A General Uniqueness Criterion for Gibbs Measures with Non-compact Spins and Some Applications

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The Setting

In this presentation we will focus on a joint work with Yu. Kondratiev, Ju. Kozicki and T. Pasurek. The main theorem we present is a refinement of a result obtained by Dobrushin and Pechersky in 1982.

The setting:

- *an infinite countable, connected, simple graph* $\mathbb{G} = (\mathbb{E}, \mathbb{V})$ *with finite maximum degree* n *and chromatic number* m;
- *a standard Borel space* $(\Xi, \mathcal{E})$;
- *the product space* $(X, \mathcal{F}) := (\Xi, \mathcal{E})^{\mathbb{V}}$;
- *a family* $\pi = (\pi_l^x)_{l,x}$ *s.t.* $x \mapsto \pi_l^x(A)$ *is measurable* $\forall A \in \mathcal{E}$;
- *the set* $\mathcal{M}(\pi)$ *of probability measures consistent with* π

$$\mu(f) = \int_X \left(\int_\Xi f(z_l \times x_{l^c}) \pi_l^x(dz_l) \right) \mu(dx).$$

The Total Variation Distance

For $\pi, \pi' \in \mathcal{P}(\Xi)$ define

$$d(\pi, \pi') := \inf_{\rho \in \mathcal{C}(\pi, \pi')} \int_{\Xi^2} \mathbb{1}_{\neq}(\xi, \eta) \rho(d\xi, d\eta),$$

where $\mathcal{C}(\pi, \pi')$ is the set of all couplings of π and π'.
In our case, there exists $\rho_l^{x,y}$ such that

$$d(\pi_l^x, \pi_l^y) = \int_{\Xi^2} \mathbb{1}_{\neq}(\xi, \eta) \rho_l^{x,y}(d\xi, d\eta),$$

Contraction Condition

We denote by $\Pi_1(h, K, \kappa)$ the class of all families π such that

$$d(\pi_l^x, \pi_l^y) \leq \sum_{l' \in \partial l} \kappa_{ll'} \mathbb{1}_{\neq}(x_{l'}, y_{l'}), \ \forall l \in \mathbb{V} \text{ and } x, y \in X(h, K),$$

where $\kappa = (\kappa_{ll'})_{l,l' \in \mathbb{V}}$ has positive entries and null diagonal such that

$$\bar{\kappa} := \sup_{l \in \mathbb{V}} \sum_{l' \in \partial l} \kappa_{ll'} < 1,$$

and for a constant $K > 0$, $l \in \mathbb{V}$ and a measurable function $h : \Xi \to \mathbb{R}_+ := [0, +\infty)$, we set

$$X(h, K) = \{x \in X : h(x_l) \leq K \text{ for all } l \in \mathbb{V}\}.$$

Integrability Condition

Denote by $\Pi_2(h, C, c)$ the class of families π such that

$$\pi_l^x(h) \leq C + \sum_{l' \in \partial l} c_{ll'} h(x_{l'}), \forall l \in \mathbb{V} \text{ and } x \in X,$$

where $C > 0$ and $c = (c_{ll'})_{l,l' \in \mathbb{V}}$ has positive entries and null diagonal such that

$$\bar{c} := \sup_{l \in \mathbb{V}} \sum_{l' \in \partial l} c_{ll'} < 1/n^m.$$

We introduce a new set of measures $\mathcal{M}^t(\pi)$ as the set of measures $\mu \in \mathcal{M}(\pi)$ for which

$$\sup_l \int_X h(x_l) \mu(dx) < \infty.$$

The Uniqueness Result

Theorem

There exists $K_0 > 0$ *dependent only on* n, m, C *such that, for all* $K > K_0$ *and any* κ *such that* $\bar{\kappa} < 1$*, for each* $\pi \in \Pi_1(h, K, \kappa) \cap \Pi_2(h, C, c)$*, the set* $\mathcal{M}^t(\pi)$ *is a singleton at most.*

Idea of the Proof

The proof of the theorem follows immediately from

Lemma

Let $\mu_1, \mu_2 \in \mathcal{M}^t(\pi)$ *and* $\nu \in \mathcal{C}(\mu_1, \mu_2)$ *such that*

$$\int_X \int_X \mathbb{1}_{\neq}(x_l, y_l)\nu(dx, dy) = 0, \forall l \in \mathbb{V}.$$

Then $\mu_1 = \mu_2$.

Auxiliary Functionals

For $l \in \mathbb{V}$ and $(x^1, x^2) \in X^2$, set

$$I_l(x^1, x^2) := \mathbb{1}_{\neq}(x_l^1, x_l^2)$$

and

$$\gamma(\nu) := \sup_{l \in \mathbb{V}} \nu(I_l)$$

for $\nu \in \mathcal{C}(\mu_1, \mu_2)$. In order to control the behaviour of $\gamma(\nu)$ one needs to also introduce

$$\lambda(\nu) := \max_{i=1,2} \sup_{l,l' \in \mathbb{V}} \nu(I_l H_{l'}^i),$$

where

$$H_l^i(x^1, x^2) := h(x_l^i), \; i = 1, 2.$$

The Reconstruction Transformation

We start from an arbitrary coupling $\nu_0 \in \mathcal{C}(\mu_1, \mu_2)$ and apply to it succesively transformations R_l for each $l \in \mathbb{V}$, which will yield a coupling ν_* with $\gamma(\nu_*) = 0$.

$$(R_l \nu)(f) = \int_{X^2} \left(\int_{\Xi^2} f(\xi \times x_{l^c}, \eta \times y_{l^c}) \rho_l^{x,y}(d\xi, d\eta) \right) \nu(dx, dy),$$

It is easy to see that

$$(R_l \nu)(I_{l_1}) = \nu(I_{l_1}), \qquad (R_l \nu)(I_{l_1} H^i_{l_2}) = \nu(I_{l_1} H^i_{l_2}) \quad \text{for } l \neq l_1,\ l \neq l_2.$$

Reconstruction in One Site

Lemma

For an arbitrary $\nu \in \mathcal{P}(X^2)$, $l, l_1 \in \mathbb{V}$, with $l \neq l_1$ and $i = 1, 2$ the following estimates hold

$$(R_l\nu)(I_l H^i_{l_1}) \leq \sum_{l_2 \in \partial l} \nu(I_{l_2} H^i_{l_1}),$$

$$(R_l\nu)(I_{l_1} H^i_l) \leq C\nu(I_{l_1}) + \sum_{l_2 \in \partial l} c_{ll_2}\nu(I_{l_1} H^i_{l_2}),$$

$$(R_l\nu)(I_l H^i_l) \leq C \sum_{l_1 \in \partial l} \nu(I_{l_1}) + \sum_{l_1, l_2 \in \partial l} c_{ll_2}\nu(I_{l_1} H^i_{l_2}),$$

$$(R_l\nu)(I_l) \leq \sum_{l' \in \partial l} \kappa_{ll'}\nu(I_{l'}) + K^{-1} \sum_{i=1,2} \sum_{l_1, l_2 \in \partial l} \nu(I_{l_1} H^i_{l_2}).$$

The Dobrushin-Pechersky Matrix

By applying the reconstruction transformation once at every site $l \in \mathbb{V}$ one obtains a coupling $\nu_1 \in \mathcal{C}(\mu_1, \mu_2)$ such that

$$\begin{pmatrix} \gamma(\nu_1) \\ \lambda(\nu_1) \end{pmatrix} \leq M(K) \begin{pmatrix} \gamma(\nu_0) \\ \lambda(\nu_0) \end{pmatrix},$$

where

$$M(K) = \begin{pmatrix} \bar{\kappa} + P(K^{-1}) & Q(K^{-1}) \\ nC\frac{n^m - 1}{n-1} + P(K^{-1}) & n^m \bar{c} + Q(K^{-1}) \end{pmatrix},$$

where P, Q are polynomial functions of order m of K^{-1} with non-negative coefficients depending on n, C, and m and null free coefficients.

Decay of Correlations of Gibbs measures

Theorem

Let $\mu \in \mathcal{M}^t(\pi)$, $\Lambda, \tilde{\Lambda} \Subset \mathbb{V}$ disjoint and $f, g : X \to \mathbb{R}$ with $f \in \mathcal{B}(\Xi^{\tilde{\Lambda}})$ and $g \in \mathcal{B}(\Xi^{\Lambda})$ such that $|g(x)| \leq \sum_{l \in \Lambda} h(x_l)$. Then there exists a constant K_0 such that if $\pi \in \Pi_1(h, K_0, \bar{\kappa}) \cap \Pi_2(h, C, \bar{c})$ for some $\bar{\kappa}$, one can find a constant $D = D(n, m, C, \kappa) > 0$ for which one has

$$|\mu(fg) - \mu(f)\mu(g)| \leq D|\Lambda|^2 \exp\left(-\alpha d(\Lambda, \tilde{\Lambda})\right) \int_X |f(x)|\tilde{h}(x)\mu(dx)$$

whenever $\mu \in \mathcal{M}^t(\pi)$ and α and $\tilde{h}$ are defined as follows

$$\alpha := -\frac{1}{r} \log \left[\frac{1}{2}(\max(\{\kappa, \bar{c}n^m\}) + 1)\right],$$

$$\tilde{h}(y) := \sup_{l \in \Lambda} \max \left\{ \int_\Xi h(x_l)\mu_l(dx|y), \int_\Xi h(x_l)\mu_l(dx), 1 \right\}, \forall y \in X.$$

Classical Lattice Systems: The Model

IPS on a lattice $\mathbb{Z}^d$ are mathematical models of *anharmonic crystals:* infinite collection of spins $(x_l)_l \in X := (\mathbb{R}^N)^{\mathbb{Z}^d}$, governed by the following formal Hamiltonian

$$H(x) := \sum_l V_l(x_l) + \frac{1}{2}\sum_{l,l'} W_{ll'}(x_l, x_{l'}).$$

Assumptions on the interaction

(W) There exist constants $R \geq 2$, $I_W \geq 0$ and a symmetric matrix $\mathbb{J} = (J_{ll'})_{\mathbb{Z}^d \times \mathbb{Z}^d}$ with non-negative entries and zero diagonal, such that for all $x_l, x_{l'} \in \mathbb{R}^N$

$$|W_{ll'}(x_l, x_{l'})| \leq J_{ll'}(I_W + |x_l|^R + |x_{l'}|^R),\ l \neq l'.$$

(FR) The potential has finite range, i.e there exists $r > 0$ such that

$$J_{ll'} = 0 \ \text{ for any } |l' - l| > r.$$

Hence $W_{ll'} \equiv 0$ for $|l - l'| > r$. We write $l' \sim l$ if $|l' - l| \leq r$.

(V) For given $P \geq R$, there exist positive $A_1 \leq A_2$ and real $B_1 \leq B_2$ such that the estimate

$$A_1 |x_l|^P + B_1 \leq V_l(x_l) \leq A_2 |x_l|^P + B_2$$

holds for all $x_l \in \mathbb{R}^N$.

For $\Lambda \Subset \mathbb{Z}^d$ and $y \in X$ and $A \in \mathcal{B}(X)$, we can define

$$\pi_\Lambda(A|y) = \frac{1}{Z_\Lambda^\beta(y)} \int_{(\mathbb{R}^N)^\Lambda} \mathbb{1}_A(x_\Lambda \times y_{\Lambda^c}) \exp\left(-\beta H_\Lambda(x_\Lambda|y)\right) \times_{l\in\Lambda} dx_l,$$

$$Z_\Lambda^\beta = \int_{(\mathbb{R}^N)^\Lambda} \exp\left(-\beta H_\Lambda(x_\Lambda|y)\right) \times_{l\in\Lambda} dx_l$$

as being the *local Gibbs specification* of the model.
Here, the local energy is defined to be

$$H_\Lambda(x_\Lambda|y) = \sum_{l\in\Lambda} V_l(x_l) + \frac{1}{2} \sum_{l\sim l':l,l'\in\Lambda} W(x_l, x_{l'}) + \sum_{l\sim l':l\in\Lambda,l'\in\Lambda^c} W(x_l, y_{l'}),$$

Tempered configurations

We introduce the set of *tempered configurations*

$$X^t := \cup_{p>d} X_p = \{x \in X | \exists p = p(x) : ||x||_p < \infty\},$$

where

$$X_p := \left\{ x \in X \middle| ||x||_p := \left[\sum_l (1+|l|)^{-p} |x_l|^R \right]^{1/R} < \infty \right\}, \quad p > d,$$

and also the set of *tempered Gibbs measures*

$$\mathcal{M}^t := \{\mu \in \mathcal{M}(\pi) | \exists p = p(\mu) > d : \mu(X_p) = 1\}.$$

One-point estimate

The key technical result is the following *exponential bound* for the one-point kernels $\pi_l(dx|y)$.

Lemma

Assuming the assumptions on the interaction hold, for some positive τ there exists a corresponding $\Upsilon = \Upsilon(\beta, \tau) > 0$ such that for all $l \in \mathbb{Z}^d$ and $y \in X^t$

$$\int_X \exp\{\beta\tau|x_l|^R\}\pi_l(dx|y) \leq \exp\left\{\beta\left(\Upsilon + \sum_{l' \neq l} J_{ll'}|y_{l'}|^R\right)\right\}.$$

Classical Lattice Systems: High Temperature Case

Assume (V), (W) and (FR).

Theorem (Uniqueness by small interaction)

For every $\beta_0 > 0$ one finds $\mathcal{J} := \mathcal{J}(\beta_0) > 0$, such that the set $\mathcal{M}^t(\pi)$ is a singleton at all values of $\beta \leq \beta_0$ and $||J||_0 \leq \mathcal{J}$.

Theorem (Uniqueness by high temperature)

One finds β_0 such that, for any $\beta \leq \beta_0$, the set $\mathcal{M}^t(\pi)$ is a singleton.

Classical Lattice Systems: Low Temperature Case

Strengthening the conditions on the interaction potentials, one can get

Theorem (Uniqueness by small interaction)

For every $\beta^0 > 0$ one finds $\mathcal{J} := \mathcal{J}(\beta^0)$ such that the set $\mathcal{M}^t(\pi)$ is a singleton at all values $\beta \geq \beta^0$ and $||J||_0 \leq \mathcal{J}$.

Theorem (Uniqueness by low temperature)

For each $\beta^0 > 0$ and $\mathcal{J}_0 <$some given constant one finds a proper $\zeta_0 := \zeta_0(\beta^0, \mathcal{J}_0) > 0$ such that the corresponding set $\mathcal{M}^t(\pi)$ is a singleton at all values of $\beta \geq \beta^0$ and $||J||_0 \leq \mathcal{J}_0$ related by

$$\beta^{1-R/2}||J||_0 =: \zeta \leq \zeta_0.$$

Further Applications

The main result can also be applied to

- Gibbs measures in continuum,
- Gibbs measures on the cone of discrete measures,
- Gibbs measures on marked configuration spaces.

Strategy of proof for these models: show equivalence of these models to new ones, lying on the lattice to which one can apply the uniqueness criterion.

Thank you for your attention!