Limiting Spectral Distribution of Large Sample Covariance Matrices

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Introduction and Motivation

- Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^N$ be i.i.d centered random vectors with covariance matrix $\Sigma = \mathbb{E}(\mathbf{X}_1 \mathbf{X}_1^T) = \ldots = \mathbb{E}(\mathbf{X}_n \mathbf{X}_n^T)$.

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- For fixed N, the strong law of large numbers implies

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- What happens once $N := N_n \to \infty$ as $n \to \infty$?

Marčenko-Pastur Theorem

- Let $(X_{ij})_{i,j\geqslant 1}$ be a family of i.i.d. random variables such that $\mathbb{E}(X_{11}) = 0$ and $\mathrm{Var}(X_{11}) = \sigma^2$.
- Let $\mathbf{B}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{X}_k \mathbf{X}_k^T = \frac{1}{n}\mathcal{X}_n\mathcal{X}_n^T$ where

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- The empirical spectral measure of $\mathbf{B}_n$ is defined by

$$\mu_{\mathbf{B}_n} = \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_k}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $\mathbf{B}_n$.

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- We suppose that $c_n := \frac{N}{n} \xrightarrow[n\to+\infty]{} c \in (0, \infty)$.

Marčenko-Pastur Theorem

Theorem

For any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$*,*

$$\int f \, d\mu_{\mathbf{B}_n} \xrightarrow[n \to +\infty]{} \int f \, d\mu_c \quad a.s.$$

where μ_c *is the Marčenko-Pastur law*

$$\left(1 - \frac{1}{c}\right)_+ \delta_0 + \frac{1}{2\pi c \sigma^2 x} \sqrt{(b-x)(x-a)} \mathbf{1}_{[a,b]}(x) dx$$

with $.+ := \max(0, .)$*,* $a = \sigma^2(1 - \sqrt{c})^2$ *and* $b = \sigma^2(1 + \sqrt{c})^2$*.*

The Stieltjes Transform

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- For a sequence of measures $(\nu_n)_n$ on $\mathbb{R}$, we have

$$\left(\nu_n \xrightarrow[n\to\infty]{\mathcal{L}} \nu\right) \Leftrightarrow \left(\forall z \in \mathbb{C}_+,\; S_{\nu_n}(z) \xrightarrow[n\to\infty]{} S_\nu(z)\right).$$

Non linear Stationary Process

- $X = (X_k)_{k \in \mathbb{Z}}$ is said to be stationary if $\forall k \in \mathbb{Z}$ and $\forall \ell \in \mathbb{N}$,

$$(X_k, \ldots, X_{k+\ell}) \overset{\mathcal{D}}{\sim} (X_0, \ldots, X_\ell)$$

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$$X_k = g(\xi_k) \text{ with } \xi_k = (\ldots, \varepsilon_{k-1}, \varepsilon_k)$$

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Theorem: Banna and Merlevède (2013)

Suppose that $lim_{n\to\infty}\ N/n = c \in (0,\infty)$ and

$$\sum_{k\geq 0} \|P_0(X_k)\|_2 < \infty \text{ where } P_0(X_k) = \mathbb{E}(X_k|\xi_0) - \mathbb{E}(X_k|\xi_{-1})$$

Then with probability one $\mu_{\mathbf{B}_n}$converges weakly to a non-random probability measure whose Stieltjes transform $S = S(z)$ satisfies the equation

$$z = -\frac{1}{\underline{S}} + \frac{c}{2\pi}\int_0^{2\pi} \frac{1}{\underline{S} + \left(2\pi f(\lambda)\right)^{-1}} d\lambda\,,$$

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$$f(x) = \frac{1}{2\pi}\sum_k \text{Cov}\,(X_0, X_k)e^{ixk}\;,\; x\in\mathbb{R}$$

Applications

- $X_k = \sum_{i\geq 0} a_i \varepsilon_{k-i}$ where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of i.i.d. centered random variables then

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- Functions of linear processes (ex: Riesz-Raikov sums)
- ARCH models

Strategy of Proof

Our aim: $\lim_{n\to\infty} S_{\mu_{\mathbf{B}_n}}(z) = S(z), \;\; \forall z \in \mathbb{C}^+$.

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Technic of blocks associated with the Lindeberg method

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Thank you for your attention!