Directed random graphs and convergence to the Tracy-Widom distribution

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Outline

Directed Random Graphs
  Directed Random Graph on $\mathbb{Z}$
  Directed Random Graph on $\mathbb{Z} \times \{1, 2, \ldots, m\}$

Convergence to the Tracy-Widom Distribution
  Last-Passage Directed Percolation
  Directed Random Graph on $\mathbb{Z} \times \mathbb{Z}$
Directed Random Graphs

Consider a random graph on vertex set $\mathbb{Z}$ with edges between any pair of vertices $(i, j), i, j \in \mathbb{Z}$, present with probability $p$ independently of the other edges.
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Direct each edge $(i, j)$ from $\min(i, j)$ to $\max(i, j)$. 
A path $\pi$ is an increasing subsequence of vertices $\pi = (i_0, i_1, \ldots, i_\ell)$ successively connected by edges. The number of edges, $\ell = |\pi|$, is the length of the path.

Define

$$L(i, j) := \text{the maximum length of all paths with vertices between } i \text{ and } j.$$
**Skeleton points**

**Definition:** A vertex $i$ of the directed random graph $G$ is called skeleton point if for any $i' < i < i''$, there is a path from $i'$ to $i$ and a path from $i$ to $i''$.

Let $S$ be the set of all skeleton points. Denote its elements as

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \Gamma_2 < \cdots.$$ 

$\{\Gamma_{r+1} - \Gamma_r, r \in \mathbb{Z}\}$ are independent random variables, whereas $\{\Gamma_{r+1} - \Gamma_r, r \neq 0\}$ are i.i.d.

The sequence forms a stationary renewal process with rate

$$\lambda := \frac{1}{E(\Gamma_2 - \Gamma_1)} = \prod_{k=1}^{\infty} (1 - (1 - p)^k)^2.$$
For all integers $m < n$,

$$L(\Gamma_m, \Gamma_n) = L(\Gamma_m, \Gamma_{m+1}) + L(\Gamma_{m+1}, \Gamma_{m+2}) + \cdots + L(\Gamma_{n-1}, \Gamma_n).$$
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Let $\Phi(n) = \max\{k \in \mathbb{Z} : \Gamma_k \leq n\}$. Then we can write 

$$L(0, n) = L(0, \Gamma_1) + \sum_{i=2}^{\Phi(n)} L(\Gamma_{i-1}, \Gamma_i) + L(\Gamma_{\Phi(n)}, n).$$
**Theorem:** (Denisov *et al.*, 2012)

Let

\[ C = \lim_{n \to \infty} \frac{L(0, n)}{n} \text{ a.s.} \]

and

\[ \sigma^2 = \text{Var}[L(\Gamma_1, \Gamma_2) - C(\Gamma_2 - \Gamma_1)]. \]

Then

\[
\left( \frac{L(0, \lfloor nt \rfloor) - Cnt}{\sigma \sqrt{n\lambda}}, \ t \geq 0 \right) \overset{d}{\to} (B_t, \ t \geq 0) \text{ as } n \to \infty,
\]

where \((B_t, \ t \geq 0)\) is standard Brownian motion.
Random Directed Slab Graph

For a fixed integer $m$, let $G_m$ be a random graph with vertices $\mathbb{Z} \times \{1, 2, \ldots, m\}$ and with edge probability $p$.

Direct the edges according to the product order of the labels: $(i_1, i_2) < (j_1, j_2)$ if the two pairs are distinct and $i_1 \leq i_2, j_1 \leq j_2$. 
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Skeleton points

The restriction of $G_m$ onto $\mathbb{Z} \times \{j\}$ is a directed random graph.

**Definition:** Point $i$ is a skeleton “point” if $(i, j)$ is a skeleton point of the restriction of $G_m$ onto $\mathbb{Z} \times \{j\}$ for all $j \in \{1, 2, \ldots, m\}$ and if for all $j \in \{1, 2, \ldots, m - 1\}$ there is an edge between $(i, j)$ and $(i, j + 1)$. 
Skeleton points

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Denote the points of the skeleton by 

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \Gamma_2 < \cdots.$$
Denote by $L_{n,m}$ the maximum length of all paths of the graph $G_m$ restricted to $\{0, \ldots, n\} \times \{1, \ldots, m\}$. 

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Denote by $L_{n,m}$ the maximum length of all paths of the graph $G_m$ restricted to $\{0, \ldots, n\} \times \{1, \ldots, m\}$.

$$L_{n,m}^* := \max_{1=i_1<i_2<\cdots<i_m<i_{m+1}=\Phi(n)} \sum_{j=1}^{m} L^{(j)}[\Gamma_{i_j}, \Gamma_{i_{j+1}}].$$
Theorem: (Denisov et al., 2012)

Let

$$C = \lim_{n \to \infty} \frac{L_{n,1}}{n} \text{ a.s.}$$

and

$$\sigma^2 = \text{Var}[L^{(1)}(\Gamma_1, \Gamma_2) - C(\Gamma_2 - \Gamma_1)].$$

Then

$$\frac{L_{n,m} - Cn}{\sigma \sqrt{n\lambda}} \xrightarrow{d} Z_{1,m} \quad \text{as} \quad n \to \infty,$$

where $Z_{*,m}$ is a random variable defined in terms of $m$ independent standard Brownian motions, $B^{(1)}, \ldots, B^{(m)}$, via the formula

$$Z_{1,m} := \sup_{0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1} \sum_{j=1}^{m} [B_{t_j}^{(j)} - B_{t_{j-1}}^{(j)}], \quad t \geq 0.$$
Brownian Directed Percolation

Let \((B^{(r)}, r \geq 1)\) be a sequence of independent standard Brownian motions and for any \(t \geq 0\) and \(m \geq 1\) define

\[
Z_{t,m} := \sup_{0 = t_0 < t_1 \ldots < t_{m-1} < t_m = t} \sum_{j=1}^{m} [B_{t_j}^{(j)} - B_{t_{j-1}}^{(j)}].
\]

By Brownian scaling, \(Z_{t,m} / \sqrt{t}\) has the same law as \(Z_{1,m}\).
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\[
Z_{t,m} := \sup_{0=t_0 < t_1 \cdots < t_{m-1} < t_m = t} \sum_{j=1}^{m} [B^{(j)}_{t_j} - B^{(j)}_{t_{j-1}}].
\]

By Brownian scaling, \(Z_{t,m}/\sqrt{t}\) has the same law as \(Z_{1,m}\).
Denote by $\lambda_m$ the largest eigenvalue of the random $m \times m$ matrix from GUE.

**Theorem:** (Baryshnikov, 2001) The random variable $Z_{1,m}$ has the same law as $\lambda_m$.

**Theorem:** (Tracy and Widom, 1994)

\[
m^{1/6} \left( \lambda_m - 2 \sqrt{m} \right) \xrightarrow{d} F_{TW} \quad \text{as} \quad m \to \infty.
\]

Using that for arbitrary $t > 0$ it holds $Z_{t,m}/\sqrt{t} \xrightarrow{d} \lambda_m$, we get

\[
m^{1/6} \left( \frac{Z_{t,m}}{\sqrt{t}} - 2 \sqrt{m} \right) \xrightarrow{d} F_{TW} \quad \text{as} \quad m \to \infty.
\]

Upon setting $m = \lfloor t^a \rfloor$, we have

\[
t^{a/6} \left( \frac{Z_{t,\lfloor t^a \rfloor}}{\sqrt{t}} - 2 \sqrt{t^a} \right) \xrightarrow{d} F_{TW} \quad \text{as} \quad t \to \infty.
\]
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Theorem: (Konstantopoulos and T., 2013)
Consider the directed random graph on $\mathbb{Z} \times \mathbb{Z}$ and let $L_{n,m}$ be the maximum length of all paths between two vertices in $\{0, 1, \ldots, n\} \times \{1, 2, \ldots, m\}$. Then, for all $0 < a < 3/14$,

$$n^{a/6} \left( \frac{L_{n,\lfloor n^a \rfloor} - Cn}{\sqrt{\lambda} \sigma^2 n} - 2 \sqrt{n^a} \right) \overset{d}{\to} F_{TW} \quad \text{as} \quad n \to \infty.$$
Let $\Pi(n, k)$ be the set of all up-right paths $\pi$ in $\mathbb{Z}_+^2$ from $(1, 1)$ to $(n, k)$ and let $\{\omega_i^{(r)}, i \geq 1, r \geq 1\}$ be i.i.d. random variables.
Let $\Pi(n, k)$ be the set of all up-right paths $\pi$ in $\mathbb{Z}_+^2$ from $(1, 1)$ to $(n, k)$ and let $\{\omega_i^{(r)}, i \geq 1, r \geq 1\}$ be i.i.d. random variables.

The last passage time to the point $(n, k)$ is defined by

$$T(n, k) = \max_{\pi \in \Pi(n, k)} \sum_{(i, r) \in \pi} \omega_i^{(r)}.$$
Examples

- If \( \{ \omega_i^{(r)} \}, i \geq 1, r \geq 1 \) are exponentially or geometrically distributed random variables. Then for any \( \gamma \geq 1 \), \( T(n, \lfloor \gamma n \rfloor) \) appropriately rescaled/centered converges to the Tracy-Widom distribution. (Johansson, 2000)

- If \( \{ \omega_i^{(r)} \}, i \geq 1, r \geq 1 \) are i.i.d. random variables such that \( E|\omega_1^{(1)}|^p < \infty \) for some \( p > 2 \). Then for all \( a \) such that \( 0 < a < \frac{6}{7} (1/2 - 1/p) \), \( T(n, \lfloor n^a \rfloor) \) appropriately rescaled/centered converges to the Tracy-Widom distribution. (Bodineau and Martin, 2005)
Skeleton points

Let $G$ be the random graph on $\mathbb{Z} \times \mathbb{Z}$ and $G^{(j)}$ its restriction on $\mathbb{Z} \times \{j\}$.

**Definition:** A vertex $(i, j)$ of the directed random graph $G$ is called skeleton point if it is a skeleton point for $G^{(j)}$ (for any $i' < i < i''$, there is a path from $(i', j)$ to $(i, j)$ and a path from $(i, j)$ to $(i'', j)$) and if there is an edge from $(i, j)$ to $(i, j + 1)$.

Denote the skeleton points on line $j$ as

$$\cdots < \Gamma_{-1}^{(j)} < \Gamma_{0}^{(j)} \leq 0 < \Gamma_{1}^{(j)} < \Gamma_{2}^{(j)} < \cdots .$$
Upper bound

Let $X^{(j)}(t) := \Gamma^{(j)}_{\Phi(j)}(t)$ and $Y^{(j)}(t) := \Gamma^{(j)}_{\Phi(j)}(t)+1$.

It holds

$$L_{n,m} \leq \overline{L}_{n,m}$$

where

$$\overline{L}_{n,m} := \sup_{0 = t_0 < t_1 \ldots < t_{m-1} < t_m = n} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] + m.$$
Upper bound

Let $X^{(j)}(t) := \Gamma^{(j)}_{\phi(j)}(t)$ and $Y^{(j)}(t) := \Gamma^{(j)}_{\phi(j)}(t) + 1$.

It holds

$\bar{L}_{n,m} \leq \bar{L}_{n,m}$

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$\bar{L}_{n,m} := \sup_{0 = t_0 < t_1 \ldots < t_{m-1} < t_m = n} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] + m.$
Lower bound

\begin{align*}
L_{n,m} & \geq \underline{L}_{n,m} \\
\text{where} \\
\underline{L}_{n,m} & := \sup_{0=t_0<t_1\ldots<t_{m-1}<t_m=n} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)]
\end{align*}
Lower bound

It holds

\[ L_{n,m} \geq L^{-}_{n,m} \]

where

\[ L^{-}_{n,m} := \sup_{0=t_0<t_1<\cdots<t_{m-1}<t_m=n} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)] - \sum_{j=1}^{m} \max_{0 \leq i \leq \Phi^{(j)}(n)} (\Gamma^{(j)}_{i+1} - \Gamma^{(j)}_i) \]
Centering

We introduce the quantity

\[ S_{n,m} := \sup_{0=t_0 < t_1 < \ldots < t_{m-1} < t_m = n} \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - C[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right\}. \]
Centering

We introduce the quantity

\[ S_{n,m} := \sup_{0=t_0 < t_1 < \cdots < t_{m-1} < t_m = n} \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - C[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right\}. \]

which can be rewritten as

\[
\frac{1}{\sigma} S_{n,m} = \sup_{0=t_0 < t_1 < \cdots < t_{m-1} < t_m = n} \sum_{j=1}^{m} \sum_{k=\Phi^{(j)}(t_{j-1})+1}^{\Phi^{(j)}(t_j)} \chi^{(j)}_k,
\]

where

\[
\chi^{(j)}_k := \frac{1}{\sigma} \left\{ L^{(j)}[\Gamma^{(j)}_{k-1}, \Gamma^{(j)}_k] - C(\Gamma^{(j)}_k - \Gamma^{(j)}_{k-1}) \right\}.
\]

The term \( \frac{1}{\sigma} S_{n,m} \) resembles a last passage percolation path weight, except that random indices are involved.
Lemma: For $a < 3/7$

\[
\frac{S_{n,\lfloor n^a \rfloor} - (L_{n,\lfloor n^a \rfloor} - Cn)}{n^{1/2-a/6}} \xrightarrow{p} 0 \text{ as } n \to \infty.
\]
Recall that

\[ n^{a/6} \left( \frac{Z_{n,\lfloor n^a \rfloor}}{\sqrt{n}} - 2 \sqrt{n^a} \right) \xrightarrow{d} F_{TW} \text{ as } n \to \infty. \]

Thus, to show

\[ n^{a/6} \left( \frac{L_{n,\lfloor n^a \rfloor} - Cn}{\sqrt{\lambda \sigma^2 n}} - 2 \sqrt{n^a} \right) \xrightarrow{d} F_{TW} \text{ as } n \to \infty \]

it remains to prove

\[ \sigma^{-1} S_{n,\lfloor n^a \rfloor} - Z_{\lambda n,\lfloor n^a \rfloor} \xrightarrow{p} 0 \text{ as } n \to \infty. \]
Coupling with Brownian motion

The difference between $\sigma^{-1} S_{n,m}$ and $Z_{\lambda n,m}$ can be bounded by

$$|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \leq 2 \sum_{j=1}^{m} U_n^{(j)} + 2 \sum_{j=1}^{m} V_n^{(j)}$$

where

$$U_n^{(j)} := \max_{0 \leq i \leq n} \left| \sum_{k=1}^{i} \chi_{k}^{(j)} - B_{i}^{(j)} \right|, \quad V_n^{(j)} := \sup_{0 \leq s \leq n} \left| B_{\Phi(j)(s)}^{(j)} - B_{\lambda s}^{(j)} \right|.$$  

Using Komlós-Major-Tusnády strong approximation result we construct jointly the RW’s and BM’s such that

$$\frac{1}{n^{1/2-a/6}} \sum_{j=1}^{\lfloor n^a \rfloor} U_n^{(j)} \to 0.$$  

To show $\frac{1}{n^{1/2-a/6}} \sum_{j=1}^{\lfloor n^a \rfloor} V_n^{(j)} \to 0$ we used a version of the Baum-Katz theorem for the counting process.
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Thank you for your attention!