Invariance principle for variable speed random walks on trees

Anita Winter, Universität Duisburg-Essen

(with **Siva Athreya** and **Wolfgang Löhr**)

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Offen im Denken

Brownian motion: The central object in stochastic analysis

The **Brownian motion** on $\mathbb{R}$ starting in $x \in \mathbb{R}$ is a **continuous path** Markov process with

$$\mathbb{P}_x\{B_t \le z\} := \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{z} \mathrm{d}y \; e^{-\frac{(y-x)^2}{2t}}, \qquad z \in \mathbb{R}, t \ge 0.$$

Scaling Property. Assume that $B := (B_t)_{t \ge 0}$ is a Brownian motion starting in 0. Define for all $n \in \mathbb{N}$ the process $\tilde{B} := (\tilde{B}_t)_{t \ge 0}$ given by

$$\tilde{B}_t := \frac{1}{\sqrt{n}} B_{nt}.$$

Then $\tilde{B} := (\tilde{B}_t)_{t \ge 0}$ is also a standard Brownian motion starting in 0.

Central limit theorem

Let X_1, X_2, ... be independent, identically distributed $\{-1,1\}$-valued random variables with $\mathbb{P}\{X_n = \pm 1\} = \frac{1}{2}$. Then for all $z \in \mathbb{R}$, $t \geq 0$,

$$\mathbb{P}\Big\{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i \leq z\Big\} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{z} \mathrm{d}y \; e^{-\frac{y^2}{2t}} = \mathbb{P}_0\{B_t \leq z\}.$$

Equivalently, for **all bounded and continuous functions** $f : \mathbb{R} \to \mathbb{R}$, $t \geq 0$,

$$\mathbb{E}\Big[f\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i\Big)\Big] \underset{n\to\infty}{\longrightarrow} \mathbb{E}_0\big[f(B_t)\big].$$

This convergence is referred to as **weak convergence** and abbreviated by $\underset{n\to\infty}{\Longrightarrow}$, i.e., we write

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i \underset{n\to\infty}{\Longrightarrow} B_t.$$

Functional central limit theorem

Denote by $\mathcal{D}([0,\infty))$ the space of **cadlag-functions** (continuous from the right $+$ limits from the left) equipped with the **Skorokhod-topology** (extending uniform convergence on compacta for continuous fcts).

The random trajectories of the random walks $(\sum_{i=1}^{\lfloor t\rfloor} X_i)_{t\geq 0}$ and Brownian motion $(B_t)_{t\geq 0}$ are elements of $\mathcal{D}([0,\infty))$, a.s.

Functional CLT.

$$\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor} X_i\Big)_{t\geq 0} \underset{n\to\infty}{\Longrightarrow} (B_t)_{t\geq 0}.$$

Once more this means that **all bounded and continuous functionals (and even more)** $f:\mathcal{D}([0,\infty))\to\mathbb{R}$ **converge**.

Example. We can conclude from the functional CLT that

$$\mathbb{P}\Big\{\frac{1}{\sqrt{n}}\max_{k=1,\dots,n}\sum_{i=1}^{k} X_i \geq z\Big\} \underset{n\to\infty}{\longrightarrow} \mathbb{P}_0\Big\{\max_{s\in[0,1]} B_s \geq z\Big\} = \frac{\sqrt{2}}{\sqrt{\pi}}\int_z^\infty \mathrm{d}y\, e^{-\frac{y^2}{2}}.$$

Shared properties of random walk and Brownian motion

The **random walk** $(S_t)_{t\geq 0}$ where $S_t := \sum_{i=1}^{\lfloor t \rfloor} X_i$ and the Brownian motion $B := (B_t)_{t\geq 0}$ share the following properties:

- **Strong Markov property.** Both are strong Markov processes.
- **Skip-free.** Their random trajectories “do not jump over points”.
- **On natural scale.** They are both on “natural scale”.

  Let for a stochastic process X with values in $E \subseteq \mathbb{R}$, $\tau_z := \inf\{t \geq 0 : X_t = z\}$. We say that X is on **natural scale** iff for all $x, y, z \in E$ with $y < x < z$ and $\tau_y \wedge \tau_z < \infty$,

$$\mathbb{P}_x\{\tau_y < \tau_z\} = \frac{z-x}{z-y} = \frac{r(x,z)}{r(y,z)},$$

  where here $r(\cdot, \cdot)$ denotes the **Euclidian distance**.

Stone's PhD

Charles Stone (1963), Limit theorems for random walks, birth and death processes, and diffusion processes, Illinois Journal of Mathematics

``As defined here, classes of Markov processes have in common that the basic state space is a subset of the reals, and the **random trajectories do not jump over points** in the state space. ... In the discrete-time setting, any such process has a discrete state space and is a random walk. In the continuous-time setting, if the state space is an interval, the path functions are continuous ...; if the state space is discrete, the process is a birth and death process.''

Any such process X on **"natural scale"** shares the following **occupation time formula**: for all $x < z$,

$$\mathbb{E}_x\Big[\int_0^{\tau_z} \mathrm{d}s\, f(X_s)\Big] = 2\int \nu(\mathrm{d}y)\, f(y)\big(z\wedge y - x\vee y\big).$$

The measure ν is referred to as **speed measure**.

Stone's invariance principle

Charles Stone (1963), Limit theorems for random walks, birth and death processes, and diffusion processes, Illinois Journal of Mathematics

``As defined here, classes of Markov processes have in common that the basic state space is a subset of the reals, and the **random trajectories do not jump over points** in the state space. ... In the discrete-time setting, any such process has a discrete state space and is a random walk. In the continuous-time setting, if the state space is an interval, the path functions are continuous ...; if the state space is discrete, the process is a birth and death process.

We include both possibilities by allowing the **state space to be any closed subset of the reals**. These processes are all very similar in their analytic and probabilistic structure. When put in their **"natural scale"**, they are determined by a **speed measure** $\nu(dx)$ ...

It is fairly obvious that in some sense the processes depend continuously on $\nu(dx)$ . ...''

Example: Recovering the functional CLT from Stone

The continuous time **random walk** is such a Markov process on $\mathbb{Z}$ whose speed measure equals the **counting measure** $q := \sum_{z\in\mathbb{Z}} \delta_z$; and after Brownian rescaling (**rescaling edge length** by a factor $\frac{1}{\sqrt{n}}$ and **speeding up time** by a factor n)

$$\nu_n := \tfrac{1}{\sqrt{n}} \cdot q\big(\sqrt{n}\,\cdot\,\big).$$

On the other hand the **Brownian motion** is such a process whose speed measure equals the **Lebesgue measure**.

According to Stone. As "in some sense" $(\nu_n)_{n\in\mathbb{N}}$ converges to the Lebesgue measure, the suitably rescaled random walk converges in path space to standard Brownian motion.

Another example which falls in Stone's class

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be "smooth enough" such that the following SDE has a unique strong solution:

$$\mathrm{d}X_t = \sigma\big(X_t\big)\mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}.$$

Then $X := (X_t)_{t \geq 0}$ is a **skip-free, strong Markov process on natural scale**. Its **speed measure** equals

$$\nu(\mathrm{d}x) = \sigma^{-2}(x)\mathrm{d}x.$$

The main goal

Generalize Stone's invariance principle from skip-free strong Markov processes on natural scale on $\mathbb{R}$ (equipped with **Euclidian metric**) to **skip-free strong Markov processes on natural scale** on **TREES**, e.g.,

- discrete trees,
- so-called $\mathbb{R}$**-trees**,
- $\mathbb{R}$ with a metric **different** from the Euclidian,
- and many more, e.g. $T_q := \{\pm q^n; n \in \mathbb{Z}\} \cup \{0\}.$ with the Euclidian metric.

Outline of the talk

1. Metric measure trees (T, r, ν)
2. Speed-ν motion on (T, r)
3. Gromov-weak convergence of metric measure trees
4. Invariance principle
5. Main steps in the proof
6. Related work and examples

Metric measure trees

Pointed metric spaces

A **pointed metric space** (X, r, ρ) consists of

- a **metric space** (X, r) and
- a **distinguished point** $\rho \in X$, called **root**.

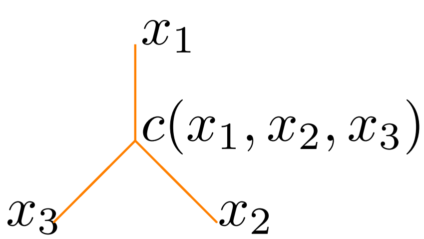
Rooted metric trees

A **rooted metric tree** (T, r, ρ) is a pointed metric space (T, r, ρ) which is **0-hyperbolic**, i.e., for all $x_1, x_2, x_3, x_4 \in T$,

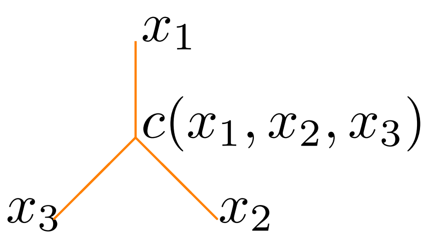
$$\begin{aligned} &r(x_1, x_2) + r(x_3, x_4) \\ &\le \max\left\{r(x_1, x_3) + r(x_2, x_4), r(x_1, x_4) + r(x_2, x_3)\right\}, \end{aligned}$$

and for all $x_1, x_2, x_3 \in T$, there is a point $c(x_1, x_2, x_3) \in T$ such that for all $i \neq j \in \{1, 2, 3\}$,

$$r\big(x_i, c(x_1, x_2, x_3)\big) + r\big(x_j, c(x_1, x_2, x_3)\big) = r(x_i, x_j).$$



Intervals = Arcs



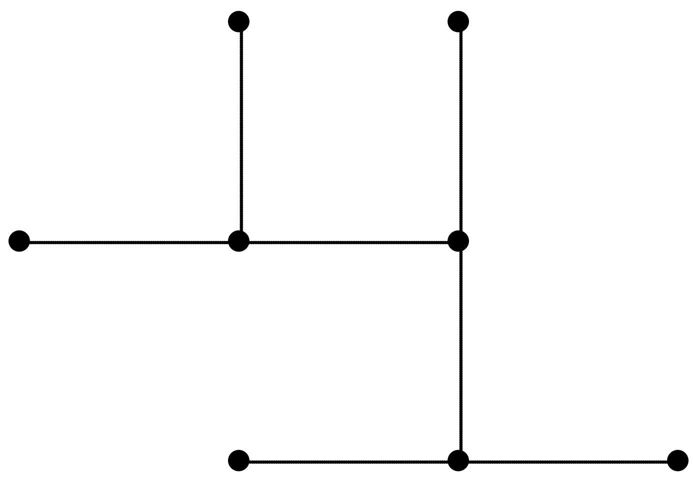
As usual, we define **intervals/arcs** by

$$[a,b] := \{v \in T : r(a,v) + r(v,b) = r(a,b)\},$$

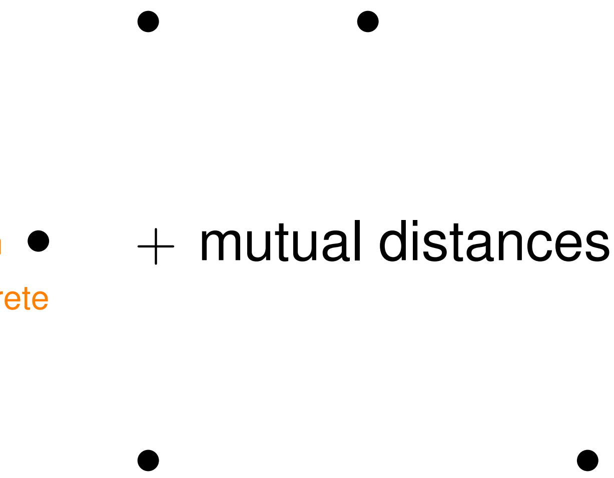
and analogously $(a,b) := [a,b] \setminus \{a,b\}$, $[a,b) := [a,b] \setminus \{b\}$ and $(a,b] := [a,b] \setminus \{a\}$

Rooted $\mathbb{R}$-trees

An **rooted $\mathbb{R}$-tree** (T, r, ρ) is a **rooted metric tree** (T, r, ρ) which is in addition **path-connected**.



rather than in the discrete

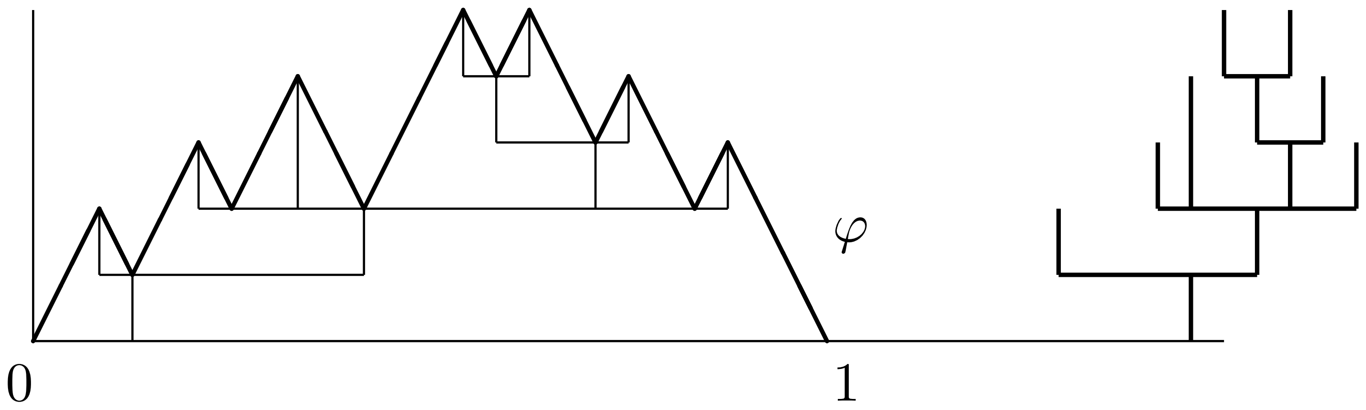


$+$ mutual distances

Prominent example: Rooted $\mathbb{R}$-tree "below" an excursion

$\varphi \in C([0,1])$, $\varphi\big|_{\{0,1\}} \equiv 0$, $\varphi\big|_{(0,1)} > 0$

pseudo-metric on $[0,1]$. $r_\varphi(s,t) := \varphi(s) + \varphi(t) - 2 \cdot \inf_{u \in [s,t]} \varphi(u)$.



Fact. $T\big|_\varphi = [0,1]_{/\equiv_\varphi}$ is a **compact** real tree with root 0.

Example. CRT $=$ "below" $2\cdot$ **Brownian excursion**;

Lèvy trees "via" **excursions of Lèvy processes**

Rooted compact metric (finite) measure trees

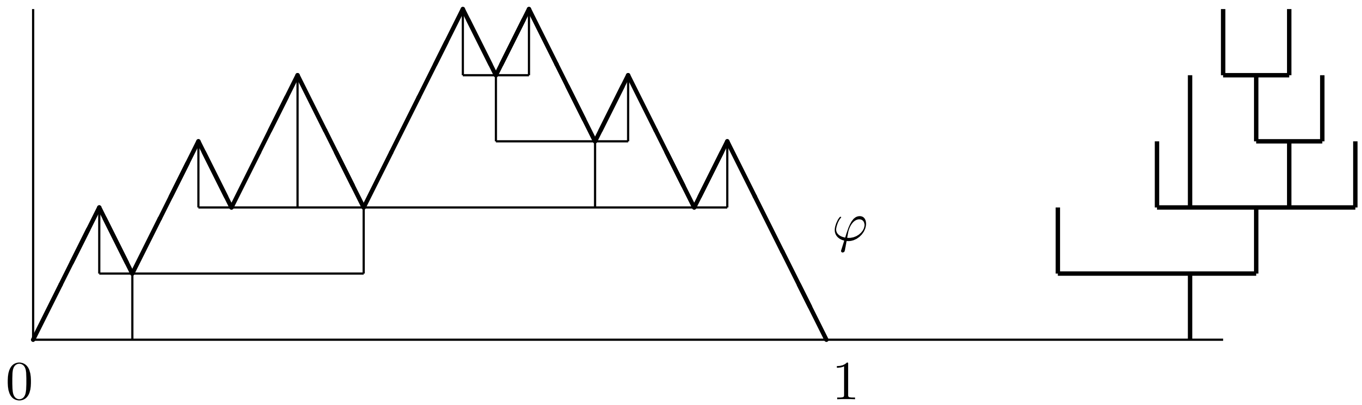
A **rooted compact metric (finite) measure tree** (T, r, ρ, ν) consists of

- **a rooted compact metric tree** (T, r, ρ) and
- **a finite measure** ν on $(T, \mathcal{B}(T))$ of **full support**.

Prominent example: Rooted metric measure tree “below” an excursion

$\varphi \in C([0,1])$, $\varphi\big|_{\{0,1\}} \equiv 0$, $\varphi\big|_{(0,1)} > 0$

pseudo-metric on $[0,1]$. $r_\varphi(s,t) := \varphi(s) + \varphi(t) - 2 \cdot \inf_{u \in [s,t]} \varphi(u)$.



Fact. The $\mathbb{R}$-tree $T\big|_\varphi = [0,1]_{/\equiv_\varphi}$ can be **turned into a metric measure tree** if additionally equipped with the **image measure of the Lebesgue measure on** $[0,1]$ under the map which sends $x \in [0,1]$ into the tree $T\big|_\varphi = [0,1]_{/\equiv_\varphi}$.

Variable speed RM and BM via the same Dirichlet form

Variable speed random walk on graph trees with edge lengths

Assume we are given **graph trees** $T=(V,E)$ with **edge lengths** $\{w_e;\, e\in E\}$. Moreover, we are given **jump rates** $\{\gamma_v;\, v\in V\}$.

The **variable speed random walk** associated with $(T,\{w_e;\, e\in E\},\{\gamma_v;\, v\in V\})$ is a V-valued Markov chain in continuous time which has the following dynamics: given the MC is currently in $v\in V$,

- it waits an exponential time with mean γ_v^{-1} until it jumps away.
- at the jump time, it pick the neighboring vertex $v'\sim v$ with a probability I N V E R S E L Y proportional to $w_{\{v,v'\}}$ as its next position.

Variable speed random walk on graph trees with edge lengths

One to one correspondence between graph trees with edge lengths and jump rates with metric measure spaces, i.e.,

$$(T=(V,E),\{w_e;\,e\in E\},\{\gamma_v;\,v\in V\}) \qquad \Leftrightarrow \qquad (V,r,\nu).$$

- Associate V with the **metric** and a **measure**

$$r(v,v') := \sum\nolimits_{e\in v\mapsto v'} w_e,\quad \forall v,v'\in V,$$
$$\nu(A) := \tfrac{1}{2}\cdot\sum\nolimits_{v\in A}\gamma_v^{-1}\sum\nolimits_{v'\sim v} r^{-1}(v,v'),\quad \forall A\subseteq V.$$

- Conversely, **the speed-ν random walk on (V,r)** is a V-valued Markov chain with jumps from $v\mapsto v'\sim v$ at rate $\frac{1}{2\nu(\{v\})r(v,v')}$.

Metatheorem. The Markov chains $(X^n)_{n\in\mathbb{N}}$ converge weakly on path space provided that the underlying metric measure spaces $(V^n,r^n,\nu^n)_{n\in\mathbb{N}}$ “converge”.

Example: Simple RW on $\mathbb{Z}$

For each $n \in \mathbb{N}$, put

$$T_n :\equiv \mathbb{Z}, \quad r_n\big(v, v \pm 1\big) := \tfrac{1}{\sqrt{n}}, \quad \nu_n(\{v\}) := \tfrac{1}{\sqrt{n}}, \qquad \forall v \in \mathbb{Z}.$$

The ν_n **-speed random walk on** (T_n, r_n) is the **SRW** on $\mathbb{Z}$ with **edge length re-scaled** by $\frac{1}{\sqrt{n}}$ and **speeded up** (in each vertex v) by a factor of

$$\gamma_n(v) := \tfrac{1}{2\nu_n(\{v\})} \sum_{v'=v\pm 1} r_n^{-1}(v, v') = \tfrac{1}{2} \cdot \sqrt{n} \cdot 2\sqrt{n} = n.$$

Dirichlet form heuristics

For the construction of our processes on general **rooted metric measure trees**, (T, r, ρ, ν), we will rely on **Dirichlet forms**.

The **Dirichlet form** of the previous Markov chain is given through its generator A acting on bounded functions, i.e.,

$$\begin{aligned}
\mathcal{E}(f,g) &:= -(Af,g)_\nu \\
&= -\sum_{v\in T}\nu(\{v\})\tfrac{1}{2\nu(\{v\})}\sum_{v'\sim v}\tfrac{1}{r(v,v')}\big(f(v')-f(v)\big)g(v) \\
&= \tfrac{1}{2}\sum_{v\in T}\tfrac{1}{2}\sum_{v'\sim v}\tfrac{1}{r(v,v')}\big(f(v')-f(v)\big)\big(g(v')-g(v)\big) \\
&= \tfrac{1}{2}\sum_{v\in T}\tfrac{1}{2}\sum_{v'\sim v}r(v,v')\tfrac{f(v')-f(v)}{r(v,v')}\tfrac{g(v')-g(v)}{r(v,v')} \\
&\approx \tfrac{1}{2}\int \mathrm{d}\lambda \nabla f\nabla g.
\end{aligned}$$

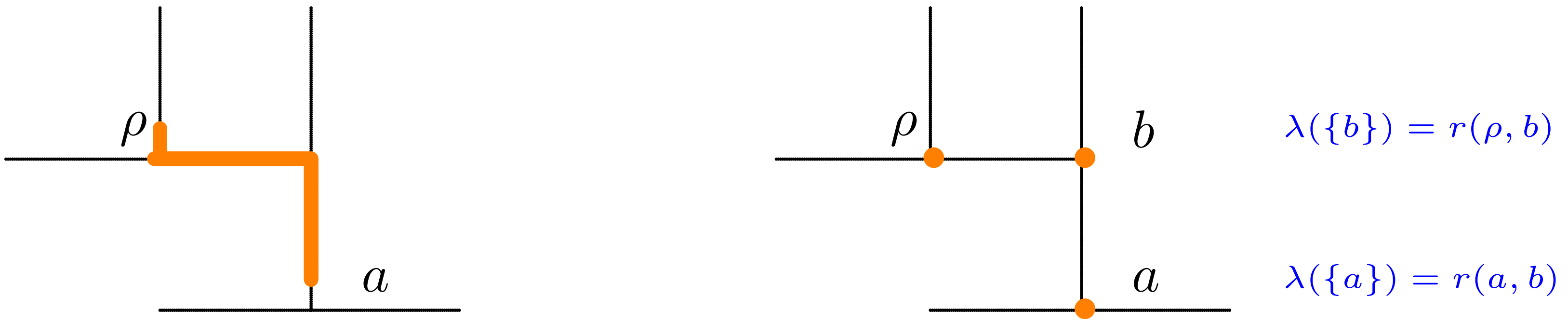
Our strategy. Define universal **length measure** and **gradient**.

The (universal) length measure

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

(T, r, ρ) **rooted metric tree**, $T' \subset T$ countably dense

$$T^o := \bigcup_{a,b \in T'} (a, b)$$



Then $\exists!$ σ-finite Borel measure $\lambda = \lambda^{(T,r,\rho)}$ s.t.

- $\lambda((\rho, a]) = r(\rho, a)$.
- $\lambda(T \setminus T^o) = 0$

Notice. λ depends on the choice of the root ρ.

Absolute continuity

We say $f \in \mathcal{C}(T)$ is **absolutely continuous** iff $\forall\, \varepsilon > 0$ and $\forall\, S \subseteq T$ $\exists\, \delta = \delta(\varepsilon, S)$, such that for arcs all $[x_1, y_1], ..., [x_n, y_n] \in S$ with $\sum_{i=1}^{n} r(x_i, y_i) < \delta$, $\sum_{i=1}^{n-1} \big|f(x_i) - f(y_i)\big| < \varepsilon$.

$$\mathcal{A} := \big\{f \in \mathcal{C}(T) :\ f \text{ absolutely continuous}\big\}.$$

Notice. If (T, r) is a **discrete tree** then **each bounded function is absolutely continuous**.

The (universal) gradient

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

Proposition. **(Athreya, Löhr & W.)** For all $f \in \mathcal{A}$, there exists a (unique up to $\lambda^{(T,r,\rho)}$-zero sets) $g \in L^1(\lambda^{(T,r,\rho)})$ such that for all $x, y \in T$,

$$f(y) - f(x) = \int_x^y \lambda^{(T,r,\rho)}(\mathrm{d}z)\, g(z)$$

$$:= -\int_{(\rho,x]} \lambda^{(T,r,\rho)}(\mathrm{d}z)\, g(z) + \int_{[\rho,y]} \lambda^{(T,r,\rho)}(\mathrm{d}z)\, g(z).$$

We refer to g as the **gradient** and write $\nabla f = g$.

The (universal) Dirichlet form

Siva Athreya, Michael Eckhoff & Anita Winter (2013), Brownian motion on $\mathbb{R}$-trees, Transaction of AMS

Let (T, r, ρ, ν) be a **rooted compact metric (finite) measure tree**. Put

$$\mathcal{E}(f, g) := \tfrac{1}{2} \int \mathrm{d}\lambda^{(T,r,\rho)} \nabla f \nabla g,$$

and

$$\mathcal{D}(\mathcal{E}) := \{ f \in L^2(\nu) \cap \mathcal{A} : \nabla f \in L^2(\lambda^{(T,r,\rho)}) \}.$$

Proposition. **(AEW2013; Athreya, Löhr & W.)** The bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet form and there is a strong Markov process $(X^x)_{x \in T}$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Speed-ν motion on (T, r)

Siva Athreya, Michael Eckhoff & Anita Winter (2013), Brownian motion on $\mathbb{R}$-trees, Transaction of AMS

We want to refer to this strong Markov process $X = (X_t)_{t \geq 0}$ whose Dirichlet form is $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as **speed-ν motion on (T, r).**

Particular cases.

- If (T, r) is discrete, then the speed-ν motion on (T, r) is the **speed-ν random walk on (T, r)**.
- If (T, r) is an $\mathbb{R}$-tree, then the speed-ν motion on (T, r) is the **ν-Brownian motion on (T, r)** which was constructed in **[AEW2013]**.

Occupation time formula

Siva Athreya, Michael Eckhoff & Anita Winter (2013), Brownian motion on $\mathbb{R}$-trees, Transaction of AMS

First hitting time of x.

$$\tau_z := \inf\{t \geq 0 : X_t = z\}$$

Proposition. (AEW 2013 & Athreya, Löhr & W.) If (T, r) is compact, then for all $x, z \in T$, and for all bounded $f : T \to \mathbb{R}$,

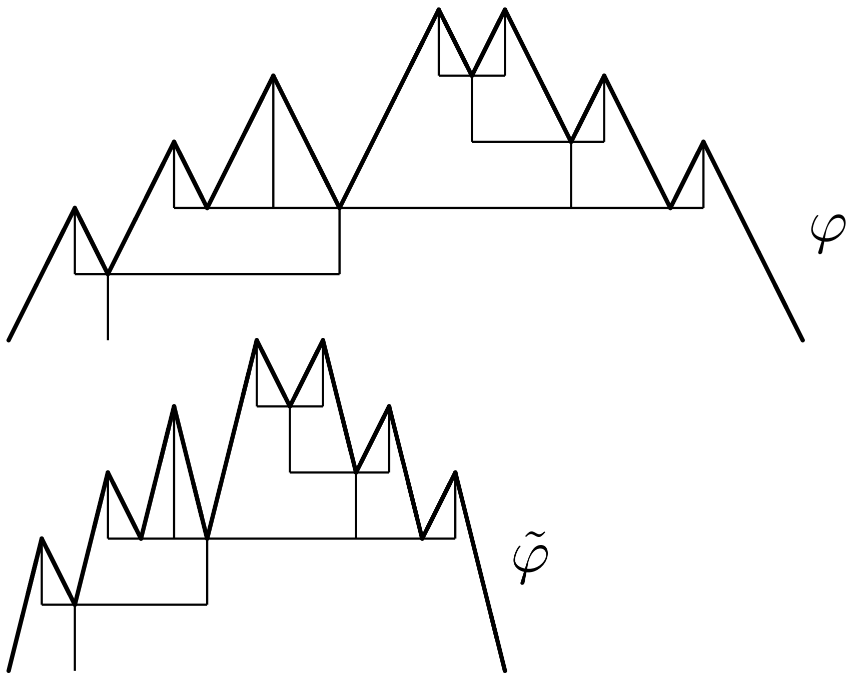
$$\mathbb{E}_x\Big[\int_0^{\tau_z} f(X_t)\mathrm{d}t\Big] = 2\int_T f(y) \cdot r\big(z, c(x, z, y)\big)\,\nu(\mathrm{d}y).$$

Convergence of metric measure trees

Be careful with excursions and uniform topology ...

... as notion of convergence of trees

There is a tradition to encode **trees via excursions**, and to consider excursions a elements of the space of continuous functions equipped with the **uniform topology on compacta**.



φ and $\tilde{\varphi}$ encode the same tree

Weak convergence of finite measures

Given a metric space (E, d).

We say that a sequence of finite measure ν_n **converges weakly** towards the finite measure ν if and only of for all bounded and continuous functions $f : E \to \mathbb{R}$,

$$\int_E \nu_n(\mathrm{d}y)\, f(y) \underset{n\to\infty}{\longrightarrow} \int_E \nu(\mathrm{d}y)\, f(y).$$

In that case we write $\nu_n \underset{n\to\infty}{\Longrightarrow} \nu$.

Notice that if $\nu_n \underset{n\to\infty}{\Longrightarrow} \nu$, then in particular the **total masses converge**.

Gromov-weak topology

We call two **pointed compact metric measure spaces** (X, r, ρ, ν) and (X', r', ρ', ν') **equivalent** iff there is a isometry $\varphi : X \to X'$ with $\varphi(\rho) = \rho'$ and $\nu \circ \varphi^{-1} = \nu'$.

Let

$$\mathbb{M} := \text{ the space of all equivalence classes}$$

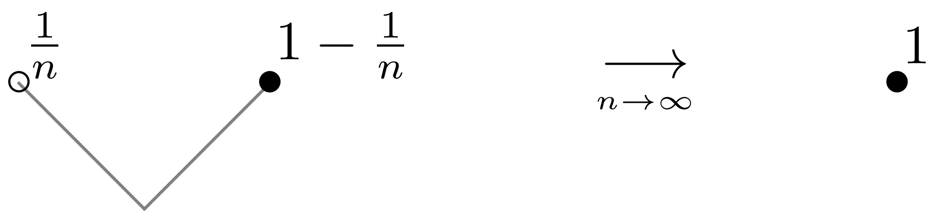
Let $\mathcal{X} := \overline{(X, r, \rho, \nu)}$, $\mathcal{X}_1 := \overline{(X_1, r_1, \rho_1, \nu)}$, $\mathcal{X}_2 := \overline{(X_2, r_2, \rho_2, \nu)}$, ... be in $\mathbb{M}$. We say that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ converges to $\mathcal{X}$ in **pointed Gromov-weak topology** if and only if there exists a metric space (E, d_E, ρ_E) and isometries $\varphi : X \to E$ with $\varphi(\rho) = \rho_E$, $\varphi_1 : X_1 \to E$ with $\varphi_1(\rho_1) = \rho_E$, $\varphi_2 : X_2 \to E$ with $\varphi_2(\rho_2) = \rho_E$, ... such that

$$\nu_n \circ \varphi_n^{-1} \underset{n \to \infty}{\Longrightarrow} \nu \circ \varphi^{-1}.$$

Gromov-weak versus convergence of the supports

Example. Let $\mathcal{X}_n$ be represented by

$$\Big(\{1,2\}, r_n(1,2) \equiv 1, \rho_n :\equiv 1, \tfrac{1}{n}\delta_1 + \big(1-\tfrac{1}{n}\big)\delta_2\Big)$$



Obviously, $\mathcal{X}_n \underset{n\to\infty}{\longrightarrow} \mathcal{X} := \overline{(\{1,2\}, \rho = 1, \delta_2)}$ Gromov-weakly. However, the **supports do not converge**.

How can we characterize convergence of supports?

Hausdorff distance

Let (X, r) be a **compact** metric space.

Hausdorff-distance. For $A_1, A_2 \subseteq_{\text{closed}} X$,

$$d_{\text{H}}(A_1, A_2) := \inf\{\varepsilon > 0 : \; A_1 \subseteq A_2^\varepsilon \text{ and } A_2 \subseteq A_1^\varepsilon\},$$

where A^ε is the ε-neighborhood of A.

Gromov-Hausdorff-weak convergence

Romain Abraham, Jean-Francois Delmas & Patrick Hoscheit, A note on the Gromov-Hausdorff-Prohorov distance between (locally) compact metric measure spaces, EJP 2013

Let $\mathcal{X} := \overline{(X, r, \rho, \nu)}$, $\mathcal{X}_1 := \overline{(X_1, r_1, \rho_1, \nu)}$, $\mathcal{X}_2 := \overline{(X_2, r_2, \rho_2, \nu)}$, ... be in $\mathbb{M}$. We say that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{X}$ in **pointed Gromov-Hausdorff-weak topology** if and only if there exists a metric space (E, d_E, ρ_E) and isometries $\varphi : X \to E$ with $\varphi(\rho) = \rho_E$, $\varphi_1 : X_1 \to E$ with $\varphi_1(\rho_1) = \rho_E$, $\varphi_2 : X_2 \to E$ with $\varphi_2(\rho_2) = \rho_E$, ... such that

$$\nu_n \circ \varphi_n^{-1} \underset{n\to\infty}{\Longrightarrow} \nu \circ \varphi^{-1} \quad \underline{\text{AND}} \quad d_H\big(\varphi_n(X_n), \varphi(X)\big) \underset{n\to\infty}{\longrightarrow} 0.$$

The global lower-mass bound property

Athreya, Löhr & W., The gap between Gromov-vague and Gromov-Hausdorff-vague topology, in preparation..

For a **compact** pointed mm-space $\mathcal{X} = \overline{(X, r, \rho, \nu)}$ and $\delta > 0$, let

$$m_\delta(\mathcal{X}) := \inf \left\{ \nu(\bar{B}(x, \delta)) : x \in \mathrm{supp}(\nu) \right\} > 0.$$

We say that a family $\Gamma \subseteq \mathbb{M}$ satisfies the **global lower-mass bound property** iff for all $\delta > 0$,

$$\inf_{\mathcal{X} \in \Gamma} m_\delta(\mathcal{X}) > 0.$$

Proposition. (Athreya, Löhr & W.) Let $\mathcal{X} = (X, r, \rho, \nu)$ and $\mathcal{X}_n = (X_n, r_n, \rho_n, \nu_n)$, $n \in \mathbb{N}$, be such that $(\mathcal{X}_n)_{n \in \mathbb{N}} \to \mathcal{X}$ **pointed Gromov-weakly**. Then the following are equivalent.

1. $(\mathcal{X}_n)_{n \in \mathbb{N}}$ satisfies the **uniform global lower mass-bound property**.
2. $(\mathrm{supp}(\nu_n))_{n \in \mathbb{N}} \to \mathrm{supp}(\nu)$ in Gromov-**Hausdorff** topology.

A perturbation result

Athreya, Löhr & W., The gap between Gromov-vague and Gromov-Hausdorff-vague topology, in preparation..

Proposition. Consider $\mathcal{X} = (X, r, \rho, \mu)$, $\mathcal{X}_1 = (X_1, r_1, \rho_1, \mu_1)$, $\mathcal{X}_1 = (X_2, r_2, \rho_2, \mu_2)$, ... in $\mathbb{M}$, and finite measures μ'_n on X_n, $n \in \mathbb{N}$. Assume that $\mathcal{X}_n \underset{n\to\infty}{\longrightarrow} \mathcal{X}$ Gromov-weakly, and

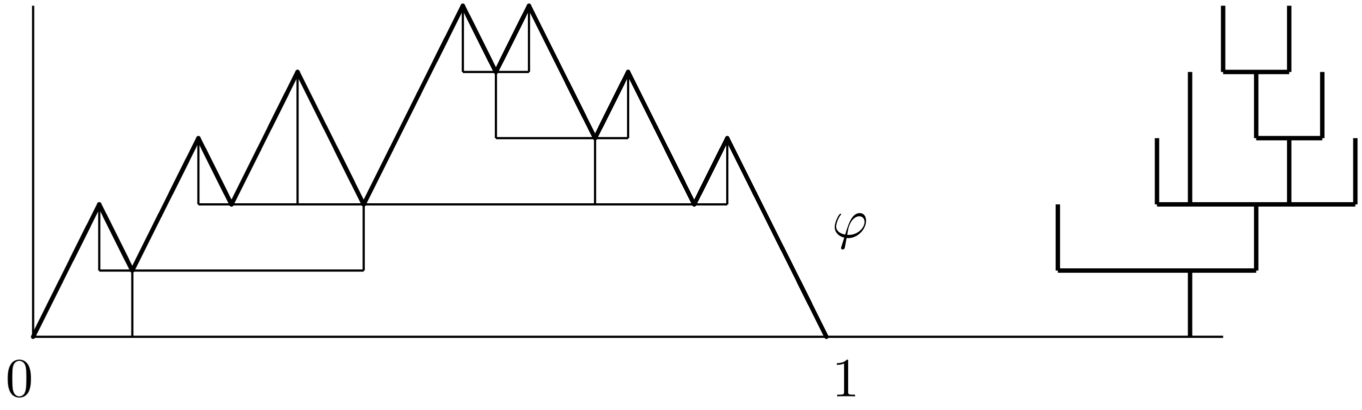
$$d_{Pr}(\mu_n, \mu'_n) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad d_H(\mathrm{supp}(\mu_n), \mathrm{supp}(\mu'_n)) \underset{n\to\infty}{\longrightarrow} 0.$$

Then $(X_n, r_n, \rho_n, \mu'_n)$ **also converges Gromov-Hausdorff-weakly** to $\mathcal{X}$.

The gluing map is continuous

$\varphi \in C([0,1])$, $\varphi\big|_{\{0,1\}} \equiv 0$, $\varphi\big|_{(0,1)} > 0$

pseudo-metric on $[0,1]$**.** $r_\varphi(s,t) := \varphi(s) + \varphi(t) - 2 \cdot \inf_{u \in [s,t]} \varphi(u)$.



Löhr, Equivalence of Gromov-Prohorov- and Gromov's $\underline{\Box}_\lambda$ -metric on the space of metric measure spaces, (2013)

Proposition. The map which sends an excursion to the $\mathbb{R}$-tree $T\big|_\varphi = [0,1]_{/\equiv_\varphi}$ equipped with the **image measure of the Lebesgue measure on** $[0,1]$ under the map which sends $x \in [0,1]$ into the tree $T\big|_\varphi = [0,1]_{/\equiv_\varphi}$ is **continuous with respect to** the **Gromov-Hausdorff-weak** topology.

A typical application of the perturbation result

Given a discrete tree (T, r, ρ) with edge length equal to 1, consider

- **Uniform on skeleton.** $\bar{\lambda}$ the normalized length measure.
- **Degree measure.** $\nu(\{x\}) := \frac{\deg(x)}{2}$, $x \in T$.

Fix a sequence $(a_n) \downarrow 0$, and assume that there is a limit tree $\mathcal{X}$ such that

$$(T_n, a_n^{-1} r_n, \rho_n, \lambda^{(T_n, a_n^{-1} r_n, \rho_n)}) \underset{n\to\infty}{\longrightarrow} \mathcal{X},$$

Gromov-Hausdorff-weakly. Then as $d_{\mathrm{Pr}}\big(\bar{\lambda}^{(T_n, a_n^{-1} r_n, \rho_n)}, \nu\big) \le a_n$, we might conclude that also

$$(T_n, a_n^{-1} r_n, \rho_n, \nu_n^{(T_n, r_n)}) \underset{n\to\infty}{\longrightarrow} \mathcal{X},$$

Gromov-Hausdorff-weakly.

Notice that the **measure read off the contour is uniform on the skeleton**.

The invariance principle

Heading towards path-wise convergence

Problem. The speed-ν_n motions are taking values in different spaces (T_n, r_n).

Notion of convergence in path space. For every $n \in \mathbb{N} \cup \{\infty\}$, let X^n be a càdlàg process with values in a metric space T_n. We say that $(X^n)_{n\in\mathbb{N}}$ converges to X in path space if there exists a metric space E and isometric embeddings $\phi_n : T_n \to E$, $n \in \mathbb{N} \cup \{\infty\}$, such that $(\phi_n \circ X^n)_{n\in\mathbb{N}}$ converges to $\phi_\infty \circ X$ in Skorohod path space.

The invariance principle

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

Theorem. **(Athreya, Löhr & W.)**

Assume that for all $n \in \mathbb{N}$, $(T_n, r_n, \rho_n, \nu_n)$ is a rooted compact metric measure tree. Let (T, r, ρ, ν) be a rooted compact metric measure tree. Assume that the sequence

$((T_n, r_n, \rho_n, \nu_n))_{n\in\mathbb{N}}$ **converges to** (T, r, ρ, ν) **pointed Gromov-Hausdorff-weakly**.

Let X^n be the speed-ν_n motion on $(T_n, r_n)_{n\in\mathbb{N}}$ started in ρ_n, and X the speed-ν motion on (T, r) starting in ρ. **Then X^n converges weakly in path-space to X**.

What if the global lower mass-bound property fails?

Let r_n be the Euclidian distance on $[0,1]$.

- Consider the MC X^n with values in $T_n \equiv \{0,1\}$ which jump from 0 to 1 at unit rate 1 ($\nu_n(\{0\}) := \frac{1}{2}$) but from 1 to 0 at rate n ($\nu_n(\{1\}) := \frac{1}{2n} \downarrow 0$). As $n \to \infty$ we will mostly see the process in 0 while at a countable number of times it is in 1. Thus X^n does not convergence in path-space (to the constant path) but all **finite dimensional distributions** do.

- Let $T_n := [0,1]$ and $\nu_n := \frac{n-1}{2n}(\delta_0 + \delta_1) + \frac{1}{n}\lambda_{[0,1]}$. Then $(T_n, \nu_n) \underset{n\to\infty}{\longrightarrow} (\{0,1\}, \frac{1}{2}(\delta_0 + \delta_1))$ Gromov-weakly. As X^n is sticky Brownian motion and has continuous paths, while X does not; we don't have convergence in path space but all **finite dimensional distributions converge**.

What if the lower mass-bound property fails?

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

Theorem. (Athreya, Löhr & W.)

Assume that for all $n \in \mathbb{N}$, $(T_n, r_n, \rho_n, \nu_n)$ is a rooted compact metric measure tree. Let (T, r, ρ, ν) be a rooted compact metric measure tree. Assume that the sequence

$((T_n, r_n, \rho_n, \nu_n))_{n\in\mathbb{N}}$ **converges to** (T, r, ρ, ν) **pointed Gromov-weakly**.

Let X^n be the speed-ν_n motion on $(T_n, r_n)_{n\in\mathbb{N}}$ started in ρ_n, and X the speed-ν motion on (T, r) starting in ρ. **Then the finite dimensional distributions of X^n converge to those of X.**

From compact to locally compact, complete metric measure spaces

The **invariance principle** can be stated **also for locally compact, complete trees** provided that we

- assume the **speed measure** to be **finite on bounded sets**
- replace the Gromov-weak topology by the **Gromov-vague** topology
- replace the global lower mass-bound property by a **local lower mass-bound property**
- be **careful** when the potential limit motion hits the boundary as limit points loose their Markov property the moment they hit the boundary

Main steps in the proof

From the PhD thesis of David Aldous

David Aldous (1989), Stopping times and tightness II, Annals of Probability

``One may draw a loose distinction between two methods of proving weak convergence results for stochastic processes. The classical method ... starts by proving f.d.d.-convergence and then verifies a tightness condition. The modern approach ... starts with a **characterization of the limit process**, then shows the characterization is asymptotically true for the approximating processes, and then argues this must imply weak convergence. One result which is sometimes useful in the modern approach is the following. Let X^n be $\mathbb{R}$-valued processes. Regard X^n as a random element of the usual function space equipped with Skorokhod-topology. Let τ^n denote a natural stopping time for X^n. Then the condition for all $\delta_n \to 0$ and all uniformly bounded (τ_n),

$$X^n_{\tau_n+\delta_n} - X^n_{\tau_n} \to 0 \qquad \text{in probability}$$

implies **tightness** of the sequence $(X^n)_{n\in\mathbb{N}}$.

Steps in the proof: a short summary

1. **Simplifying.** Any compact measure tree can be approximated by discrete trees. Thus we may assume w.l.o.g. that the approximating speed-ν_n motions on (T_n, r_n) are in fact continuum time random walks on discrete trees.

2. **Existence of limit processes** is ensured once we can show that

   it becomes unlikely that the walks have moved more than a certain distance in a sufficiently small amount of time, uniformly in $n \in \mathbb{N}$ and in the initial points.

3. **Identify the limit.** Verifying that any limit point satisfies the

   - **strong Markov property** (equi-continuity in initial state; coupling)
   - **occupation time formula** (semi-continuities of hitting times)

Tightness

Corollary of Aldous's criterion. Let (E,r) be a compact metric space. For $n\in\mathbb{N}$, let $T_n\subseteq E$ and (X^n) a cadlag strong Markov process on T_n. Then $(X^n)_{n\in\mathbb{N}}$ is tight provided that for every $R>0$,

$$\lim_{t\to 0}\lim_{n\to\infty}\sup_{x\in T_n}\mathbb{P}_x\big\{r(x,X^n_t)>R\big\}=0.$$

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

Lemma. **(Athreya, Löhr & W.)** Let (T,r,ν) be a discrete measure tree, $x\in T$, and X the **speed-ν random walk on (T,r) started in x**. Then for every $\varepsilon>0$, $0<\delta<\varepsilon$ and $t<(\varepsilon-\delta)\nu(B(\rho,\delta))$,

$$\mathbb{P}\big\{\sup_{s\in[0,t]}r(X_s,x)>2\varepsilon\big\}\le 2\mathrm{deg}_\varepsilon(T)\Big(1-\tfrac{\varepsilon-\delta}{\varepsilon+\delta}\exp\big(-\tfrac{2t}{\varepsilon\nu(B(x,\delta))}\big)\Big),$$

$\mathrm{deg}_\varepsilon(T)$ is the maximal number of edges which start inside a ball $B(x,\varepsilon)$ and end outside the ball $B(x,2\varepsilon)$.

The strong Markov property of the limit: Strategy

Denote by $\mathbb{P}^n_x$ the law of X^n started in $x \in T_n$. Assume the limiting tree is **compact**, and let $\{\tilde{\mathbb{P}}_x;\, x \in T\}$ a limit point. Denote by $\tilde{S} = (\tilde{S}_t)_{t \geq 0}$ the operator which sends $f \in \mathcal{C}(T)$ to

$$\tilde{S}_t f(x) := \tilde{\mathbb{E}}_x[f(X_t)].$$

We need to show that $\tilde{S}$ is a **semi-group** using the semi-group property S^n corresponding to X^n.

The main tool will be to verify **equi-continuity** of $\{S^n;\, n \in \mathbb{N}\}$.

Equi-continuity

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

Proposition. (Athreya, Löhr & W.) Let X^n be the **speed-ν_n random walk on** (T_n, r_n). If (T_n, r_n, ν_n) converges Gromov-weakly to a **compact** (T, r, ν) and the global lower mass-bound property holds, then the family of functions

$$\mathbb{P}^n : T_n \to \mathcal{M}_1(\mathcal{D}_E([0,\infty))), \quad x \mapsto \mathcal{L}_x(X^n),$$

is uniformly equicontinuous, where $\mathcal{D}_E([0,\infty))$ is equipped with the **Skorohod metric** and $\mathcal{M}_1(\mathcal{D}_E([0,\infty)))$ with the **Prohorov metric**.

Idea behind proof. Fix $\varepsilon > 0$. We have to find $\delta = \delta(\varepsilon) > 0$ (independent of n) such that $d_{Pr}(\mathbb{P}^n_x, \mathbb{P}^n_y) < \varepsilon$ whenever $d_E(x,y) \le \delta$.

- **Let run $X^{n,x}$ and $X^{n,y}$ until $\tau_y(X^{n,x})$, and put $X^{n,x}_{\tau_y(X^{n,x})+\cdot} := X^{n,y}_\cdot$.**
- **To estimate the Skorohod distance between paths, use the "time deformation" $\lambda(t) := t + (\tau_y(X^{n,x}) \wedge \varepsilon t)$ which is of dilatation $\mathrm{dil}(\lambda) \le \varepsilon$, and $X^{n,x}_t = X^{n,y}_{\lambda(t)}$ for all $t \ge \frac{1}{\varepsilon}\tau_y(X^{n,x})$.**
- **$\mathbb{P}^n_x\{\tau_y(X^{n,x}) > c\} \le c^{-1}\mathbb{E}[\tau_y(X^{n,x})] < 2 \cdot c^{-1}\delta \cdot \nu_n(T_n)$, whenever $d_E(x,y) < \delta$.**

Characterization Markov processes via occupation times

Athreya, Löhr & W., Invariance principle for variable speed random walks on trees, ArXiv:math.PR/1404.6290.

General abstract non-sense.

Assume that (T,r) is a **compact metric** (finite) **measure tree**, and that we are given two T-valued **strong Markov processes** X and Y such that for all $x,y \in T$,

$$\mathbb{E}_x\Big[\int_0^{\tau_y} f(X_t)\mathrm{d}t\Big] = \mathbb{E}_x\Big[\int_0^{\tau_y} f(Y_t)\mathrm{d}t\Big] < \infty.$$

Then the laws of X and Y agree.

The occupation time formula holds also for the limit

Lemma. Let E be a Polish space and $\mathcal{D}_E([0,\infty))$ the corresponding Skorohod space. For a subset $A \subseteq E$, define

$$\sigma_A : \mathcal{D}_E([0,\infty)) \to \mathbb{R}_+ \cup \{\infty\}, \quad w \mapsto \inf\big\{t \in \mathbb{R}_+ : w(t) \in A\big\}$$

$$F_A : \mathcal{D}_E([0,\infty)) \to \mathbb{R}_+, \quad w \mapsto \int_0^{\sigma_A(w)} \mathrm{d}s\, f\big(w_s\big)$$

Every **continuous path** is a **lower semi-continuity point** of F_A whenever A **is closed**, and an **upper semi-continuity point** of F_A whenever A **is open**.

We apply the latter by making use of

- $F_{\{z\}} \geq F_{B(z,\varepsilon)}$ but $F_{\{z\}} = \sup_{\varepsilon>0} F_{\bar{B}(z,\varepsilon)}$.
- Starting in x, **on trees** there is a **unique point** at which we enter the balls $\bar{B}(z,\varepsilon)$.

Related work and examples

Brownian motion on disconnected sets

Shankar Bhamidi, Steve Evans, Ron Peled, and Peter Ralph (2008), Brownian motion on disconnected sets, basic hypergeometric functions, and some continued fractions of Ramanujan

- Equip $\mathbb{R}$ with the Euclidian distance.
- Put $T_q := \{\pm q^k;\ k \in \mathbb{Z}\} \cup \{0\} \subseteq \mathbb{R}$.
- Obviously $\{T_q;\ q > 1\}$ is **dense** in $\mathbb{R}$ and **length measure** is **boundedly finite**.
- Consequently, $\{(T_q, 0, \lambda^{T_q});\ q > 1\}$ converges **Gromov-Hausdorff-vaguely** to $(\mathbb{R}, 0, \lambda)$.
- **Thus** the **speed-λ^{T_q} motion on T_q converges in path space towards standard BM as $q \downarrow 1$**.

Croydon: Random Walks on Galton-Watson trees

The **GW-process** models a population, in which individuals independently at constant rate 1 either die or split into 2 individuals. It is known that the population gets extinct in finite time.

David Aldous (1993), The continuum random tree III, Annals of probability

Let T_n denote the corresponding family **tree conditioned on total population size** n. Then $\frac{1}{\sqrt{n}} T_n \underset{n\to\infty}{\Longrightarrow} T$ for a continuum tree T.

David Croydon (2008), Convergence of simple random walks on random discrete trees to Brownian motion on the continuum random tree, Annales de línstitut Henri Poincaré (B)

Consider $X^n = (X_t^n)_{t\ge 0}$ the SRW which jumps at **constant rate** 1 **to each** of the **neighboring vertices** in T_n **with equal probability**. Then there exists a strong Markov process $B = (B_t)_{t\ge 0}$ with continuous paths such that

$$\Big(\frac{1}{\sqrt{n}} X_{n^{\frac{3}{2}}t}\Big)_{t\ge 0} \underset{n\to\infty}{\Longrightarrow} \big(B_t\big)_{t\ge 0}.$$

Croydon: Simple RW on Galton-Watson trees

For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ **be the GW-tree conditioned on total population size** n and put

$$T_n := \mathcal{T}_n \quad r_n(v, v') := \tfrac{1}{\sqrt{n}}, \quad \nu_n(\{v\}) := \tfrac{\deg(v)}{2n}, \qquad\qquad v, v' \in T_n; v \sim v'.$$

The ν_n **-speed random walk on** (T_n, r_n) is the **SRW** on T_n with **edge length re-scaled** by $\frac{1}{\sqrt{n}}$ and **speeded up** (in each vertex v) by a factor of

$$\gamma_n(v) = \frac{1}{2\nu_n(\{v\})} \sum_{v' \sim v} r_n^{-1}(v, v') = \frac{1}{2} \cdot \frac{2n}{\deg(v)} \cdot \deg(v)\sqrt{n} = n^{\frac{3}{2}}.$$

Croydon's homogeneous invariance principle

David Croydon (2010), Scaling limits for simple random walks on random ordered graph trees, Advances in Applied Probability

``Consider a family of random ordered graph trees $(T_n)_{n\in\mathbb{N}}$, where T_n has n vertices. It has previously been established that if the associated search-depth processes converge to the normalised Brownian excursion when re-scaled appropriately, then the simple random walks on the graph trees have the Brownian motion on the continuum random tree as their scaling limit. ... this result is extended to demonstrate the existence of a **diffusion** scaling limit whenever the **volume measure** on the **limiting real tree** is **non-atomic, supported on the leaves of the limiting tree, and satisfies a polynomial lower bound for the volume of balls**.''

Note that in contrast to Croydon's invariance principle we allow for **inhomogenous**, and even **state-dependent rescaling**.

Random walks on size-biased Galton Watson trees

Harry Kesten (1986), Subdiffusive behavior of random walk on a random cluster, Poincare

Let $\mathcal{T}_{\text{Kesten}}$ **be the GW-tree conditioned to never die out**, and X the discrete time nearest neighbor random walk on $\mathcal{T}_{\text{Kesten}}$. Consider the **re-scaled height process**

$$Z_t^n := n^{-\frac{1}{3}} \cdot r\big(\rho, X_{\lfloor nt \rfloor}\big).$$

Kesten showed that under the **annealed law** the family $\{Z^{(n)};\, n \in \mathbb{N}\}$ converges weakly in path space to a non-trivial diffusion $(Z_t)_{t\geq 0}$.

Barlow and Kumagai showed that under the **quenched law** the family $\{Z^{(n)};\, n \in \mathbb{N}\}$ is NOT tight (=does not have limit points) for almost all realizations T_{Kesten} of $\mathcal{T}_{\text{Kesten}}$.

RW on size-biased GW-trees; The annealed regime

Harry Kesten (1986), Subdiffusive behavior of random walk on a random cluster, Poincare

Let $\mathcal{T}_{\text{Kesten}}$ **be the GW-tree conditioned to never die out**, and put for each realization T_{Kesten},

$$T_n := T_{\text{Kesten}} \quad r_n(v,v') := \frac{1}{n^{\frac{1}{3}}}, \quad \nu_n(\{v\}) := \frac{\deg(v)}{2n^{\frac{2}{3}}}, \qquad v,v' \in T_n; v \sim v'.$$

The ν_n**-speed random walk on** (T_n, r_n) is the **SRW** on T_n with **edge length re-scaled** by $\frac{1}{n^{\frac{1}{3}}}$ and **speeded up** by a factor of

$$\gamma_n(v) = \frac{1}{2\nu_n(\{v\})} \sum_{v' \sim v} r_n^{-1}(v,v') = \frac{1}{2} \cdot \frac{2n^{\frac{2}{3}}}{\deg(v)} \cdot \deg(v) n^{\frac{1}{3}} \equiv n.$$

As it is known that there is a limit measure $\mathbb{R}$-tree (T,r,ν) such that $(T_n, r_n, \nu_n) \underset{n\to\infty}{\Longrightarrow} (T,r,\nu)$. That is, for a.a. realizations of (T_n, r_n, ν_n) and (T,r,ν) the **speed-**ν_n **random walk on** (T_n, r_n) **converges in path space to the** ν**-speed Brownian motion on** (T,r). Moreover, as ν-speed Brownian motion on (T,r) is **recurrent**, so is $(Z_t := r(\rho, X_t))_{t\geq 0}$.

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RW on size-biased GW-trees; The quenched regime

Martin Barlow and Takashi Kumagai (2006), Random walk on the incipient infinite cluster on trees

Barlow and Kumagai show that for each typical realization T_{Kesten} of the **GW-tree conditioned to never die out**,

$$\liminf_{n\to\infty} \nu_n\big(B(\rho,R)\big) = 0, \text{ and } \limsup_{n\to\infty} \nu_n\big(B(\rho,R)\big) = \infty,$$

and thus that the sequence $\{\nu_n;\, n \in \mathbb{N}\}$ does NOT have vague limit points.

Consequently, the assumptions on our invariance principle FAIL for almost all realizations of $\mathcal{T}_{\text{Kesten}}$. For a quenched statement to hold you **need to rather work with a state-dependent rescaling**.

Many thanks