A Brief Introduction to McKean-Vlasov Processes
and non-linear diffusions in general

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Young Women in Probability
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Roughly speaking, **McKean-Vlasov processes** or McKean-Vlasov diffusions are stochastic processes which can be described by SDEs of the form

\[
\begin{cases}
    dX_t = \int \alpha(X_t, u)\mu_t(du)dB_t + \int \beta(X_t, u)\mu_t(du)dt, \\
    \mu_t = \mathcal{L}(X_t),
\end{cases}
\]

\[X_0 \text{ given}\]  \hspace{1cm} (1)

where $B$ is a standard $d$-dimensional Brownian motion and $\mathcal{L}(X_t)$ denotes the marginal distribution of the process $X$ at the time $t$.

In general one can think of processes which satisfy SDEs of the following form

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\[X_0 \text{ given}\]  \hspace{1cm} (2)

These processes are called **non-linear diffusions**.
What is a McKean Vlasov process?

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A little bit of history

The story of these processes started with a stochastic toy model for the Vlasov equation of plasma proposed by Mark Kac in his paper "Foundations of kinetic theory (1956)".
In 1966 Henry P. McKean published his seminal paper "A class of Markov processes associated with non-linear parabolic equations".
Why these processes are interesting?

- **Theoretical interest**
  - Existence, Uniqueness and Properties
  - Mean Fields
  - Stochastic Control

- **Connection with non-linear parabolic PDEs**
  - Vlasov equation of plasma
  - Granular media equation

- **Applications in several areas**
  - Physics
  - Finance
  - Social Interactions
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For the rest of this talk we are going to assume that the diffusion coefficient is constant.

Consider non-linear SDE

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    dX_t &= \sqrt{2} dB_t + \int \beta(X_t, u) \mu_t(du) dt, \\
    \mu_t &= \mathcal{L}(X_t),
\end{align*} \tag{3} \]

where \( \beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) is bounded and Lipschitz continuous.
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Family of generators of the form

\[ L_t = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=1}^{d} \int \beta_i(x, y) \mu_t(dy) \frac{\partial}{\partial x_i}, \]

for all \( t \geq 0 \).

Martingale Formulation

PDE of the form

\[ \frac{\partial u}{\partial t}(t, x) = L_t u(t, x), \quad t > 0, \]

\[ u(0, x) = u_0. \]
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\]
A natural way of associating a particle system is to consider one with mean field interaction.

For each $N \in \mathbb{N}$ consider the particle system

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\begin{cases}
    dX^{i,N}_t = \sqrt{2} dB^i_t + \int \beta(X^{i,N}_t, y) \prod_{j=1}^N(dy) \, dt & i = 1, \ldots, N \\
    X^{i,N}_0 = X_0,
\end{cases}
$$

where

$$
\prod^N_t = \frac{1}{N} \sum_{j=1}^N \delta_{X^{j,N}_t}(dx)
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Clearly this can be written as follows

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Observe that (4) has a unique solution for every $N \in \mathbb{N}$.
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  dX_{t}^{i,N} &= \sqrt{2}dB_{t}^{i} + \int \beta(X_{t}^{i,N}, y)\Pi_{t}^{N}(dy)dt \\
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Observe that (4) has a unique solution for every $N \in \mathbb{N}$.
Properties

- Existence and Uniqueness
- Moment Control
- Behaviour at Infinity
  - Existence of a stationary distribution
  - Uniqueness of the stationary distribution
  - Speed of convergence towards the invariant distribution

- All these properties depend on the assumptions on the coefficient $\beta$!
  - Bounded and Lipschitz continuous
  - Bounded and Locally Lipschitz
  - What about unbounded coefficients?
  - Linear growth, Polynomial growth?
  - We require additional assumptions!
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Consider equations of the form

\[
\begin{align*}
\frac{dX_t}{dt} &= \sqrt{2} dB_t - [\nabla V(X_t) + \nabla W \ast \mu_t(X_t)]dt, \\
\mu_t &= \mathcal{L}(X_t),
\end{align*}
\]

where \( \ast \) denotes the convolution operator.

Provided some regularities on \( V \) and \( W \) the existence of these processes can be proved.

Moreover, it is not difficult to prove that the laws \( \mu_t, t \geq 0 \) are absolutely continuous and their densities \( u_t, t \geq 0 \) satisfy the so-called granular media equation

\[
\frac{\partial u}{\partial t} = \nabla \cdot \left[ \nabla u + u \nabla V + u(\nabla W \ast u) \right].
\]
Some work done on this case

- 1998 - Benachour et al. studied equation (5) with $V = 0$ in the one-dimensional case.
- 2001 - Malrieu studied equation (5) by using a particle system and propagation of chaos approach.
- 2008 - Herrmann et al. generalised Benachour et al. results to the multidimensional case.
- 2008 - Cattiaux et al. generalised Malrieu’s work.
- 2014 - Pierre del Moral and Tugaut proved uniform propagation of chaos for processes of the form (5) with $V = 0$. 
Theorem

Let $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a function satisfying assumptions (I)- (III) and $\xi$ a probability measure which belongs to $\mathcal{P}_q$ with $q = \max\{m + m_1 + 1, m_2 + 1\}$. Then there exists a unique strong solution to the non-linear stochastic differential equation

$$
\begin{aligned}
&dX_t = \sqrt{2}dB_t + \int \beta(X_t, u)\mu_t(du)dt, \\
&\mu_t = \mathcal{L}(X_t).
\end{aligned}
$$

Moreover, we have

$$
\sup_{0 \leq t \leq T} \mathbb{E}[|X_t|^q] < \infty,
$$

for all $T > 0$. 

Our approach consist in the application of a fixed-point argument in an appropriate space of curves of probability measures.

It was inspired by the work of V. Kolokoltsov [2].

Assumptions (I) and (II) are more or less standard and easy to prove.

Assumption (III) might be difficult to check though.

It is possible to extended this approach to more general non-linear diffusions.

Work in progress

- Behaviour at infinity
- Other kinds of non-linear diffusions
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Thank you very much for listening!

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