosh(t_i - t_j) - 1)} (\det(-\Delta_{J(t)}^0))^{+3/2}$$

Mermin-Wagner Theorem for H^{012} .

$$\frac{1}{\langle Z_0 \rangle_{\beta, h}} \geq 1 + \frac{1}{N} \sum_{p \in \Lambda^*} \frac{1}{-\hat{\Delta}(p) + h}.$$

$$\Rightarrow \limsup_{N \rightarrow \infty} \limsup_{\Lambda \uparrow \mathbb{Z}^d} \langle Z_0 \rangle_{\beta, h} = 0.$$

$\nwarrow$
 $P_{\beta, h}(0 \leftrightarrow \infty)$

Proof. In hyperspherical coordinates,

$$\langle Z_i \rangle = \langle e^{ti} \rangle = \langle e^{2ti} \rangle$$

$$\langle \bar{\psi}_i \psi_j \rangle = \langle s_i s_j e^{ti + tj} \rangle$$

Let $T_j = \frac{\partial}{\partial s_j}$ and define

$$\hat{T}(p) = \sum_j e^{-ip \cdot j} T_j$$

$$\hat{S}(p) = \sum_j e^{-ip \cdot j} s_j.$$

Proceeding as in usual MW proof, get

$$\sum e^{ip \cdot j} \langle \bar{\psi}_0 \psi_j \rangle \geq \frac{\langle Z_0 \rangle}{-\hat{\Delta}(p) + h}$$

$$\Rightarrow \underbrace{\langle \bar{\psi}_0 \psi_0 \rangle}_{1 - \langle Z_0 \rangle} \geq \langle Z_0 \rangle \int \frac{1}{-\hat{\Delta}(p) + h} \frac{dp}{(2\pi)^d}$$