GRADUATE SEMINAR ON PDE: KINETIC THEORY AND THE BOLTZMANN EQUATION

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1. Organizational comments

- Duration of talks: $\sim$ 80 minutes talk + 10 minutes for questions and discussion.
- Prepare a short handout with the most important statements of the talk.
- A lecture-style blackboard presentation is recommended.
- For all talks, it is sufficient to present the main ideas and intuition, rather than detailed proofs.
- Speakers should arrange for a meeting with Marvin roughly two weeks before the talk. Please prepare a preliminary version of the handout before the meeting.

Talks 8, 9, and 10 are still available! The possible dates are July 15, July 22.

No.	**Talk**	**Date**	**Speaker**
	First meeting	2026-04-15	
1	Regularity for linear kinetic equations and applications	2026-04-22	K. W.
2	Boltzmann collision operator	2026-04-29	B. G. C.
	— *no seminar* —	2026-05-06	—
3	Convergence to equilibrium	2026-05-13	E. R.
	— *no seminar* —	2026-05-20	—
	— *no seminar* —	2026-05-27	—
4	Regularizing effect of the Boltzmann equation	2026-06-03	S. F. S.
5	Existence of generalized solutions	2026-06-10	C. P.
	— *no seminar* —	2026-06-17	—
6	Conditional regularity (1): pointwise decay	2026-06-24	T. A. A.
7	Conditional regularity (2): C^∞ smoothness	2026-07-01	L. C. G.
	— *no seminar* —	2026-07-08	—
8 / 9	Short time well-posedness / Global existence near equilibrium	2026-07-15	???
10	Monotonicity of the Fisher information	2026-07-22	???

- Talks will be distributed in February.
- Attendance of each talk is mandatory.
- Grades will be based on mathematical quality and clarity of the presentation.
- Speakers will be informed about their grades after all talks have been presented.

2. Overview of the topic

Kinetic theory emerged in the 19th century through the fundamental works of Maxwell and Boltzmann to statistically describe large systems of particles. The Boltzmann equation is one of the central equations in this field and it models the evolution of a dilute gas at a mesoscopic scale [Vil02]. Rather than tracking the individual positions and velocities of every particle, the equation describes the evolution of a distribution function, which represents the density of particles in phase space (position and velocity). Furthermore, in the appropriate macroscopic limits, one recovers standard hydrodynamic models, such as the Euler or Navier-Stokes equations.

The unknown in the Boltzmann equation is a probability density $f(t, x, v)$ which keeps track of the "number" of particles that have velocity $v \in \mathbb{R}^n$ at time $t \in (0, \infty)$ at the point $x \in \mathbb{R}^n$

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \quad \text{in } (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n.$$

Here, $Q(f, f)$ is the Boltzmann collision operator, which acts only on v and models the interaction between particles.

As one of the fundamental equations from statistical mechanics and kinetic theory, the Boltzmann equation has received substantial interest from the mathematical community through the years. The goal of this seminar is to give an introduction to the Boltzmann equation and to explore the topic from various angles. We will discuss classical results in the field, as well as some of the most important developments from the past years:

- **hypoellipticity:** The linearized Boltzmann equation falls into the class of kinetic Fokker-Planck equations (special case: Kolmogorov equation). These are kinetic equations that admit smooth solutions in all variables, which makes them hypoelliptic.
  → Talk 1 explores the phenomenon of hypoellipticity in the linear(ized) setting.

- **regularizing effect:** The Boltzmann collision operator is regularizing in the velocity variable, i.e. $Q(f, f)$ shares some properties with the Laplacian Δ_v in v, or rather with an integro-differential version, the fractional Laplacian $-(-\Delta_v)^s$ for some $s \in (0, 1)$.
  → Talks 2, 4 establish this connection. Talks 6, 7 deal with conditional smoothness of solutions.

- **well-posedness:** The holy grail for any evolutionary equation from mathematical physics is to establish a well-posedness theory, i.e. to show the existence of a unique classical solution that attains its initial datum. (Due to the regularizing effect, there is the hope that solutions are classical and smooth). Proving it is incredibly hard, (arguably more difficult than solving the Millenium problem for Navier Stokes), but there exist partial results, approaching this question from various angles.
  → Talks 5 (existence of weak solutions), 6, 7 (existence of classical solutions conditional to macroscopic bounds), 4, 10 (existence of solutions to the x-homogeneous equation), 9 (short time existence), 9 (existence near equilibrium) address this topic.

- **long time behavior:** If a solution exists for all times, it is a natural question to ask, if the solution converges to some equilibrium state (Maxwellian), and if yes, how fast does it converge? The answer to this question is one of the main results for which Villani was awarded the Fields medal in 2010.
  → Talk 3 deals with this question. Talks 7, 9 apply or generalize this result.

3. List of talks

1. **Regularity for linear kinetic Fokker-Planck equations and applications.** The following class of kinetic equations is a linearized version of the Boltzmann equation, and it also generalizes the Kolmogorov equation

$$\partial_t f + v \cdot \nabla_x f - a_{ij} \partial_{v_i v_j} f = h.$$

Here, (a_{ij}) is a uniformly elliptic diffusion matrix with coefficients belonging to some Hölder space. Proving a Schauder-type regularity theory for these linear equation can be crucial to establish well-posedness of nonlinear kinetic equations, such as the Boltzmann or Landau equation. This connection is explained in [IM21].

Task: Explain Schauder estimates for linear kinetic Fokker-Planck equations with Hölder continuous coefficients and how it can be used to establish well-posedness of the nonlinear toy model in [IM21]. You can decide whether you rather want to focus on the proof of Schauder estimates, or on the proof of global well-posedness.

Literature: [IM21]. Another reference for a kinetic Schauder-type estimate: [HS20].

2. **Boltzmann collision operator.** There are many models for collisions in kinetic theory, and they are captured by the collision operator $Q(f, g)$, which models the interactions between particles. The Boltzmann collision operator was derived by Boltzmann in 1872 under the assumptions that particles interact via binary elastic collisions that are localized in space and time and microreversible. The goal of this talk is to provide a rough explanation on how to derive the Boltzmann equation from these principles and to give an overview of the most common interaction potentials that will be relevant for this course.

Task: Explain how to derive the Boltzmann equation (you can use [Vil02, Chapter 1] as the main reference). Explain the role of the gain and loss term in the Boltzmann collision operator and give an overview of the family of non-cutoff collision kernels that are derived from inverse-power law potentials. Moreover, comment on the terms "grazing collisions", "soft" / "hard" / "Mawellian potentials" (see [Vil02, Chapter 1.3]), and comment on how the Landau equation is related to the Boltzmann equation. You may also explain Carleman's representation and how it allows to write the Boltzmann operator as a fractional Laplacian up to lower order terms.

Literature: [Vil02]. For a more detailed explanation of Carleman coordinates, you can look at [IS20a] and the references therein.

3. **Convergence to equilibrium.** One of the most significant mathematical results for the Boltzmann equation is the convergence to equilibrium for smooth solutions, established by Desvillettes and Villani in [DV05]. Their result states that any solution f to the Boltzmann equation with appropriate decay for large velocities and such that f stays C^∞ uniformly for all $t > 0$ converges to equilibrium faster than any algebraic rate $O(t^{-k})$, $k \in \mathbb{N}$ as $t \to \infty$. It is one of the main results for which Villani received the Fields Medal in 2010. Before [DV05], Desvillettes and Villani proved the same result in the much simpler case of a linear kinetic Fokker-Planck equation [DV01].

Task: Explain the proof of the convergence to equilibrium for linear Fokker-Planck equations (see [DV01, Theorem 1.1]). Then, present the generalization of the result on the Boltzmann equation [DV05] (it is sufficient to explain the main additional difficulties arising in the nonlinear case).

Literature: [DV01, DV05]. See also [Vil02, Chapter 3].

4. **Regularizing effect of the Boltzmann equation.** One of the main themes in the study of the (non-cutoff) Boltzmann equation is its "regularizing effect". This means that the Boltzmann collision operator $Q(f,g)$ behaves like a singular integral operator, i.e. $Q(f,f) \asymp -(-\Delta)^s f$ (see the previous talk), which means that solutions to the space-homogeneous equation

$$\partial_t f = Q(f,f) \quad \text{in } (0,\infty) \times \mathbb{R}^n$$

automatically become C^∞ for positive times. This effect is captured by the so-called entropy dissipation. A sharp version of this estimate is proved in the celebrated article [ADVW00].

Task: Explain the entropy dissipation estimate and give an overview of how it implies regularity of solutions to the space-homogeneous Boltzmann equation (Theorem 1.1 in [ADVW00]). Give a short overview of the applications of this result to the space-homogeneous Boltzmann equation. You may also sketch how it implies instantaneous smoothness of solutions. Finally, you might comment on the limitation of the estimate for soft potentials ($\gamma < 0$).

Literature: [ADVW00] (and the references therein). See also [Vil02, Chapter 4]. For applications of the entropy dissipation and coercivity estimates to instantaneous smoothness, you can take a look at [DW04, He12].

5. **Existence of generalized solutions.** For any evolutionary equation that comes from mathematical physics, we are interested in whether the problem is well posed or not. In the case of the Boltzmann equation, we consider the Cauchy problem for a given initial data $f_0 \geq 0$ and we would like to determine the existence (and uniqueness) of a classical solution f to the Boltzmann equation so that $f(0,\cdot) = f_0$. This problem is arguably more complicated to solve than the Millenium problem for Navier Stokes!

A natural first step in establishing a well-posedness theory is to prove the existence of weak solutions. In the context of the Boltzmann equation (without cut-off), the existence of generalized solutions was established by Alexandre, Villani in [AV02].

Task: Explain the weak solution concept in [AV02] and present their main results [AV02, Theorem 2.1 and Corollary 2.2]. Outline the main ideas of their proof, focusing on the role of the entropy dissipation and the cancellation effect, and comment on how they manage to get rid of Grad's angular cut-off assumption.

Literature: [AV02]. See also [Vil02, Chapter 2.4].

6. **Conditional regularity (1): Macroscopic bounds and pointwise decay.** Although the Boltzmann collision operator has a regularizing effect, proving C^∞ regularity of solutions seems completely out of reach with current techniques (and it might actually be false). This raises the natural question of whether solutions might be smooth, and therefore do not possess any singularity, under some mild additional assumptions. For instance, one could ask, whether any singularity for solutions to Boltzmann must be observable on a macroscopical scale. This question was answered affirmatively in a series of groundbreaking work by Imbert, Silvestre, (and Mouhot) [Sil16, IMS20a, IS20b, IS21, IS22], where they establish C^∞ regularity and decay estimates of solutions conditional to the assumption that certain hydrodynamic quantities (mass, energy, entropy) remain bounded.

The goal of this talk is to explain the first step of the conditional regularity program, which consists in proving pointwise moment bounds for the Boltzmann equation.

Task: Explain how the boundedness of mass, energy, and entropy implies pointwise decay and moment estimates of solutions to the Boltzmann equation in the velocity variable. To do so, explain the result and sketch the proof of [IMS20a, Theorem 1.3] (see also [Sil16, Theorem 1.2], which establishes pointwise decay (and moment estimates) of solutions to the space-inhomogeneous Boltzmann equation under macroscopic bounds.. You might restrict yourself to the easier case of hard potentials $\gamma > 0$.

Literature: [Sil16, IMS20a]. See also the survey articles [IS20a, Sil23].

7. **Conditional regularity (2): C^∞ regularity of solutions.** This talk is a continuation of the previous one but focuses on the regularity of solutions, rather than on their decay. The goal of this talk is to outline how regularity results for linear kinetic equations with integral diffusion yield smoothness of solutions to the nonlinear Boltzmann equation satisfying macroscopic bounds. These bounds allow to write the Boltzmann collision operator as a uniformly elliptic integro-differential operator in the velocity variable.

Task: Explain the main result on conditional regularity for Boltzmann (see [IS22, Theorem 1.2]). Outline the key steps in the proof focusing on the bootstrap mechanism that allows to improve the smoothness of solutions in each step. You may use the C^α- and Schauder-type regularity results for linear equations from [IS20b, IS21] as a black-box (there is no need to comment on their proofs), but you should explain in what way the macroscopic bounds are needed in order to apply them. You might restrict yourself to the easier case of hard potentials $\gamma > 0$.

Literature: [IS22]. See also the survey articles [IS20a, Sil23].

8. **Short time well-posedness.** Proving global well-posedness of the Boltzmann equation is arguably more complicated than solving the Millenium problem for Navier-Stokes and it is currently completely out of reach. Yet, it is an intriguing question to check whether a well-posedness theory can be established for short times: Given an initial datum f_{in}, is there a time $T > 0$ such that there is a unique solution to the Boltzmann equation for $t \in (0, T)$? Since the Boltzmann equation regularizes solutions, it is important to develop such a theory under minimal assumptions on the initial data f_{in}. The paper [HST25] gives an affirmative answer to this question!

Task: Explain Theorem 1.1 of [HST25] and discuss its relevance for the field. Give an overview of the main ideas of their proof and how the authors overcome the difficulties of the problem, in particular via the propagation of polynomial bounds in Lemma 4.1 and by application of the conditional regularity results due to Imbert-Silvestre. Finally, (shortly) mention and comment on their uniqueness result (Theorem 1.4).

Literature: [HST25] and the references therein.

9. **Global existence of solutions close to equilibrium.** Although a well-posedness theory for the Boltzmann equation is currently completely out of reach, it is a natural question to ask, whether the existence of global classical solutions can be established under the assumption that their initial data is already close to the equilibrium state. There exist many positive results on this question in the literature and there are various ways to prove them. One of the first results on this topic is the celebrated paper by Gressmann, Strain [GS11], where the authors prove a sharp coercivity estimate with respect to an anisotropic fractional Sobolev norm. However, there is also a very elegant (and more recent) proof, combining the convergence to equilibrium [DV05] with the conditional regularity program developed in [IS22], and short time existence results (e.g. [HST25]).

Task: The goal of this talk is to present the results on global existence of solutions to Boltzmann close to equilibrium that were obtained in [SS23, Theorem 1.1 and 1.2] (see also [HST25, Corollary 1.5]), and to explain their elegant strategy of proof. You may use the short-existence results established in [SS23, HST25] as a black box. The results in [DV05, IS22] have already been discussed in previous talks. If time permits, you might comment on the proof of Gaussian lower bounds [IMS20b]. Finally, you may also (shortly) review the results in the following celebrated articles: [GS11, AMSY23].

Literature: Focus on [SS23, HST25]. You may comment on the results in [GS11, AMSY23]. See also the discussion in [HST25, Section 1.4.1; Close-to-equilibrium solutions].

10. **Monotonicity of the Fisher information.** The regularizing effect of the Boltzmann collision operator has been a core theme in the study of the Boltzmann equation during the past decades due to its importance in order to establish existence of classical solutions. Despite huge progress in the field, the case of very soft potentials (which is the most singular case) has remained elusive for a long time. In a recent breakthrough paper by Imbert, Silvestre, and Villani [ISV26], they proved that the Fisher-information decreases along the flow of the (space homogeneous) Boltzmann equation. This allowed them to establish for the first time the existence of global smooth solutions to the homogeneous Boltzmann equation with very soft potentials.

Task: Explain the main results on the monotonicity of the Fisher information (see [ISV26, Theorem 1.2]) and the global existence of smooth solutions (see [ISV26, Theorem 1.6]). You may restrict your presentation to the class of collision kernels given by [ISV26, (1.8)]. Explain the main ideas of the proof of [ISV26, Theorem 1.2] and make a few comments on how it implies [ISV26, Theorem 1.6].

Literature: [ISV26]. You can also look at the recent survey article [Vil25].

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