

SELECTED TOPICS LECTURE: INTRODUCTION TO NONLOCAL PDES

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The classical theory of second-order elliptic partial differential equations deals with differential operators like the Laplacian and models phenomena governed by purely local, continuous interactions, such as the diffusion of heat. However, many systems in nature or finance, such as gas dynamics or option pricing, demand models that account for long-range dependencies. Here, the dynamics at any given point are directly influenced by distant events. These phenomena are captured by nonlocal PDEs. This lecture provides an introduction to this topic, starting with the fundamental properties of the simplest nonlocal operator (the fractional Laplacian).

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1. The fractional Laplacian

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1. THE FRACTIONAL LAPLACIAN

Let us start by introducing the fractional Laplacian.

Definition 1.1. Let $0 < s < 1$ and $u \in C_c^\infty(\mathbb{R}^n)$. We define the fractional Laplacian as follows

$$(-\Delta)^s u(x) = c_{n,s} \operatorname{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz, \quad (1.1)$$

where

$$c_{n,s} = \frac{2^{2s} s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} \Gamma(1-s)} > 0, \quad (1.2)$$

and the principal value means

$$\operatorname{PV} \int_{\mathbb{R}^n} \cdots dz := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \cdots dz.$$

The constant (1.2) is chosen so that the Fourier multiplier is exactly $|\xi|^{2s}$ (we will see this later).

Sometimes it is convenient to write the integrand of the fractional Laplacian in terms of a second-order incremental quotient. The following lemma says that these definitions are equivalent.

Lemma 1.2. For $u \in C_c^2(\mathbb{R}^n)$, the principal value in (1.1) exists and

$$(-\Delta)^s u(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} dz. \quad (1.3)$$

The integral on the right converges absolutely.

Proof. Fix $x \in \mathbb{R}^n$. For $\delta > 0$ put

$$I_\delta(x) := \int_{|z|>\delta} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz.$$

Note that $I_\delta(x)$ is finite, since u is bounded and the kernel is integrable away from the origin. Changing variables $z \mapsto -z$ gives

$$I_\delta(x) = \int_{|z|>\delta} \frac{u(x) - u(x-z)}{|z|^{n+2s}} dz.$$

Summing up these two identities, we obtain

$$I_\delta(x) = \frac{1}{2} \int_{|z|>\delta} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} dz.$$

Once we show that the limit as $\delta \rightarrow 0$ exists, the proof is complete. To do so, note that Taylor's formula yields, for $|z| < 1$,

$$|u(x+z) + u(x-z) - 2u(x)| = \left| \int_{-1}^1 (1-|t|) z^T D^2 u(x+tz) z dt \right| \leq \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2.$$

since $\int_{-1}^1 (1-|t|) dt = 1$. Thus, writing $\omega_{n-1} = |\mathbb{S}^{n-1}|$ and using polar coordinates, as well as $s < 1$,

$$\begin{aligned} & \int_{|z|<1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} dz \\ & \leq \|D^2 u\|_{L^\infty} \omega_{n-1} \int_0^1 r^{1-2s} dr = \frac{\omega_{n-1}}{2(1-s)} \|D^2 u\|_{L^\infty} < \infty. \end{aligned}$$

On the other hand, for $|z| \geq 1$,

$$|2u(x) - u(x+z) - u(x-z)| \leq 4\|u\|_{L^\infty},$$

and hence

$$\begin{aligned} & \int_{|z|\geq 1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} dz \\ & \leq 4\|u\|_{L^\infty} \omega_{n-1} \int_1^\infty r^{-1-2s} dr = \frac{2\omega_{n-1}}{s} \|u\|_{L^\infty} < \infty. \end{aligned}$$

Hence, $I_\delta(x)$ converges absolutely as $\delta \rightarrow 0$. This concludes the proof. $\square$

Remark 1.3. Taylor's formula shows that, for $|z| < 1$,

$$|u(x) - u(x+z)| \leq \|\nabla u\|_{L^\infty} |z|.$$

Thus, when $0 < s < \frac{1}{2}$, the following integral is absolutely convergent near $z = 0$,

$$\int_{B_1} \frac{|u(x) - u(x+z)|}{|z|^{n+2s}} dz \leq \|\nabla u\|_{L^\infty} \omega_{n-1} \int_0^1 r^{-2s} dr = \frac{\omega_{n-1}}{1-2s} \|\nabla u\|_{L^\infty} < \infty.$$

Consequently, for $s < \frac{1}{2}$ the symbol PV may be omitted in (1.1).

Definition 1.4. For $0 < s < 1$, define

$$L_{2s}^1(\mathbb{R}^n) = \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^n) : \|u\|_{L_{2s}^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(z)|}{1+|z|^{n+2s}} dz < \infty \right\}.$$

We will sometimes call the integral in this definition the *tail integral*.

Note that $|x|^\alpha \in L_{2s}^1(\mathbb{R}^n)$ if and only if $\alpha < 2s$.

The following proposition establishes fine necessary conditions on u so that $(-\Delta)^s u$ is well-defined, meaning that (1.1) and (1.3) both converge. Moreover, we will show that $(-\Delta)^s u$ has $2s$ derivatives less than u . In this sense, $(-\Delta)^s$ is a *differential operator of order $2s \in (0, 2)$* .

Proposition 1.5. *Let $0 < s < 1$ and $\varepsilon > 0$. If*

$$u \in C^{2s+\varepsilon}(B_1) \cap L_{2s}^1(\mathbb{R}^n),$$

then $(-\Delta)^s u$ is well-defined at every point of B_1 by (1.3), and

$$\|(-\Delta)^s u\|_{L^\infty(B_{1/2})} \leq C \left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L_{2s}^1(\mathbb{R}^n)} \right).$$

Moreover, if $\varepsilon \notin \mathbb{N}$, then $(-\Delta)^s u \in C^\varepsilon(B_{1/2})$ and

$$\|(-\Delta)^s u\|_{C^\varepsilon(B_{1/2})} \leq C \left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L_{2s}^1(\mathbb{R}^n)} \right). \quad (1.4)$$

The constant C depends only on n, s, ε .

Proof. We first prove the estimate for $0 < \varepsilon < 1$ and put $\beta = 2s + \varepsilon \in (0, 3)$.

Step 1. Fix $\eta \in C_c^\infty(B_{7/8})$ with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{3/4}$, and write

$$u = v + w, \quad v = \eta u, \quad w = (1 - \eta)u.$$

The idea is to estimate $(-\Delta)^s v$ and $(-\Delta)^s w$ separately. Note that by construction,

$$\|v\|_{C^\beta(\mathbb{R}^n)} \leq C\|u\|_{C^\beta(B_1)}, \quad \|w\|_{L_{2s}^1(\mathbb{R}^n)} \leq \|u\|_{L_{2s}^1(\mathbb{R}^n)}.$$

Moreover, because $w = 0$ in $B_{3/4}$, for every $x \in B_{1/2}$ we have

$$(-\Delta)^s w(x) = -c_{n,s} \int_{\mathbb{R}^n \setminus B_{3/4}} \frac{w(z)}{|x - z|^{n+2s}} dz.$$

Whenever $x \in B_{1/2}$ and $w(z) \neq 0$, we have $|x - z| \geq 1/4$. Consequently, for every multi-index γ ,

$$|D_x^\gamma |x - z|^{-n-2s}| \leq \frac{C_{n,s,\gamma}}{1 + |z|^{n+2s}}.$$

We may differentiate under the integral to obtain

$$\|(-\Delta)^s w\|_{C^{m,1}(B_{1/2})} \leq C\|w\|_{L_{2s}^1(\mathbb{R}^n)} \leq C\|u\|_{L_{2s}^1(\mathbb{R}^n)} \quad (m = 0, 1, 2, \dots).$$

Step 2. It remains to prove

$$\|(-\Delta)^s v\|_{C^\varepsilon(\mathbb{R}^n)} \leq C\|v\|_{C^\beta(\mathbb{R}^n)}. \quad (1.5)$$

We define

$$\delta_z^2 v(x) = 2v(x) - v(x + z) - v(x - z).$$

The L^∞ bound follows by observing that

$$|\delta_z^2 v(x)| \leq C\|v\|_{C^\beta} |z|^{\min\{\beta, 2\}} \quad (|z| < 1), \quad |\delta_z^2 v(x)| \leq 4\|v\|_\infty \quad (|z| \geq 1)$$

in the same way as in the proof of Lemma 1.2, using that $\min\{\beta, 2\} > 2s$ when $0 < \varepsilon < 1$ and $\beta > 2s$.

To estimate the Hölder seminorm, take $x, x' \in \mathbb{R}^n$. We set $h = x - x'$ and assume that $|h| \in (0, 1)$. Then, we have

$$|(-\Delta)^s v(x) - (-\Delta)^s v(x')| \leq \frac{c_{n,s}}{2} \int_{|z| < |h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}} dz + \frac{c_{n,s}}{2} \int_{|z| \geq |h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}} dz$$

$$=: \frac{c_{n,s}}{2} (I_{\text{near}} + I_{\text{far}}).$$

It remains to show

$$I_{\text{near}} + I_{\text{far}} \leq C \|v\|_{C^\beta} |h|^\varepsilon. \quad (1.6)$$

We distinguish between two cases, depending on β .

Case A: $0 < \beta \leq 1$. Hölder continuity of v gives

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \leq \begin{cases} C \|v\|_{C^\beta} |z|^\beta, & |z| < |h|, \\ C \|v\|_{C^\beta} |h|^\beta, & |z| \geq |h|. \end{cases}$$

Polar coordinates then yield

$$\begin{aligned} I_{\text{near}} &\leq C \|v\|_{C^\beta} \int_0^{|h|} \rho^{\beta-2s-1} d\rho = \frac{C}{\varepsilon} \|v\|_{C^\beta} |h|^\varepsilon, \\ I_{\text{far}} &\leq C \|v\|_{C^\beta} r^\beta \int_{|h|}^\infty \rho^{-2s-1} d\rho = \frac{C}{2s} \|v\|_{C^\beta} |h|^\varepsilon. \end{aligned}$$

Case B: $1 < \beta < 3$. We claim that by the fundamental theorem of calculus, for any $z \in \mathbb{R}^n$

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| = \left| \int_0^1 (\delta_z^2 \nabla v)(x' + th) \cdot h dt \right| \leq C \|v\|_{C^\beta} |h| |z|^{\beta-1}. \quad (1.7)$$

For $1 < \beta \leq 2$, the proof of (1.7) uses that by $\nabla v \in C^{\beta-1}$, we have

$$|\nabla v(w \pm z) - \nabla v(w)| \leq [\nabla v]_{C^{\beta-1}} |z|^{\beta-1},$$

and hence we conclude

$$\begin{aligned} |\delta_z^2 \nabla v(w)| &= |\nabla v(w+z) + \nabla v(w-z) - 2\nabla v(w)| \\ &\leq |\nabla v(w+z) - \nabla v(w)| + |\nabla v(w-z) - \nabla v(w)| \\ &\leq 2[\nabla v]_{C^{\beta-1}} |z|^{\beta-1}, \end{aligned}$$

which implies (1.7).

For $2 < \beta < 3$, we prove (1.7) using $\nabla v \in C^{1,\beta-2}$ and $D^2 v \in C^{\beta-2}$. Indeed, by Taylor's formula,

$$\nabla v(w \pm z) = \nabla v(w) \pm D^2 v(w)z + R_\pm(w, z), \quad \text{where } |R_\pm(w, z)| \leq C[D^2 v]_{C^\alpha} |z|^{\beta-1}.$$

Adding these two estimates and subtracting $2\nabla v(w)$, we obtain

$$|\delta_z^2 \nabla v(w)| = |2\nabla v(w) - \nabla v(w+z) - \nabla v(w-z)| \leq |R_+(w, z)| + |R_-(w, z)| \leq C[D^2 v]_{C^{\beta-2}} |z|^{\beta-1}.$$

This proves (1.7) also in this case and thus, we conclude

$$I_{\text{far}} = \int_{|z| \geq |h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}} dz \leq C \|v\|_{C^\beta} |h| \int_{|z| \geq |h|} |z|^{\beta-1-n-2s} dz \leq C \|v\|_{C^\beta} |h|^\varepsilon.$$

To estimate I_{near} , note that for $1 < \beta \leq 2$, we can use (1.7) to conclude since $|h| \leq |z|$,

$$I_{\text{near}} \leq C \|v\|_{C^\beta} \int_0^{|h|} |z|^{\beta-1-n-2s} d|z| \leq C \|v\|_{C^\beta} |z|^\varepsilon.$$

For $2 < \beta < 3$, we use the integral representation of the second difference:

$$\delta_z^2 v(x) = - \int_{-1}^1 (1 - |t|) z^T D^2 v(x + tz) z dt.$$

which implies that

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \leq |z|^2 \int_{-1}^1 (1 - |t|) |D^2 v(x + tz) - D^2 v(x' + tz)| dt \leq C[D^2 v]_{C^{\beta-2}} |h|^{\beta-2} |z|^2.$$

Hence, it follows that

$$I_{\text{near}} \leq C \|v\|_{C^\beta} |h|^{\beta-2} \int_0^{|h|} |z|^{2-n-2s} dz \leq C \|v\|_{C^\beta} |h|^{\beta-2} |h|^{2-2s} = C \|v\|_{C^\beta} |h|^\varepsilon.$$

Hence, we have shown (1.6) also in this case.

When $|x - x'| \geq 1$, the same estimate follows from the L^∞ bound. This proves (1.5) and the proof is complete in case $\varepsilon \in (0, 1)$.

Step 3. Now we consider the case $\varepsilon > 1$ with $\varepsilon \notin \mathbb{N}$. We only give a sketch of the argument. Write $\varepsilon = m + \alpha$, with $m \geq 1$ an integer and $0 < \alpha < 1$. As above, split $u = v + w$ with $v = \eta u$ and $w = (1 - \eta)u$. Translation invariance of $(-\Delta)^s$ implies that

$$D^\gamma ((-\Delta)^s v) = (-\Delta)^s (D^\gamma v), \quad |\gamma| \leq m.$$

Since $D^\gamma v \in C^{2s+\varepsilon-|\gamma|}(\mathbb{R}^n) \subset C^{2s+\alpha}(\mathbb{R}^n)$, the estimate just proved with $\alpha \in (0, 1)$ applies to each $D^\gamma v$. From here, one can conclude the proof. $\square$

Lemma 1.6. *For $u \in C_c^\infty(\mathbb{R}^n)$, the following identities hold:*

$$\begin{aligned} (-\Delta)^s [u(\cdot + x_0)](x) &= ((-\Delta)^s u)(x + x_0), \\ (-\Delta)^s [u(Q\cdot)](x) &= ((-\Delta)^s u)(Qx) && \text{for } Q \in O(n), \\ (-\Delta)^s [u(\lambda\cdot)](x) &= \lambda^{2s} ((-\Delta)^s u)(\lambda x) && \text{for } \lambda > 0. \end{aligned}$$

Proof. In (1.1), use translation invariance, the change of variables $z \mapsto Qz$, and the substitution $w = \lambda z$, respectively. $\square$

End of lecture 1. Note that there will be no lecture on Oct 20. The next lecture will take place on Oct 27.