Aspects of the dimer model, spanning trees and the Ising model

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Young Women in Probability 2014, Bonn

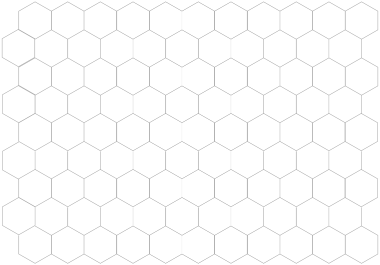
Introduction

Statistical mechanics

Understand the macroscopic behavior of a physics system based on a model describing interactions between microscopic components

Model:

- Structure is represented by a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, finite.



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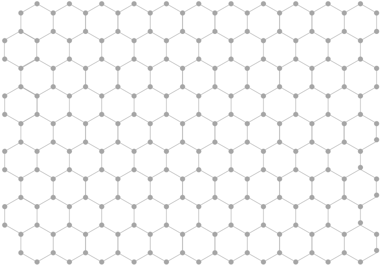

Statistical mechanics

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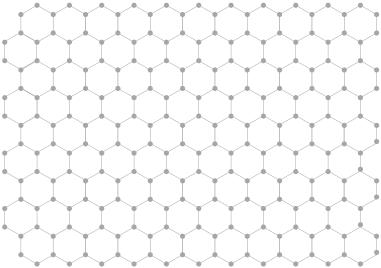
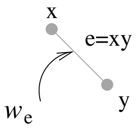


- Set of configurations on G: $\mathcal{C}(\mathsf{G})$.

Statistical mechanics

- Parameter: intensity of interactions between microscopic components and external temperature.

  Positive weight function $w = (w_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$ on the edges.



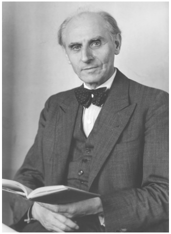
- To a configuration C, is assigned an energy $\mathcal{E}_w(\mathsf{C})$.
- Boltzmann probability on configurations:

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

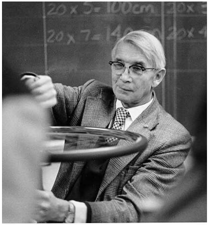
where $Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.

The Ising model

Model of ferromagnetism, mixture of two materials



Wilhelm Lenz (1888-1957)



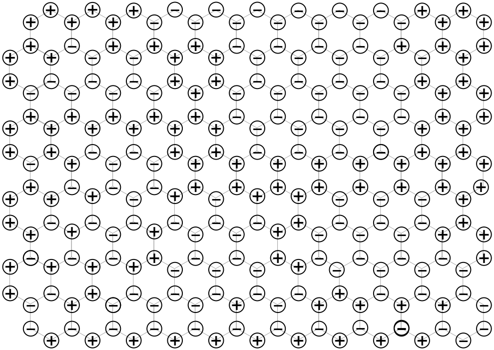
Ernst Ising (1900-1998)

- Graph $G = (V, E)$.
- A spin configuration σ assigns to every vertex x of the graph G a spin $\sigma_x \in \{-1, 1\}$.

$$\Rightarrow \mathcal{C}(G) = \{-1, 1\}^V = \text{set of spin configurations.}$$

The Ising model

- A spin configuration

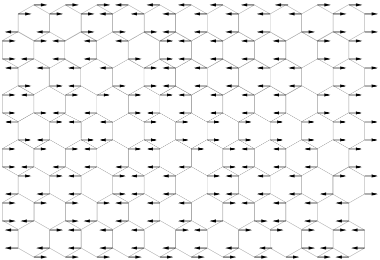


The Ising model

- A spin configuration / two interpretations.

Magnetic moments:

$+1/\rightarrow$, $-1/\leftarrow$



The Ising model

- A spin configuration / two interpretations.

Magnetic moments:
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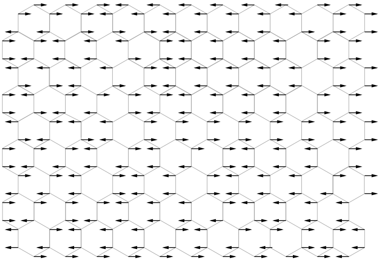
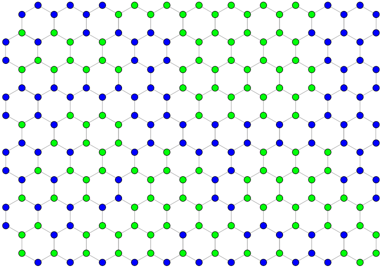


Mixture of two materials:
+1/●, -1/●.



The Ising model

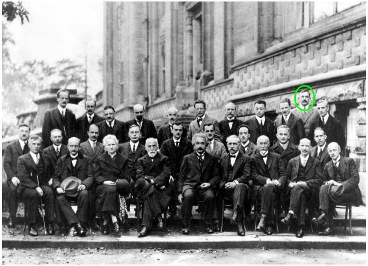
- Positive weight function: coupling constants $J = (J_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Energy of a spin configuration: $\mathcal{E}_J(\sigma) = -\sum_{\mathsf{e}=\mathsf{xy} \in \mathsf{E}} J_{\mathsf{xy}} \sigma_{\mathsf{x}} \sigma_{\mathsf{y}}$.
- Ising Boltzmann probability:

$$\forall \, \sigma \in \{-1, 1\}^{\mathsf{V}}, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{e^{-\mathcal{E}_J(\sigma)}}{Z_{\text{Ising}}(\mathsf{G}, J)}.$$

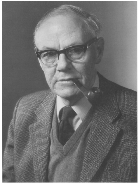
  - Two neighboring spins $\sigma_{\mathsf{x}}, \sigma_{\mathsf{y}}$ tend to align.
  - Highest is the coupling J_{xy}, stronger is this tendency.

The dimer model

Adsorption of di-atomic molecules on the surface of a crystal



Sir Ralph H. Fowler (1889-1944)
Solvay conference 1927.



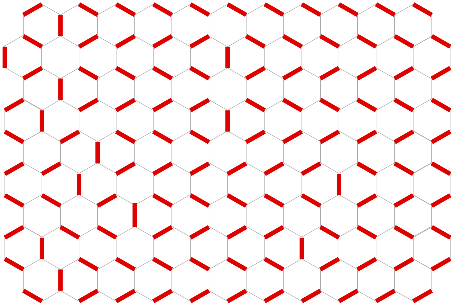
George S. Rushbrooke (1915-1995)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A dimer configuration or perfect coupling: subset of edges such that each vertex is incident to exactly one edge.

$$\Rightarrow \mathcal{C}(\mathsf{G}) = \mathcal{M}(\mathsf{G}) = \text{set of dimer configurations.}$$

The dimer model

- A dimer configuration.



The dimer model

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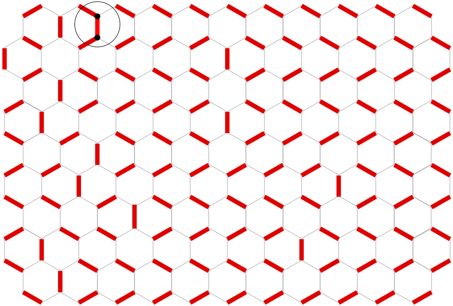

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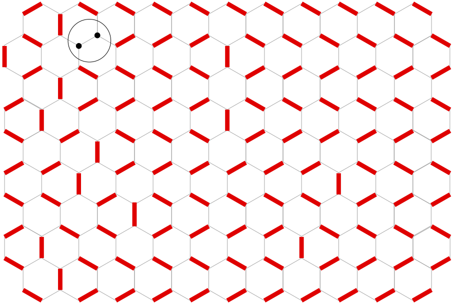

The dimer model

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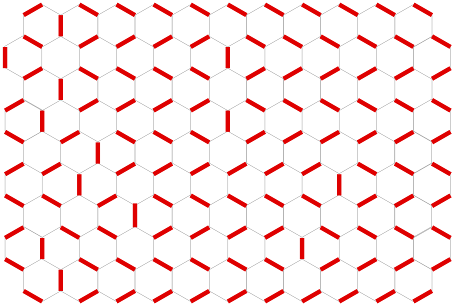


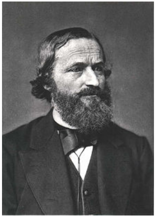
- Positive weight function: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Energy of a configuration M: $\mathcal{E}_\nu(\mathsf{M}) = -\sum_{\mathsf{e} \in \mathsf{M}} \log \nu_{\mathsf{e}}$.
- Dimer Boltzmann measure:

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\text{dimère}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\text{dimer}}(\mathsf{G}, \nu)}.$$

  - Highest is the weight, more likely is an edge to be present.

Spanning trees

Related to electrical networks



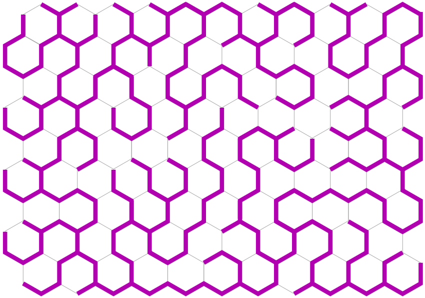
Gustav Kirchhoff (1824-1887)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A spanning tree: connected subset of edges spanning vertices of the graph, containing no cycle.

$$\Rightarrow \mathcal{C}(\mathsf{G}) = \mathcal{T}(\mathsf{G}) = \text{subset of spanning trees.}$$

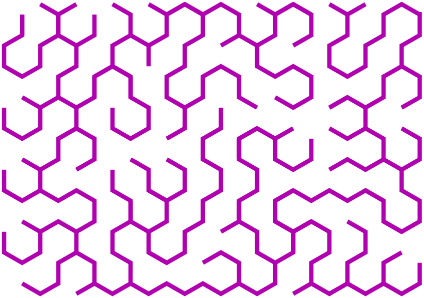
Spanning trees

- A spanning tree



Spanning trees

- A spanning tree



Spanning trees

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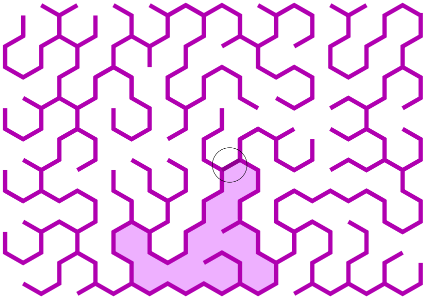

Spanning trees

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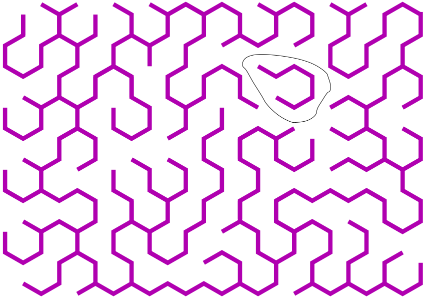
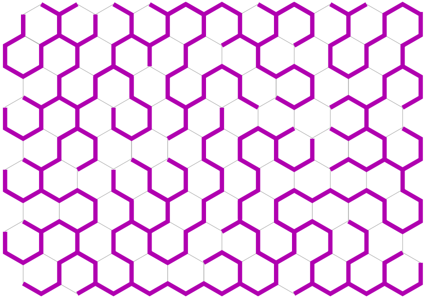


Spanning trees

- A spanning tree



- Positive weight function: $\rho = (\rho_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$.
- Energy of a tree T: $\mathcal{E}_\rho(\mathsf{T}) = -\sum_{\mathsf{e}\in\mathsf{T}} \log \rho_{\mathsf{e}}$.
- Spanning trees Boltzmann measure:

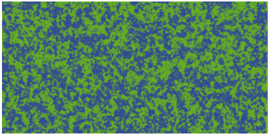
$$\forall\, \mathsf{T} \in \mathcal{T}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{tree}}(\mathsf{T}) = \frac{\prod_{\mathsf{e}\in\mathsf{T}} \rho_{\mathsf{e}}}{Z_{\mathrm{tree}}(\mathsf{G},\rho)}.$$

  - Edges with high weights are more likely to be present.

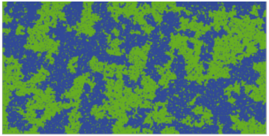
Macroscopic behavior

Let edge-lengths tend to zero
Look at a " typical configuration ".

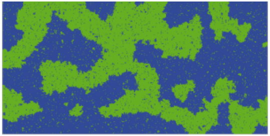
- Ising model (Simulation by R. Cerf)



J small



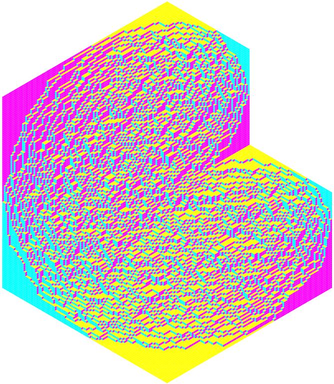
J critical



J large

Macroscopic behavior

- Dimer model (Simulation by R. Kenyon)



Macroscopic behavior

- Identification of the phase transition.
- Understanding the sub and super critical models.
- Understanding the critical model (at the phase transition):
  - Universality and phase transition.
  - Conjectures : Nienhuis, Cardy, Duplantier ...

    Proofs : G. Lawler, O. Schramm, W. Werner, D. Chelkak, S. Smirnov ...

Exactly solvable models

- Boltzmann measure on configurations:

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

  where $Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.
- The model is exactly solvable if there exists an exact, explicit expression for the partition function.
- Three exactly solvable models:
  - Ising-2d: Onsager (1944) - Fisher (1966): $Z_{\text{Ising}}(\mathsf{G}, J) = \sqrt{\det(\mathsf{K}_{\mathsf{G}^{\mathsf{F}}})}$.
  - Dimer-2d: Kasteleyn, Temperley-Fisher (1961):

    $$\text{when } \mathsf{G} \text{ is bipartite, } Z_{\text{dimer}}(\mathsf{G}, \nu) = \det(\mathsf{K}).$$
  - Spanning trees: Kirchhoff (1848): $Z_{\text{tree}}(\mathsf{G}, \rho) = \det(\Delta^{(\mathsf{r})})$.

A glimpse at some founding results of exactly solvable models

Partition function of the dimer model defined on finite, planar, bipartite graphs

- Let $\mathsf{G} = (\mathsf{W} \cup \mathsf{B}, \mathsf{E})$ be a planar, finite, bipartite graph, with $|\mathsf{W}| = |\mathsf{B}| = n$.
- Consider a dimer model on G, with weight function ν on edges.
- Orientation of the edges: Kasteleyn orientation.
- Let K be the Kasteleyn matrix, defined by:

$$\forall\, \mathsf{w} \in \mathsf{W},\ \forall\, \mathsf{b} \in \mathsf{B},\quad \mathsf{K}_{\mathsf{w},\mathsf{b}} = \begin{cases} \nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b},\ \mathsf{w} \to \mathsf{b}, \\ -\nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b},\ \mathsf{b} \to \mathsf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

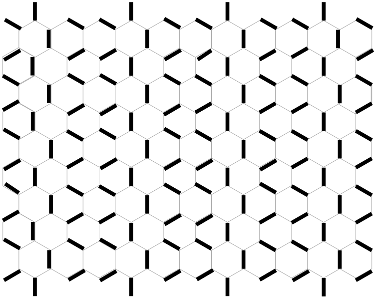
Theorem (Ka,Te-Fi)

If edges are oriented according to a Kasteleyn orientation, then:

$$Z_{\text{dimer}}(\mathsf{G}, \nu) = |\det(\mathsf{K})|.$$

Superimposition of dimer configurations

- Let M_1, M_2 be two dimer configurations of G, and $\mathsf{M}_1 \cup \mathsf{M}_2$ be their superimposition.



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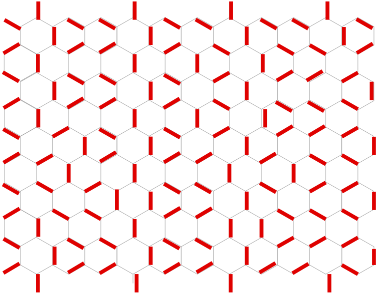

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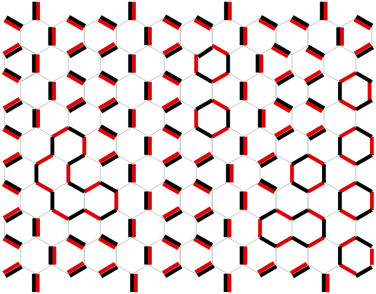

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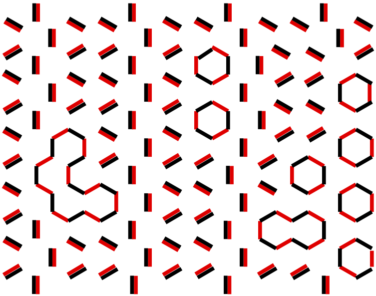


- $\mathsf{M}_1 \cup \mathsf{M}_2$ is a disjoint union of alternating cycles, where an alternating cycle consists of edges alternating between M_1 and M_2. An alternating cycle of length 2 is a doubled edge.

Partition function of spanning trees

- Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a finite graph.
- Consider spanning trees on G, with weight function ρ on edges.
- Let Δ be the Laplacian matrix defined by:

$$\forall\, \mathsf{x}, \mathsf{y} \in \mathsf{V}, \quad \Delta_{\mathsf{x},\mathsf{y}} = \begin{cases} \rho_{\mathsf{xy}} & \text{if } \mathsf{x} \neq \mathsf{y},\ \mathsf{x} \sim \mathsf{y}, \\ -\sum_{\mathsf{x}' \sim \mathsf{x}} \rho_{\mathsf{xx}'} & \text{if } \mathsf{x} = \mathsf{y} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Ki)

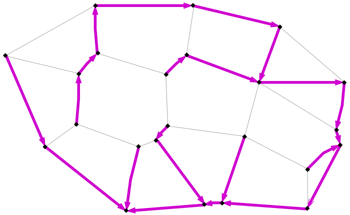
The spanning trees partition function is equal to

$$Z_{\text{tree}}(\mathsf{G}, \rho) = |\det(\Delta^{(\mathsf{r})})|,$$

where $\Delta^{(\mathsf{r})}$ is obtained from Δ by removing the line and the column corresponding to a given vertex r.

Temperley's bijection between spanning trees and the dimer model on the double graph

- Generalized form due to Kenyon-Propp-Wilson.
- Spanning tree of a graph G



Temperley's bijection between spanning trees and the dimer model on the double graph

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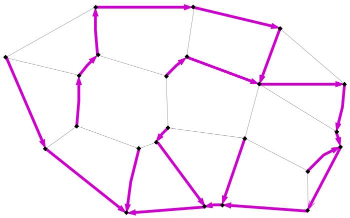
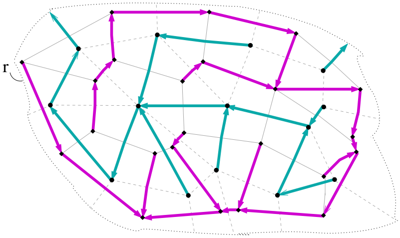
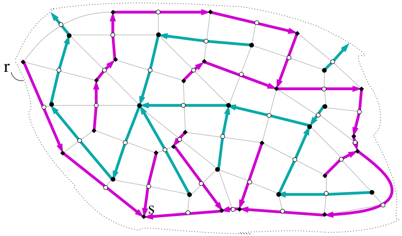
- Dual spanning tree of the dual graph G^*

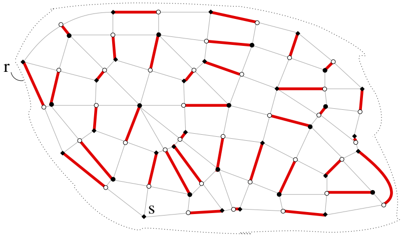




Temperley's bijection between spanning trees and the dimer model on the double graph

- Generalized form due to Kenyon-Propp-Wilson.
- Dimer configuration of the double graph G^{D}





Partition functions of the critical Ising model and critical spanning trees

Critical Ising model

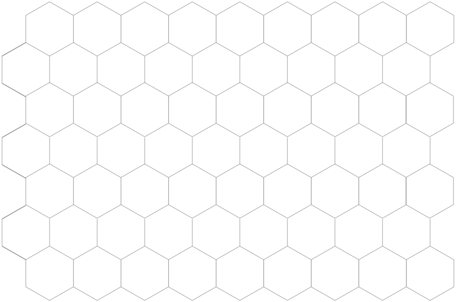
- If the graph G is $\mathbb{Z}^2$, Kramers et Wannier determine the critical coupling constants:

$$\forall\, e \in \mathsf{E}, \quad J_e = \frac{1}{2}\log(1+\sqrt{2}).$$

- Baxter generalizes the critical Ising model on $\mathbb{Z}^2$ to a large family of graphs: the isoradial graphs.

Isoradial graphs

- A graph G is isoradial if it is planar and can be embedded in the plane in such a way that all faces are inscribed in a circle of radius 1, and that the circumcenters are in the interior of the faces (Duffin-Mercat-Kenyon).



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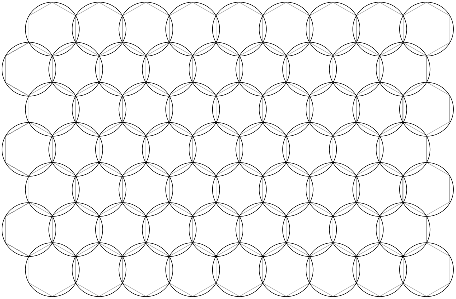

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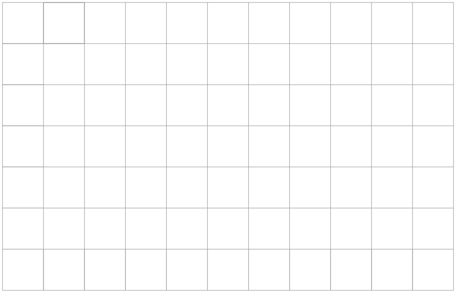

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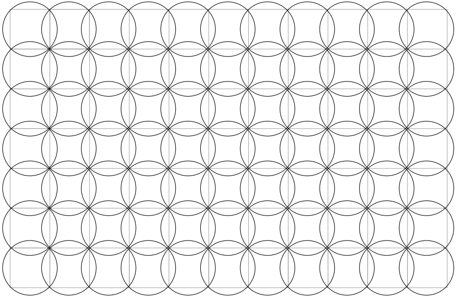

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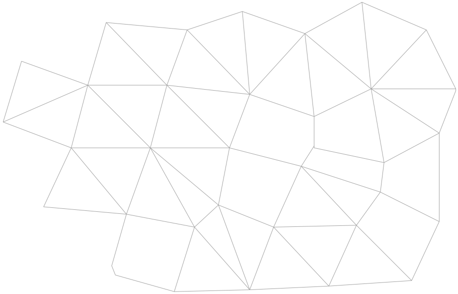

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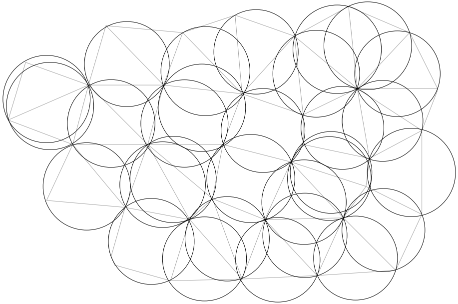
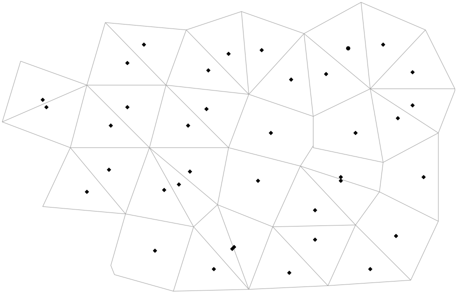


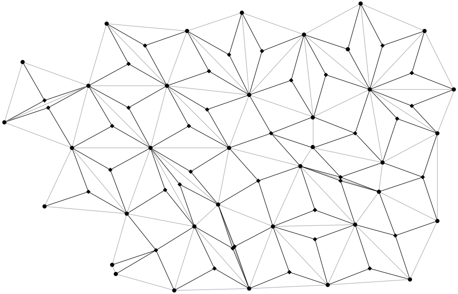
Associated rhombus graph, angles

- Take the circumcenters.



Associated rhombus graph, angles

- Join the circumcenters to the vertices of the graph G.
  $\Rightarrow$ Associated rhombus graph $\mathsf{G}^{\diamond}$.



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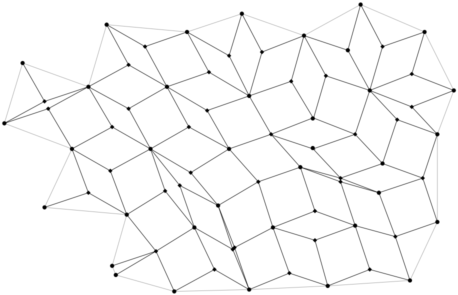
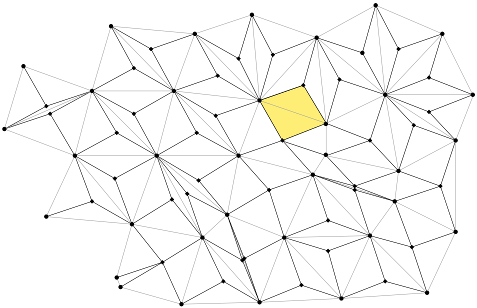
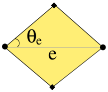
- Join the circumcenters to the vertices of the graph G.
  $\Rightarrow$ Associated rhombus graph $\mathsf{G}^{\diamond}$.



Associated rhombus graph, angles

- To every edge e is assigned the half-angle θ_e of the corresponding rhombus.





Z-invariant critical Ising model

- The Ising model is Z-invariant and critical [Ba] if the coupling constants are equal to:

$$\forall\, e \in \mathsf{E}, \quad J_e = \frac{1}{2}\log\left(\frac{1+\sin\theta_e}{\cos\theta_e}\right).$$

- Baxter determines them using:
  - Z-invariance (invariance under $\Delta - Y$ transformations),
  - generalized form of self-duality,
  - assumption of uniqueness of the critical point.
- Li and Duminil-Copin - Cimasoni, show that the Ising model is indeed critical.
- When $\mathsf{G} = \mathbb{Z}^2$, $\forall e \in \mathsf{E}$, $\theta_e = \frac{\pi}{4}$, implying that $J_e = \frac{1}{2}\log(1+\sqrt{2})$.

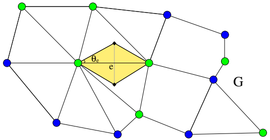
Z-invariant critical spanning trees

- Spanning trees are Z-invariant and critical [Ke] if G is isoradial and the weights are equal to:

$$\forall\, e \in \mathsf{E}, \quad \rho_e = \tan \theta_e.$$

Result

Critical Ising model on G



$$\forall\, e \in \mathsf{E}, \quad J_e = \frac{1}{2}\log\left(\frac{1+\sin\theta_e}{\cos\theta_e}\right)$$

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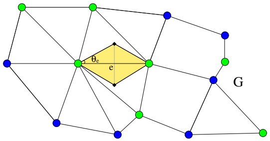
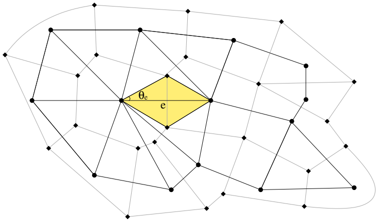


Critical spanning trees on $\bar{\mathsf{G}}$



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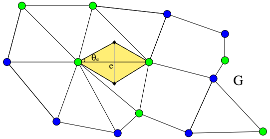

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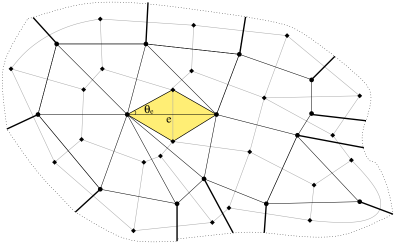

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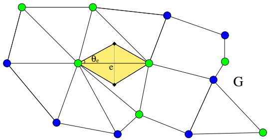

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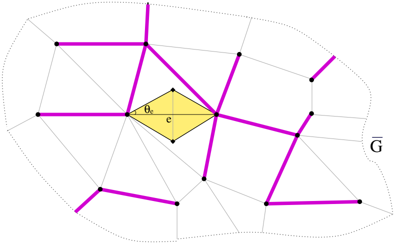
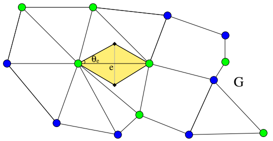


$$\forall\, e \in \mathsf{E}, \quad J_e = \frac{1}{2}\log\left(\frac{1+\sin\theta_e}{\cos\theta_e}\right)$$

$$\begin{cases} \forall\, e \in \mathsf{E}, \quad \rho_e = \tan\theta_e \\ \text{Boundary conditions } \bar{\mathsf{E}} \setminus \mathsf{E}. \end{cases}$$

Result

Critical Ising model on G



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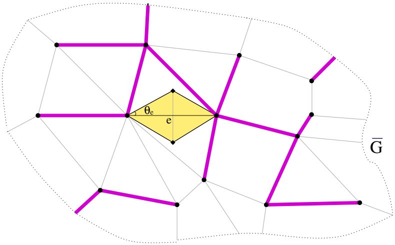

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Theorem

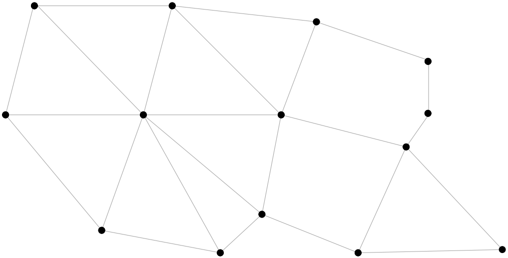
$$Z_{\text{Ising}}(\mathsf{G}, J)^2 = 2^{|\mathsf{V}|}\, |Z_{\text{tree}}(\bar{\mathsf{G}}, \rho)|.$$

Remarks

- Similar result when the graph G is embedded in the torus. The characteristic polynomial replaces the partition function:
  - Proof using Fisher's correspondence between the Ising model and the dimer model on a non-bipartite graph [B-dT], [dT].
  - Proof using Kac-Ward matrices [Ci].
- On the torus, no difficulty related to the boundary.
- Different explicit construction.

Proof 1. Towards a bipartite dimer model

- Construction of the bipartite graph G^{Q}



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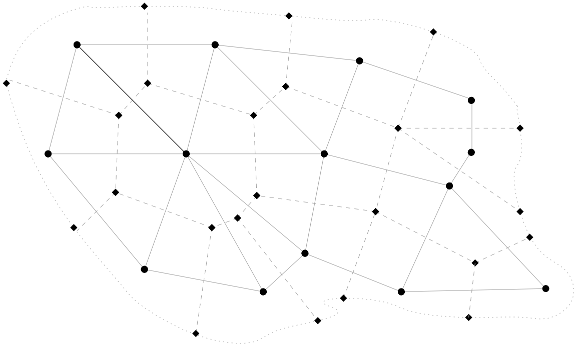

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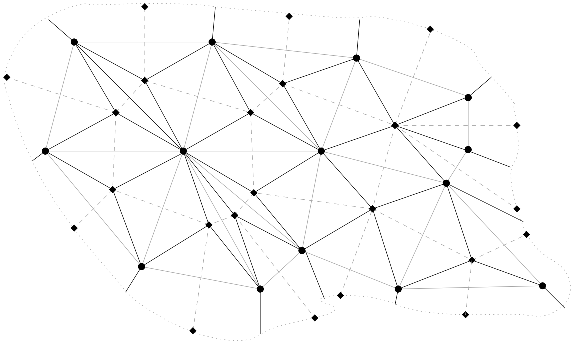

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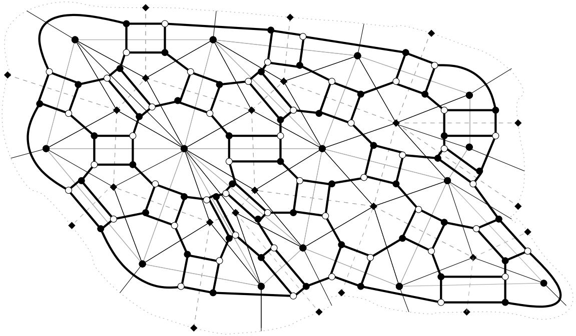

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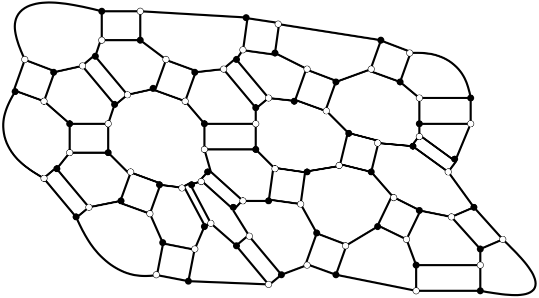

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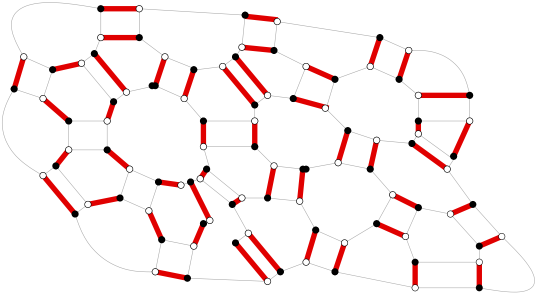
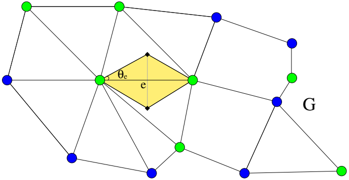


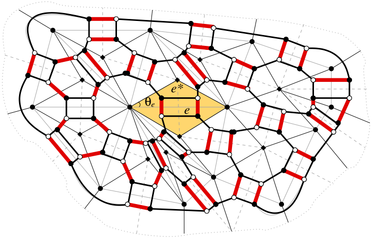
Proof 1. Towards a bipartite dimer model

Critical Ising model on G



$$\forall\, \mathsf{e} \in \mathsf{E}, \quad J_{\mathsf{e}} = \frac{1}{2}\log\left(\frac{1+\sin\theta_{\mathsf{e}}}{\cos\theta_{\mathsf{e}}}\right)$$

Dimer model on G^{Q}



$$\forall\, \mathsf{e} \in \mathsf{E}^{\mathrm{Q}}, \quad \nu_{\mathsf{e}} = \begin{cases} \sin\theta_e \text{ if } \mathsf{e} \,//\, e \\ \cos\theta_e \text{ if } \mathsf{e} \,//\, e^* \\ 1 \text{ otherwise.} \end{cases}$$

Theorem (Ni,Wu-Lin,Du,BdT)

$$Z_{\mathrm{Ising}}(\mathsf{G},J)^2 = 2^{|\mathsf{V}|}\left[\prod_{e\in\mathsf{E}}\cos^{-1}(\theta_e)\right] Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}},\nu).$$

Proof 2. Dimer partition function.

- Consider the bipartite graph $\mathsf{G}^{\mathrm{Q}} = (\mathsf{W}^{\mathrm{Q}} \cup \mathsf{B}^{\mathrm{Q}}, \mathsf{E}^{\mathrm{Q}})$, $|\mathsf{W}^{\mathrm{Q}}| = |\mathsf{B}^{\mathrm{Q}}| = k$.
- Orientation of the edges : Kasteleyn orientation.
- Let K be the matrix defined by:

$$\forall\, \mathsf{w} \in \mathsf{W}^{\mathrm{Q}},\ \forall\, \mathsf{b} \in \mathsf{B}^{\mathrm{Q}}, \quad \mathsf{K}_{\mathsf{w},\mathsf{b}} = \begin{cases} \nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b},\ \mathsf{w} \to \mathsf{b}, \\ -\nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b},\ \mathsf{b} \to \mathsf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

- $\det(\mathsf{K}) = \sum_{\sigma \in \mathcal{S}_k} \underbrace{\operatorname{sgn}(\sigma)}_{\pm 1} \quad \underbrace{\mathsf{K}_{\mathsf{w}_1,\mathsf{b}_{\sigma(1)}} \dots \mathsf{K}_{\mathsf{w}_k,\mathsf{b}_{\sigma(k)}}}_{\pm 1 \text{ contribution of dimer config.}}.$

Theorem (Ka,Te-Fi)

If edges are oriented according to a Kasteleyn orientation, then:

$$Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu) = |\det(\mathsf{K})|.$$

Proof 2. Dimer partition function.

- Consider the bipartite graph $\mathsf{G}^{\mathrm{Q}} = (\mathsf{W}^{\mathrm{Q}} \cup \mathsf{B}^{\mathrm{Q}}, \mathsf{E}^{\mathrm{Q}})$, $|\mathsf{W}^{\mathrm{Q}}| = |\mathsf{B}^{\mathrm{Q}}| = k$.
- Phases $(\phi_{\mathsf{wb}})_{\mathsf{wb}\in\mathsf{E}^{\mathrm{Q}}}$ assigned to edges : flat phasing [Ku].
- Let K be the matrix defined by:

$$\forall\, \mathsf{w} \in \mathsf{W}^{\mathrm{Q}},\ \forall\, \mathsf{b} \in \mathsf{B}^{\mathrm{Q}}, \quad \mathsf{K}_{\mathsf{w},\mathsf{b}} = \begin{cases} e^{i\phi_{\mathsf{wb}}}\ \nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b} \\ 0 & \text{otherwise.} \end{cases}$$

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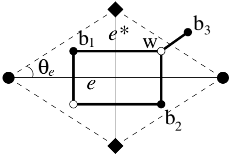
Theorem (Ku)

If the phasing of the edges is flat, then:

$$Z_{\text{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu) = |\det(\mathsf{K})|.$$

Proof 2. Dimer partition function.

- Choice of phases



$$\mathsf{K}_{\mathsf{w,b}} = \begin{cases} 1.\sin\theta_e & \text{if } \mathsf{e} \,//\, e \\ i.\cos\theta_e & \text{if } \mathsf{e} \,//\, e^* \\ e^{i(\frac{3\pi}{2}-\theta_e)}.1 & \text{if } \mathsf{e} \text{ external and in the interior} \\ e^{i[\frac{3\pi}{2}-(\theta_e+\theta^\partial)]}.1 & \text{if } \mathsf{e} \text{ external and on the boundary of } \mathsf{G}^{\mathrm{Q}}. \end{cases}$$

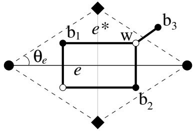
Lemma

The phasing of the edges is flat, implying that: $Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu) = |\det \mathsf{K}|$.

Proof 3. Kasteleyn matrix / Laplacian matrix

- Remark: for every external edge wb_3,

$$\mathsf{K}_{\mathsf{w},\mathsf{b}_3} = \begin{cases} -\mathsf{K}_{\mathsf{w},\mathsf{b}_1} - \mathsf{K}_{\mathsf{w},\mathsf{b}_2} & \text{if } \mathsf{wb}_3 \text{ is in the interior} \\ -\mathsf{K}_{\mathsf{w},\mathsf{b}_1} - \mathsf{K}_{\mathsf{w},\mathsf{b}_2} - ie^{-i\theta_e}(e^{-i\theta^\partial} - 1) & \text{otherwise.} \end{cases}$$



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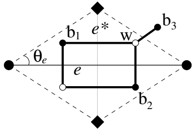
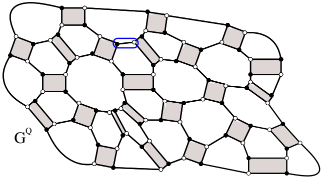
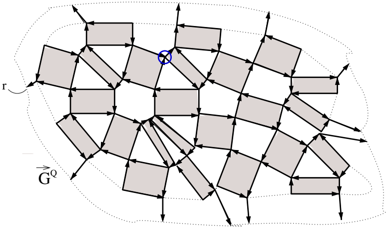


- Kasteleyn matrix / Laplacian matrix



$\{\mathsf{w} \equiv \mathsf{b}_3\}$

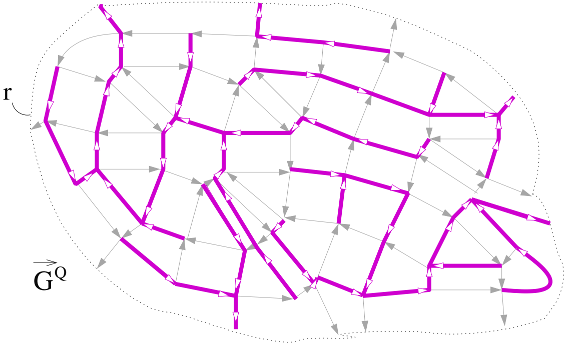


$(\mathsf{K}_{\mathsf{wb}})_{\mathsf{w}\in\mathsf{W}^Q,\, \mathsf{b}\in\mathsf{B}^Q}$ $\quad$ $(\Delta^{(\mathsf{r})}_{\mathsf{xy}})_{\mathsf{x},\,\mathsf{y}\in\vec{\mathsf{G}}^Q}$

Proof 3. Kasteleyn matrix / Laplacian matrix

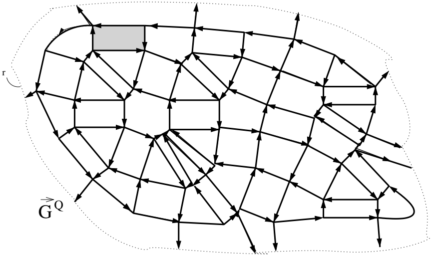
Corollary

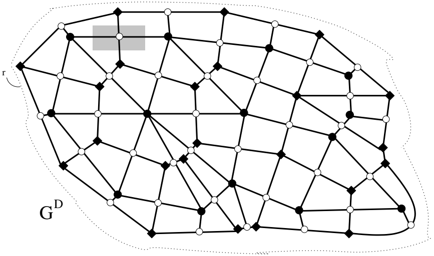
$$Z_{\text{dimer}}(\mathsf{G}^{\mathsf{Q}}, \nu) = |\det \Delta^{(\mathsf{r})}| = \left| \sum_{\mathsf{T} \in \mathcal{T}^{(\mathsf{r})}(\vec{\mathsf{G}}^{\mathsf{Q}})} \left(\prod_{\mathsf{e} \in \mathsf{T}} \rho_{\mathsf{e}} \right) \right| .$$



Proof 4. Spanning trees and dual spanning trees

- If T is a spanning tree of $\mathsf{G}^{\vec{Q}}$, then the dual configuration T^* consisting of the dual of the absent edges of T is a spanning tree of the dual graph $(\mathsf{G}^{\vec{Q}})^*$, and by extension of the double graph $\bar{\mathsf{G}}^{\mathrm{D}}$.





Bipartite version of the dual $(\mathsf{G}^{\vec{Q}})^*$ is the double graph of $\bar{\mathsf{G}}$, it is denoted $\bar{\mathsf{G}}^{\mathrm{D}}$.

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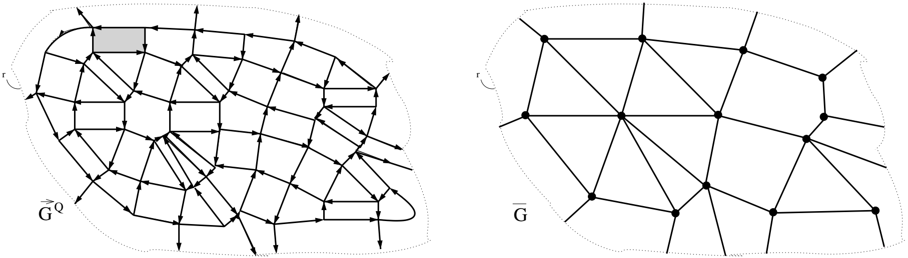

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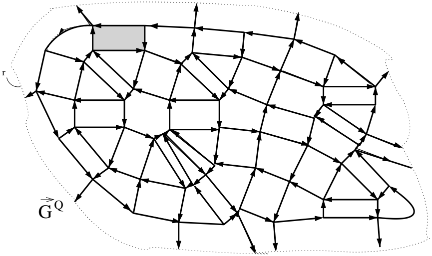

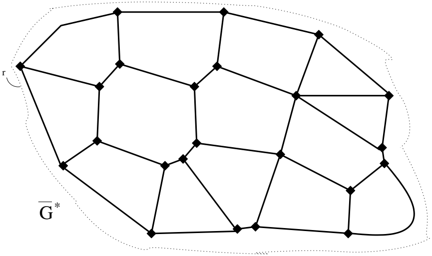

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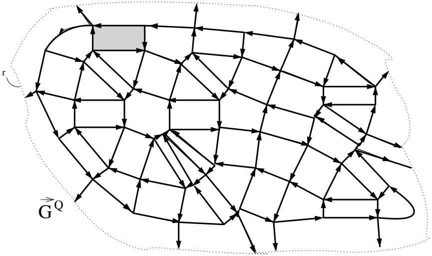

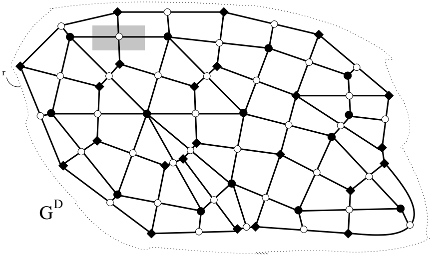

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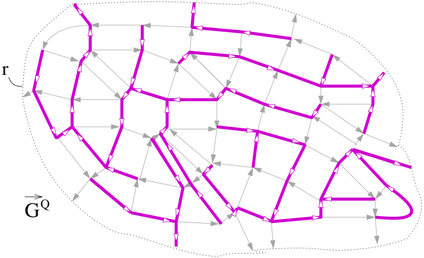

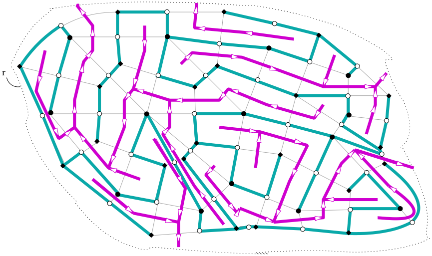

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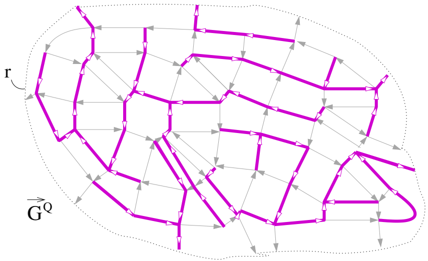
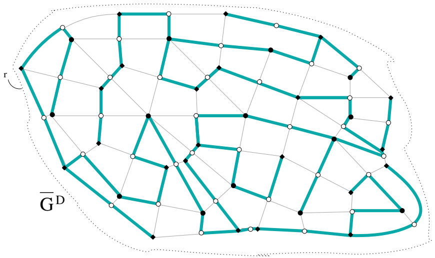
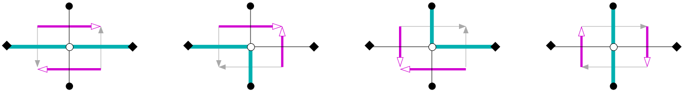




Proof 4. Characterizing dual spanning trees.

Lemma

A spanning tree of G^{D} is the dual of a spanning tree of $\vec{\mathsf{G}}^{\mathrm{Q}}$ if and only if the restriction to every white vertex (in the interior) is of the following form:

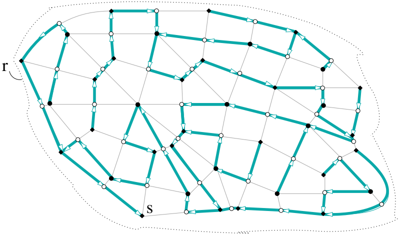


Let us denote (c) this condition, and $\mathcal{T}^{\mathsf{s}}_{(c)}(\mathsf{G}^{\mathrm{D}})$ the set of these spanning trees, oriented towards a vertex s of the boundary.

Proof 5. From dual spanning trees to the dimer model on the double graph

Proposition

Let T *be a spanning tree of* $\mathcal{T}^{\mathsf{s}}_{(c)}(\mathsf{G}^{\mathrm{D}})$*. Let* M_{T} *be the configuration consisting of edges exiting from black vertices, then* M_{T} *is a dimer configuration of* $\mathsf{G}^{\mathrm{D}}(\mathsf{s})$*.*



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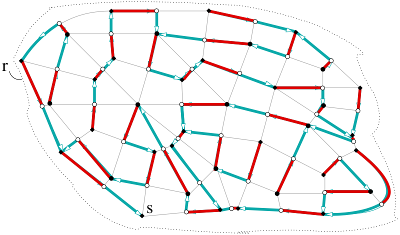

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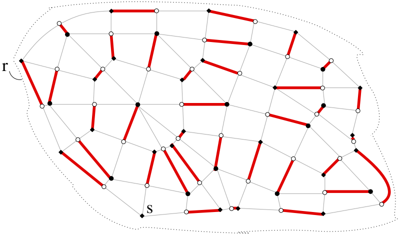
Proposition

Let T *be a spanning tree of* $\mathcal{T}^{\mathsf{s}}_{(c)}(\mathsf{G}^{\mathrm{D}})$. *Let* M_{T} *be the configuration consisting of edges exiting from black vertices, then* M_{T} *is a dimer configuration of* $\mathsf{G}^{\mathrm{D}}(\mathsf{s})$.



Thus, $\mathcal{T}^{\mathsf{s}}_{(c)}(\mathsf{G}^{\mathrm{D}}) = \bigsqcup_{\mathsf{M}\in\mathcal{M}(\mathsf{G}^{\mathrm{D}}(\mathsf{s}))} \mathcal{T}^{\mathsf{s}}_{c,\mathsf{M}}(\mathsf{G}^{\mathrm{D}})$

$$Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}},\nu) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}^{\mathrm{D}}(\mathsf{s}))} \rho(\mathcal{T}^{\mathsf{s}}_{c,\mathsf{M}}(\mathsf{G}^{\mathrm{D}})).$$

Proof 5. From dual spanning trees to the dimer model on the double graph

Proposition

Let M be a perfect matching of $\mathsf{G}^{\mathrm{D}}(\mathsf{s})$, then an edge configuration containing edges of M and satisfying condition (c) is the unoriented version of a spanning tree of $\mathcal{T}^{\mathsf{s}}_{c,\mathsf{M}}(\mathsf{G}^{\mathrm{D}})$.

As a consequence, we have:

$$\rho(\mathcal{T}^{\mathsf{s}}_{c,\mathsf{M}}(\mathsf{G}^{\mathrm{D}})) = \rho'(\mathsf{M}) = \prod_{\mathsf{e}\in\mathsf{M}} \rho'_{\mathsf{e}}.$$

$$Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}^{\mathrm{D}}(\mathsf{s}))} \prod_{\mathsf{e}\in\mathsf{M}} \rho'_{\mathsf{e}}.$$

Proof 6. Conclusion [K-P-W]

The generalized form of Temperley's bijection due to Kenyon-Propp-Wilson allows to conclude.