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Outline

Directed random graphs and convergence to the Tracy-Widom distribution

Katja Trinajstić

joint work with T. Konstantopoulos

Uppsala University

YWIP Bonn • 28 May 2014

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Outline

Directed Random Graphs

Directed Random Graph on $\mathbb{Z}$

Directed Random Graph on $\mathbb{Z} \times \{1, 2, \ldots, m\}$

Convergence to the Tracy-Widom Distribution

Convergence to the Tracy-Widom Distribution

Last-Passage Directed Percolation

Directed Random Graph on $\mathbb{Z} \times \mathbb{Z}$

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Directed Random Graphs

Consider a random graph on vertex set $\mathbb{Z}$ with edges between any pair of vertices (i, j), $i, j \in \mathbb{Z}$, present with probability p independently of the other edges.

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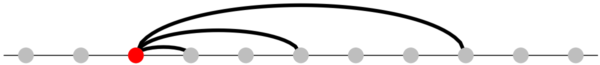

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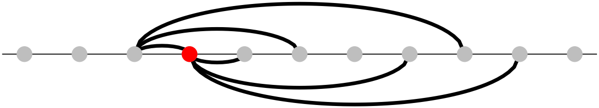

Direct each edge (i, j) from $\min(i, j)$ to $\max(i, j)$.

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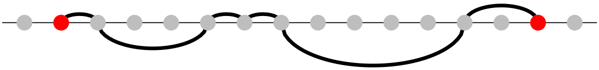
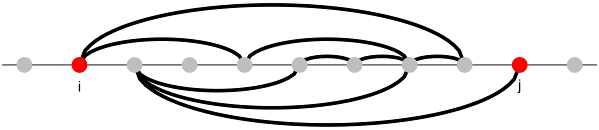
A path π is an increasing subsequence of vertices $\pi = (i_0, i_1, \ldots, i_\ell)$ successively connected by edges. The number of edges, $\ell = |\pi|$, is the length of the path.



Define

$$L(i, j) := \quad \text{the maximum length of all paths with vertices between } i \text{ and } j.$$



Skeleton points

Definition: A vertex i of the directed random graph G is called skeleton point if for any $i' < i < i''$, there is a path from i' to i and a path from i to i''.

Let $\mathcal{S}$ be the set of all skeleton points. Denote its elements as

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \Gamma_2 < \cdots .$$

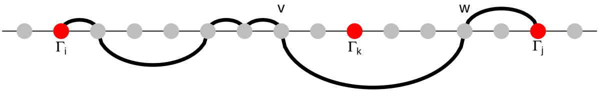
$\{\Gamma_{r+1} - \Gamma_r, r \in \mathbb{Z}\}$ are independent random variables, whereas $\{\Gamma_{r+1} - \Gamma_r, r \neq 0\}$ are i.i.d.

The sequence forms a stationary renewal process with rate

$$\lambda := \frac{1}{E(\Gamma_2 - \Gamma_1)} = \prod_{k=1}^{\infty} (1 - (1-p)^k)^2.$$

For all integers $m < n$,

$$L(\Gamma_m, \Gamma_n) = L(\Gamma_m, \Gamma_{m+1}) + L(\Gamma_{m+1}, \Gamma_{m+2}) + \cdots + L(\Gamma_{n-1}, \Gamma_n).$$



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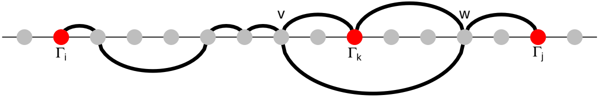

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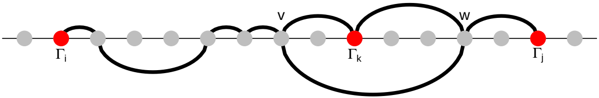


Let $\Phi(n) = \max\{k \in \mathbb{Z} : \Gamma_k \leq n\}$. Then we can write

$$L(0, n) = L(0, \Gamma_1) + \sum_{i=2}^{\Phi(n)} L(\Gamma_{i-1}, \Gamma_i) + L(\Gamma_{\Phi(n)}, n).$$

Theorem: (Denisov *et al.*, 2012)

Let

$$C = \lim_{n \to \infty} \frac{L(0, n)}{n} \text{ a.s.}$$

and

$$\sigma^2 = \mathrm{Var}[L(\Gamma_1, \Gamma_2) - C(\Gamma_2 - \Gamma_1)].$$

Then

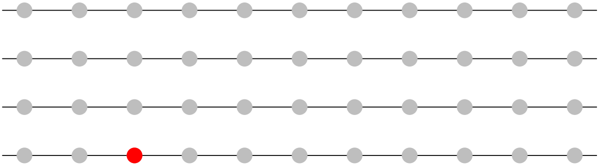
$$\left(\frac{L(0, \lfloor nt \rfloor) - Cnt}{\sigma \sqrt{n\lambda}}, \ t \geq 0 \right) \xrightarrow{d} (B_t, \ t \geq 0) \quad \text{as} \quad n \to \infty,$$

where $(B_t, \ t \geq 0)$ is standard Brownian motion.

Random Directed Slab Graph

For a fixed integer m, let G_m be a random graph with vertices $\mathbb{Z} \times \{1, 2, \ldots, m\}$ and with edge probability p.

Direct the edges according to the product order of the labels: $(i_1, i_2) \prec (j_1, j_2)$ if the two pairs are distinct and $i_1 \leq i_2$, $j_1 \leq j_2$.



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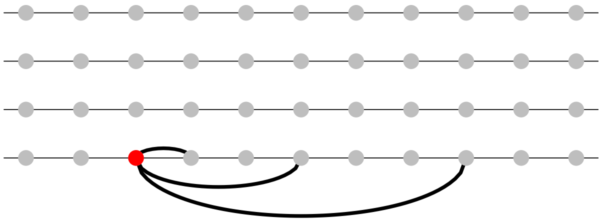

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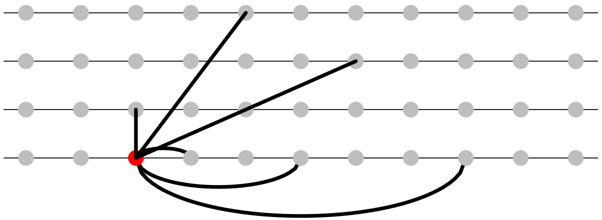

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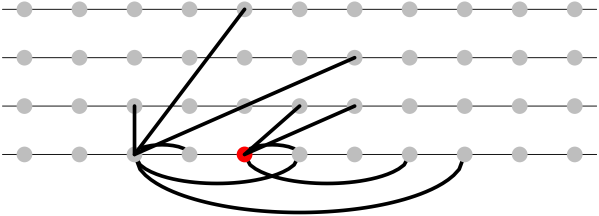

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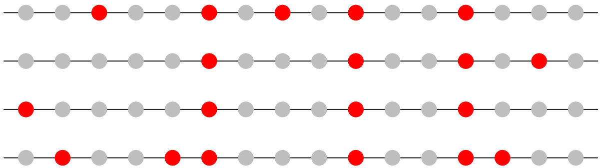
Outline

Skeleton points

The restriction of G_m onto $\mathbb{Z} \times \{j\}$ is a directed random graph.

Definition: Point i is a skeleton "point" if (i, j) is a skeleton point of the restriction of G_m onto $\mathbb{Z} \times \{j\}$ for all $j \in \{1, 2, \ldots, m\}$ and if for all $j \in \{1, 2, \ldots, m-1\}$ there is an edge between (i, j) and $(i, j+1)$.



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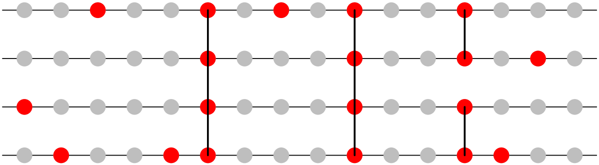

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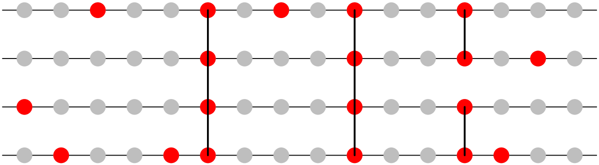
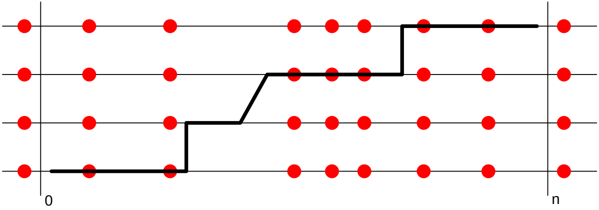


Denote the points of the skeleton by

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \Gamma_2 < \cdots .$$

Denote by $L_{n,m}$ the maximum length of all paths of the graph G_m restricted to $\{0, \ldots, n\} \times \{1, \ldots, m\}$.



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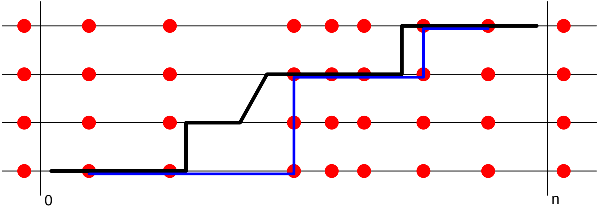


$$L^*_{n,m} := \max_{1=i_1<i_2\cdots<i_m<i_{m+1}=\Phi(n)} \sum_{j=1}^{m} L^{(j)}[\Gamma_{i_j}, \Gamma_{i_{j+1}}].$$

Theorem: (Denisov *et al.*, 2012)

Let

$$C = \lim_{n \to \infty} \frac{L_{n,1}}{n} \text{ a.s.}$$

and

$$\sigma^2 = \mathrm{Var}[L^{(1)}(\Gamma_1, \Gamma_2) - C(\Gamma_2 - \Gamma_1)].$$

Then

$$\frac{L_{n,m} - Cn}{\sigma \sqrt{n\lambda}} \xrightarrow{d} Z_{1,m} \quad \text{as} \quad n \to \infty,$$

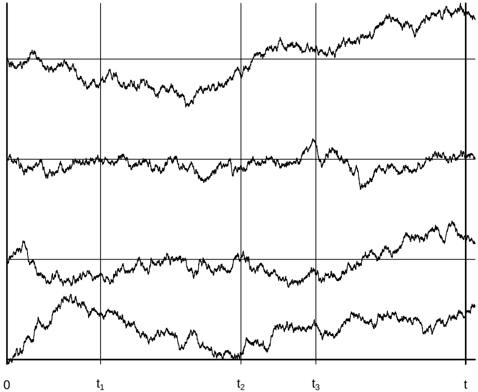
where $Z_{\bullet,m}$ is a random variable defined in terms of m independent standard Brownian motions, $B^{(1)}, \ldots, B^{(m)}$, via the formula

$$Z_{1,m} := \sup_{0=t_0<t_1 \cdots <t_{m-1}<t_m=1} \sum_{j=1}^{m} [B^{(j)}_{t_j} - B^{(j)}_{t_{j-1}}], \quad t \geq 0.$$

Brownian Directed Percolation

Let $(B^{(r)}, r \geq 1)$ be a sequence of independent standard Brownian motions and for any $t \geq 0$ and $m \geq 1$ define

$$Z_{t,m} := \sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=t} \sum_{j=1}^{m} [B^{(j)}_{t_j} - B^{(j)}_{t_{j-1}}].$$



By Brownian scaling, $Z_{t,m}/\sqrt{t}$ has the same law as $Z_{1,m}$.

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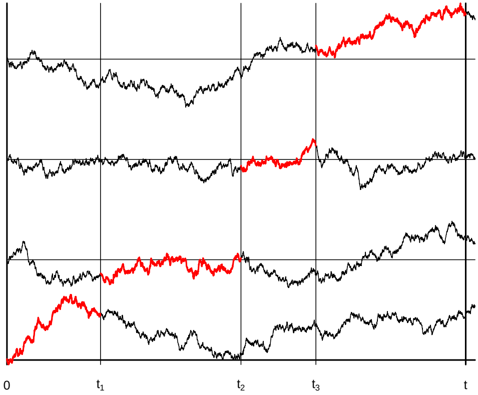

By Brownian scaling, $Z_{t,m}/\sqrt{t}$ has the same law as $Z_{1,m}$.

Denote by λ_m the largest eigenvalue of the random $m \times m$ matrix from GUE.

Theorem: (Baryshnikov, 2001) The random variable $Z_{1,m}$ has the same law as λ_m.

Theorem: (Tracy and Widom, 1994)

$$m^{\frac{1}{6}} \left(\lambda_m - 2\sqrt{m}\right) \xrightarrow{d} F_{\mathrm{TW}} \quad \text{as} \quad m \to \infty.$$

Using that for arbitrary $t > 0$ it holds $Z_{t,m}/\sqrt{t} \overset{\mathrm{d}}{=} \lambda_m$, we get

$$m^{1/6} \left(\frac{Z_{t,m}}{\sqrt{t}} - 2\sqrt{m}\right) \xrightarrow{d} F_{\mathrm{TW}} \quad \text{as} \quad m \to \infty.$$

Upon setting $m = \lfloor t^a \rfloor$, we have

$$t^{a/6}\left(\frac{Z_{t,\lfloor t^a \rfloor}}{\sqrt{t}} - 2\sqrt{t^a}\right) \xrightarrow{d} F_{\mathrm{TW}} \quad \text{as} \quad t \to \infty.$$

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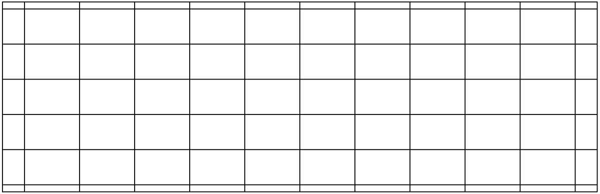
Katja Trinajstić
28 May 2014

Theorem: (Konstantopoulos and T., 2013)
Consider the directed random graph on $\mathbb{Z} \times \mathbb{Z}$ and let $L_{n,m}$ be the maximum length of all paths between two vertices in $\{0, 1, \ldots, n\} \times \{1, 2, \ldots, m\}$. Then, for all $0 < a < 3/14$,

$$n^{a/6}\left(\frac{L_{n,\lfloor n^a \rfloor} - Cn}{\sqrt{\lambda\sigma^2 n}} - 2\sqrt{n^a}\right) \xrightarrow{d} F_{\mathrm{TW}} \quad \text{as} \quad n \to \infty.$$

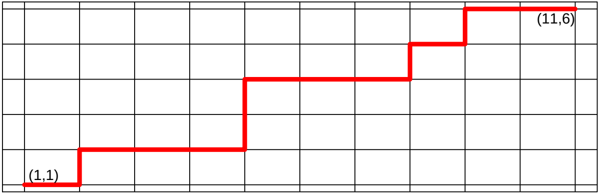
Outline

Last-Passage Directed Percolation



Let $\Pi(n, k)$ be the set of all up-right paths π in $\mathbb{Z}_+^2$ from $(1, 1)$ to (n, k) and let $\{\omega_i^{(r)}, i \geq 1, r \geq 1\}$ be i.i.d. random variables.

Last-Passage Directed Percolation



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The last passage time to the point (n,k) is defined by

$$T(n,k) = \max_{\pi \in \Pi(n,k)} \sum_{(i,r) \in \pi} \omega_i^{(r)}.$$

Examples

- If $\{\omega_i^{(r)}, i \geq 1, r \geq 1\}$ are exponentially or geometrically distributed random variables. Then for any $\gamma \geq 1$, $T(n, \lfloor \gamma n \rfloor)$ appropriately rescaled/centered converges to the Tracy-Widom distribution. (Johansson, 2000)
- If $\{\omega_i^{(r)}, i \geq 1, r \geq 1\}$ are i.i.d. random variables such that $E|\omega_1^{(1)}|^p < \infty$ for some $p > 2$. Then for all a such that $0 < a < \frac{6}{7}(1/2 - 1/p)$, $T(n, \lfloor n^a \rfloor)$ appropriately rescaled/centered converges to the Tracy-Widom distribution. (Bodineau and Martin, 2005)

Skeleton points

Let G be the random graph on $\mathbb{Z} \times \mathbb{Z}$ and $G^{(j)}$ its restriction on $\mathbb{Z} \times \{j\}$.

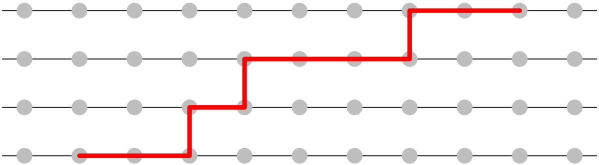
Definition: A vertex (i, j) of the directed random graph G is called skeleton point if it is a skeleton point for $G^{(j)}$ (for any $i' < i < i''$, there is a path from (i', j) to (i, j) and a path from (i, j) to (i'', j)) and if there is an edge from (i, j) to $(i, j + 1)$.

Denote the skeleton points on line j as

$$\cdots < \Gamma_{-1}^{(j)} < \Gamma_0^{(j)} \leq 0 < \Gamma_1^{(j)} < \Gamma_2^{(j)} < \cdots .$$

Upper bound

Let $X^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)}$ and $Y^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)+1}$.



It holds

$$L_{n,m} \le \overline{L}_{n,m}$$

where

$$\overline{L}_{n,m} := \sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=n} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}),\ Y^{(j)}(t_j)] + m.$$

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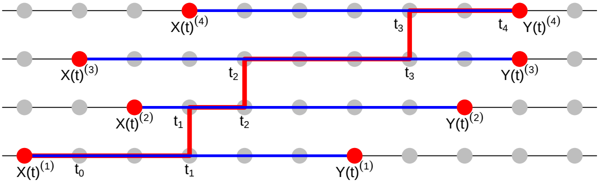

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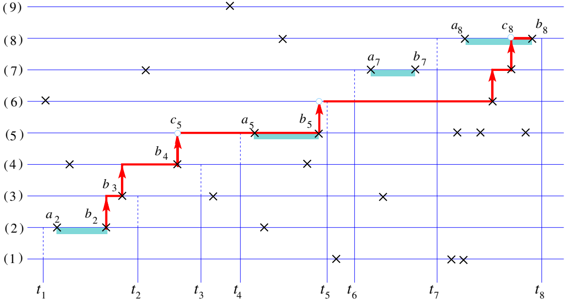
Lower bound



It holds

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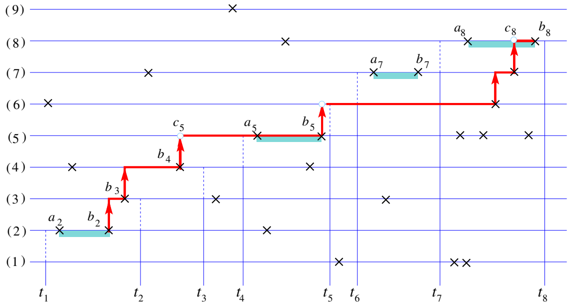
Lower bound



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$$- \sum_{j=1}^{m} \max_{0\leq i\leq \Phi^{(j)}(n)} (\Gamma^{(j)}_{i+1} - \Gamma^{(j)}_{i})$$

Centering

We introduce the quantity

$$S_{n,m} := \sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=n} \sum_{j=1}^{m} \Big\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - C[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \Big\}.$$

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which can be rewritten as

$$\frac{1}{\sigma} S_{n,m} = \sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=n} \sum_{j=1}^{m} \sum_{k=\Phi^{(j)}(t_{j-1})+1}^{\Phi^{(j)}(t_j)} \chi_k^{(j)},$$

where

$$\chi_k^{(j)} := \frac{1}{\sigma} \Big\{ L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}) \Big\}.$$

The term $\frac{1}{\sigma} S_{n,m}$ resembles a last passage percolation path weight, except that random indices are involved.

Lemma: For $a < 3/7$

$$\frac{S_{n,\lfloor n^a \rfloor} - (L_{n,\lfloor n^a \rfloor} - Cn)}{n^{1/2 - a/6}} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Recall that

$$n^{a/6}\left(\frac{Z_{n,\lfloor n^a \rfloor}}{\sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow{d} F_{\text{TW}} \quad \text{as} \quad n \to \infty.$$

Thus, to show

$$n^{a/6}\left(\frac{L_{n,\lfloor n^a \rfloor} - Cn}{\sqrt{\lambda\sigma^2 n}} - 2\sqrt{n^a}\right) \xrightarrow{d} F_{\text{TW}} \quad \text{as} \quad n \to \infty$$

it remains to prove

$$\frac{\sigma^{-1} S_{n,\lfloor n^a \rfloor} - Z_{\lambda n,\lfloor n^a \rfloor}}{n^{1/2-a/6}} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Coupling with Brownian motion

The difference between $\sigma^{-1} S_{n,m}$ and $Z_{\lambda n,m}$ can be bounded by

$$|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \leq 2 \sum_{j=1}^{m} U_n^{(j)} + 2 \sum_{j=1}^{m} V_n^{(j)}$$

where

$$U_n^{(j)} := \max_{0 \leq i \leq n} \Big| \sum_{k=1}^{i} \chi_k^{(j)} - B_i^{(j)} \Big|, \qquad V_n^{(j)} := \sup_{0 \leq s \leq n} \Big| B^{(j)}_{\Phi^{(j)}(s)} - B^{(j)}_{\lambda s} \Big|.$$

Using Komlós-Major-Tusnády strong approximation result we construct jointly the RW's and BM's such that $\frac{1}{n^{1/2-a/6}} \sum_{j=1}^{\lfloor n^a \rfloor} U_n^{(j)} \to 0$.
To show $\frac{1}{n^{1/2-a/6}} \sum_{j=1}^{\lfloor n^a \rfloor} V_n^{(j)} \to 0$ we used a version of the Baum-Katz theorem for the counting process.

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  - Directed Random Graph on $\mathbb{Z} \times \mathbb{Z}$

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Thank you for your attention!