About the geometry and regularity of largest subsolutions for a free boundary problem in $\mathbf{R}^2$: elliptic case

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In this study, we present some geometric and regularity properties of the largest subsolution of a one-phase free boundary problem under a very general free boundary condition in $\mathbf{R}^2$. Moreover, we provide density bounds for the positivity set and its complement near the free boundary.

Free Boundary Problem (FBP) that we would like to analyze is the following: for a given bounded open domain $D \subseteq \mathbf{R}^2$ and $u : \mathbf{R}^2 \setminus D \mapsto [0, +\infty)$ is a continuous function which satisfies:

$$\triangle u = 0, \quad \text{in } \Omega(u) \backslash D, \tag{1}$$

$$u = g(x), \qquad \text{on } \partial D, \tag{2}$$

$$u = 0, |\nabla u|^2 = f(x), \qquad \text{on } \partial\Omega(u), \tag{3}$$

where $\Omega(u) = \{x \in \mathbf{R}^2 | u(x) > 0\}$; $g(x)$ and $f(x)$ are positive continuous functions. For $f(x)$, there exist $\Lambda, \lambda > 0$ such that $0 < \lambda < f(x) < \Lambda$, for all $x \in \mathbf{R}^2$.

There is a wide range of physical models related to the above FBP, encompassing problems such as flame propagation and G-equations, capillary drops on a flat or inclined surface, phase transitions, and obstacle problems. There are previous results about the regularity of variational and weak solutions to these example FBPs. Most of these results require that the Free Boundary Condition (FBC) is at least Lipschitz and the media is periodic, we would like to extend these results both to viscosity solutions and to the random case since real life systems also require to work with these cases. In the random case, media can be heterogenous without any periodic setting, i.e. the FBC can be at most positive, bounded, and continuous in the space variable. In this study, we focus on regularity issues for a FBP related to these phenomena and we concentrate on the geometric description of the largest viscosity subsolution in two dimensions with weaker requirements on the data. We develop a regularity and non degeneracy theory for it's largest subsolution and give a nice geometric characterization of the free boundary. Motivated by the study of random media, we allow for the data to be highly oscillatory. Thus, we only require $f(x)$ to be positive, bounded, and measurable function. We used the continuity of $f(x)$ only to be able to obtain the continuous viscosity solutions. One can even weaken the continuity assumption on $f(x)$ by taking into account suitable viscosity solution definitions.