SELECTED TOPICS LECTURE: INTRODUCTION TO NONLOCAL PDES

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The classical theory of second-order elliptic partial differential equations deals with differential operators like the Laplacian and models phenomena governed by purely local, continuous interactions, such as the diffusion of heat. However, many systems in nature or finance, such as gas dynamics or option pricing, demand models that account for long-range dependencies. Here, the dynamics at any given point are directly influenced by distant events. These phenomena are captured by nonlocal PDEs. This lecture provides an introduction to this topic, starting with the fundamental properties of the simplest nonlocal operator (the fractional Laplacian).

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1. The fractional Laplacian

Let us start by introducing the fractional Laplacian.

Definition 1.1. Let $0 < s < 1$ and $u \in C_c^\infty(\mathbb{R}^n)$. We define the fractional Laplacian as follows

$$(-\Delta)^s u(x) = c_{n,s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z, \tag{1.1}$$

where

$$c_{n,s} = \frac{2^{2s} s\,\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)} > 0, \tag{1.2}$$

and the principal value means

$$\mathrm{PV} \int_{\mathbb{R}^n} \cdots\,\mathrm{d}z := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \cdots\,\mathrm{d}z.$$

The constant (1.2) is chosen so that the Fourier multiplier is exactly $|\xi|^{2s}$ (we will see this later).

Sometimes it is convenient to write the integrand of the fractional Laplacian in terms of a second-order incremental quotient. The following lemma says that these defintions are equivalent.

Lemma 1.2. *For $u \in C_c^2(\mathbb{R}^n)$, the principal value in (1.1) exists and*

$$(-\Delta)^s u(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z. \tag{1.3}$$

The integral on the right converges absolutely.

Proof. Fix $x \in \mathbb{R}^n$. For $\delta > 0$ put

$$I_\delta(x) := \int_{|z|>\delta} \frac{u(x) - u(x+z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Note that $I_\delta(x)$ is finite, since u is bounded and the kernel is integrable away from the origin. Changing variables $z \mapsto -z$ gives

$$I_\delta(x) = \int_{|z|>\delta} \frac{u(x) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Summing up these two identities, we obtain

$$I_\delta(x) = \frac{1}{2} \int_{|z|>\delta} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Once we show that the limit as $\delta \to 0$ exists, the proof is complete. To do so, note that Taylor's formula yields, for $|z| < 1$,

$$\big|u(x+z) + u(x-z) - 2u(x)\big| = \left| \int_{-1}^{1} (1-|t|)\, z^T D^2 u(x+tz) z \,\mathrm{d}t \right| \leq \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2.$$

since $\int_{-1}^{1} (1-|t|) \,\mathrm{d}t = 1$. Thus, writing $\omega_{n-1} = |\mathbb{S}^{n-1}|$ and using polar coordinates, as well as $s < 1$,

$$\begin{aligned} &\int_{|z|<1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} \,\mathrm{d}z \\ &\qquad \leq \|D^2 u\|_{L^\infty}\, \omega_{n-1} \int_0^1 r^{1-2s} \,\mathrm{d}r = \frac{\omega_{n-1}}{2(1-s)} \|D^2 u\|_{L^\infty} < \infty. \end{aligned}$$

On the other hand, for $|z| \geq 1$,

$$|2u(x) - u(x+z) - u(x-z)| \leq 4\|u\|_{L^\infty},$$

and hence

$$\begin{aligned} &\int_{|z|\geq 1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} \,\mathrm{d}z \\ &\qquad \leq 4\|u\|_{L^\infty}\, \omega_{n-1} \int_1^\infty r^{-1-2s} \,\mathrm{d}r = \frac{2\omega_{n-1}}{s} \|u\|_{L^\infty} < \infty. \end{aligned}$$

Hence, $I_\delta(x)$ converges absolutely as $\delta \to 0$. This concludes the proof. □

Remark 1.3. Taylor's formula shows that, for $|z| < 1$,

$$|u(x) - u(x+z)| \leq \|\nabla u\|_{L^\infty} |z|.$$

Thus, when $0 < s < \frac{1}{2}$, the following integral is absolutely convergent near $z = 0$,

$$\int_{B_1} \frac{|u(x) - u(x+z)|}{|z|^{n+2s}} \,\mathrm{d}z \leq \|\nabla u\|_{L^\infty}\, \omega_{n-1} \int_0^1 r^{-2s} \,\mathrm{d}r = \frac{\omega_{n-1}}{1-2s} \|\nabla u\|_{L^\infty} < \infty.$$

Consequently, for $s < \frac{1}{2}$ the symbol PV may be omitted in (1.1).

Definition 1.4. For $0 < s < 1$, define

$$L^1_{2s}(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \|u\|_{L^1_{2s}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(z)|}{1+|z|^{n+2s}}\,\mathrm{d}z < \infty \right\}.$$

We will sometimes call the integral in this definition the *tail integral*.

Note that $|x|^\alpha \in L^1_{2s}(\mathbb{R}^n)$ if and only if $\alpha < 2s$.

The following proposition establishes fine necessary conditions on u so that $(-\Delta)^s u$ is well-defined, meaning that (1.1) and (1.3) both converge. Moreover, we will show that $(-\Delta)^s u$ has $2s$ derivatives less than u. In this sence, $(-\Delta)^s$ is a *differential operator of order* $2s \in (0,2)$.

Proposition 1.5. *Let $0 < s < 1$ and $\varepsilon > 0$. If*

$$u \in C^{2s+\varepsilon}(B_1) \cap L^1_{2s}(\mathbb{R}^n),$$

then $(-\Delta)^s u$ is well-defined at every point of B_1 by (1.3), and

$$\|(-\Delta)^s u\|_{L^\infty(B_{1/2})} \le C\left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}\right).$$

Moreover, if $\varepsilon \notin \mathbb{N}$, *then $(-\Delta)^s u \in C^\varepsilon(B_{1/2})$ and*

$$\|(-\Delta)^s u\|_{C^\varepsilon(B_{1/2})} \le C\left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}\right). \tag{1.4}$$

The constant C depends only on n, s, ε.

Proof. We first prove the estimate for $0 < \varepsilon < 1$ and put $\beta = 2s + \varepsilon \in (0,3)$.

Step 1. Fix $\eta \in C^\infty_c(B_{7/8})$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ in $B_{3/4}$, and write

$$u = v + w, \qquad v = \eta u, \qquad w = (1-\eta)u.$$

The idea is to estimate $(-\Delta)^s v$ and $(-\Delta)^s w$ separately. Note that by construction,

$$\|v\|_{C^\beta(\mathbb{R}^n)} \le C\|u\|_{C^\beta(B_1)}, \qquad \|w\|_{L^1_{2s}(\mathbb{R}^n)} \le \|u\|_{L^1_{2s}(\mathbb{R}^n)}.$$

Moreover, because $w = 0$ in $B_{3/4}$, for every $x \in B_{1/2}$ we have

$$(-\Delta)^s w(x) = -c_{n,s} \int_{\mathbb{R}^n \setminus B_{3/4}} \frac{w(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

Whenever $x \in B_{1/2}$ and $w(z) \neq 0$, we have $|x - z| \ge 1/4$. Consequently, for every multi-index γ,

$$\left|D^\gamma_x |x-z|^{-n-2s}\right| \le \frac{C_{n,s,\gamma}}{1+|z|^{n+2s}}.$$

We may differentiate under the integral to obtain

$$\|(-\Delta)^s w\|_{C^{m,1}(B_{1/2})} \le C\|w\|_{L^1_{2s}(\mathbb{R}^n)} \le C\|u\|_{L^1_{2s}(\mathbb{R}^n)} \quad (m = 0,1,2,\ldots).$$

Step 2. It remains to prove

$$\|(-\Delta)^s v\|_{C^\varepsilon(\mathbb{R}^n)} \le C\|v\|_{C^\beta(\mathbb{R}^n)}. \tag{1.5}$$

We define

$$\delta^2_z v(x) = 2v(x) - v(x+z) - v(x-z).$$

The L^∞ bound follows by observing that

$$|\delta^2_z v(x)| \le C\|v\|_{C^\beta}|z|^{\min\{\beta,2\}} \quad (|z| < 1), \qquad |\delta^2_z v(x)| \le 4\|v\|_\infty \quad (|z| \ge 1)$$

in the same way as in the proof of Lemma 1.2, using that $\min\{\beta, 2\} > 2s$ when $0 < \varepsilon < 1$ and $\beta > 2s$.

To estimate the Hölder seminorm, take $x, x' \in \mathbb{R}^n$. We set $h = x - x'$ and assume that $|h| \in (0,1)$. Then, we have

$$\begin{aligned}|(-\Delta)^s v(x) - (-\Delta)^s v(x')| &\leq \frac{c_{n,s}}{2} \int_{|z|<|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}} \,\mathrm{d}z + \frac{c_{n,s}}{2} \int_{|z|\geq|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}} \,\mathrm{d}z \\ &=: \frac{c_{n,s}}{2} \big(I_{\mathrm{near}} + I_{\mathrm{far}} \big).\end{aligned}$$

It remains to show

$$I_{\mathrm{near}} + I_{\mathrm{far}} \leq C \|v\|_{C^\beta} |h|^\varepsilon. \tag{1.6}$$

We distinguish between two cases, depending on β.

Case A: $0 < \beta \leq 1$. Hölder continuity of v gives

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \leq \begin{cases} C\|v\|_{C^\beta} |z|^\beta, & |z| < |h|, \\ C\|v\|_{C^\beta} |h|^\beta, & |z| \geq |h|. \end{cases}$$

Indeed if $|z| < |h|$, we use that

$$|\delta_z^2 v(x)| = \big| \big(v(x) - v(x+z) \big) + \big(v(x) - v(x-z) \big) \big| \leq C \|v\|_{C^\beta} |z|^\beta,$$

and if $|z| \geq |h|$, we estimate

$$\begin{aligned}\big|\delta_z^2 v(x) - \delta_z^2 v(x')\big| &= \big| 2\big(v(x) - v(x')\big) - \big(v(x+z) - v(x'+z)\big) - \big(v(x-z) - v(x'-z)\big) \big| \\ &\leq C \|v\|_{C^\beta} |h|^\beta\end{aligned}$$

Polar coordinates then yield

$$\begin{aligned} I_{\mathrm{near}} &\leq C\|v\|_{C^\beta} \int_0^{|h|} \rho^{\beta - 2s - 1} \,\mathrm{d}\rho = \frac{C}{\varepsilon} \|v\|_{C^\beta} |h|^\varepsilon, \\ I_{\mathrm{far}} &\leq C\|v\|_{C^\beta} |h|^\beta \int_{|h|}^\infty \rho^{-2s-1} \,\mathrm{d}\rho = \frac{C}{2s} \|v\|_{C^\beta} |h|^\varepsilon. \end{aligned}$$

Case B: $1 < \beta < 3$. We claim that by the fundamental theorem of calculus, for any $z \in \mathbb{R}^n$

$$\big|\delta_z^2 v(x) - \delta_z^2 v(x')\big| = \left| \int_0^1 \big(\delta_z^2 \nabla v\big)(x' + th) \cdot h \,\mathrm{d}t \right| \leq C \|v\|_{C^\beta} |h| |z|^{\beta - 1}. \tag{1.7}$$

For $1 < \beta \leq 2$, the proof of (1.7) uses that by $\nabla v \in C^{\beta-1}$, we have

$$|\nabla v(w \pm z) - \nabla v(w)| \leq [\nabla v]_{C^{\beta-1}} |z|^{\beta-1},$$

and hence we conclude

$$\begin{aligned} |\delta_z^2 \nabla v(w)| &= \big| \nabla v(w+z) + \nabla v(w-z) - 2\nabla v(w) \big| \\ &\leq |\nabla v(w+z) - \nabla v(w)| + |\nabla v(w-z) - \nabla v(w)| \\ &\leq 2 [\nabla v]_{C^{\beta-1}} |z|^{\beta-1}, \end{aligned}$$

which implies (1.7).

For $2 < \beta < 3$, we prove (1.7) using $\nabla v \in C^{1,\beta-2}$ and $D^2 v \in C^{\beta-2}$. Indeed, by Taylor's formula,

$$\nabla v(w \pm z) = \nabla v(w) \pm D^2 v(w) z + R_\pm(w,z), \quad \text{where} \quad |R_\pm(w,z)| \leq C [D^2 v]_{C^\alpha} |z|^{\beta-1}.$$

Adding these two estimates and subtracting $2\nabla v(w)$, we obtain

$$\left|\delta_z^2 \nabla v(w)\right| = \left|2\nabla v(w) - \nabla v(w+z) - \nabla v(w-z)\right| \le \left|R_+(w,z)\right| + \left|R_-(w,z)\right| \le C[D^2 v]_{C^{\beta-2}} |z|^{\beta-1}.$$

This proves (1.7) also in this case and thus, we conclude

$$I_{\text{far}} = \int_{|z|\ge|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}}\, dz \le C\|v\|_{C^\beta} |h| \int_{|z|\ge|h|} |z|^{\beta-1-n-2s}\, dz \le C\|v\|_{C^\beta} |h|^\varepsilon.$$

To estimate I_{near}, note that for $1 < \beta \le 2$, we can use (1.7) to conclude since $|h| \le |z|$,

$$I_{\text{near}} \le C\|v\|_{C^\beta} \int_0^{|h|} |z|^{\beta-1-n-2s}\, d|z| \le C\|v\|_{C^\beta} |z|^\varepsilon.$$

For $2 < \beta < 3$, we use the integral representation of the second difference:

$$\delta_z^2 v(x) = -\int_{-1}^1 (1-|t|)\, z^T D^2 v(x+tz) z\, dt.$$

which implies that

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \le |z|^2 \int_{-1}^1 (1-|t|) |D^2 v(x+tz) - D^2 v(x'+tz)|\, dt \le C[D^2 v]_{C^{\beta-2}} |h|^{\beta-2} |z|^2.$$

Hence, it follows that

$$I_{\text{near}} \le C\|v\|_{C^\beta} |h|^{\beta-2} \int_0^{|h|} |z|^{2-n-2s}\, dz \le C\|v\|_{C^\beta} |h|^{\beta-2} |h|^{2-2s} = C\|v\|_{C^\beta} |h|^\varepsilon.$$

Hence, we have shown (1.6) also in this case.

When $|x - x'| \ge 1$, the same estimate follows from the L^∞ bound. This proves (1.5) and the proof is complete in case $\varepsilon \in (0,1)$.

Step 3. Now we consider the case $\varepsilon > 1$ with $\varepsilon \notin \mathbb{N}$. We only give a sketch of the argument. Write $\varepsilon = m + \alpha$, with $m \ge 1$ an integer and $0 < \alpha < 1$. As above, split $u = v + w$ with $v = \eta u$ and $w = (1-\eta)u$. Translation invariance of $(-\Delta)^s$ implies that

$$D^\gamma((-\Delta)^s v) = (-\Delta)^s (D^\gamma v), \qquad |\gamma| \le m.$$

Since $D^\gamma v \in C^{2s+\varepsilon-|\gamma|}(\mathbb{R}^n) \subset C^{2s+\alpha}(\mathbb{R}^n)$, the estimate just proved with $\alpha \in (0,1)$ applies to each $D^\gamma v$. From here, one can conclude the proof. □

Lemma 1.6. *For $u \in C_c^\infty(\mathbb{R}^n)$, the following identities hold:*

$$\begin{aligned}
(-\Delta)^s [u(\cdot + x_0)](x) &= ((-\Delta)^s u)(x + x_0), \\
(-\Delta)^s [u(Q\cdot)](x) &= ((-\Delta)^s u)(Qx) && \text{for } Q \in O(n), \\
(-\Delta)^s [u(\lambda\cdot)](x) &= \lambda^{2s} ((-\Delta)^s u)(\lambda x) && \text{for } \lambda > 0.
\end{aligned}$$

Proof. In (1.1), use translation invariance, the change of variables $z \mapsto Qz$, and the substitution $w = \lambda z$, respectively. □

Next, we show that the fractional Laplacian arises as a Fourier multiplier operator.

Our Fourier-transform convention is

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, \mathrm{d}x.$$

Recall that $\mathcal{F}(-\Delta u)(\xi) = |\xi|^2 \widehat{u}(\xi)$.

Lemma 1.7. *For $u \in \mathcal{S}(\mathbb{R}^n)$ and $0 < s < 1$, it holds*

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi).$$

Proof. Taking the Fourier transform of (1.3) and using that

$$\mathcal{F}[u(x+z)](\xi) = e^{i\xi\cdot z}\widehat{u}(\xi), \quad e^{it} + e^{-it} = 2\cos t$$

we obtain

$$\widehat{(-\Delta)^s u}(\xi) = \frac{c_{n,s}}{2}\left(\int_{\mathbb{R}^n} \frac{2 - e^{i\xi\cdot z} - e^{-i\xi\cdot z}}{|z|^{n+2s}}\, dz\right)\widehat{u}(\xi) = c_{n,s}\left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z\right)\widehat{u}(\xi).$$

Here $1 - \cos(\xi\cdot z) = O(|z|^2)$ near zero and is bounded at infinity, so the integral is finite. By rotation invariance and $w = |\xi| z$,

$$\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z = \int_{\mathbb{R}^n} \frac{1-\cos w_1}{(|w|/|\xi|)^{n+2s}}|\xi|^{-n}\, dw = |\xi|^{2s} J_{n,s}, \qquad J_{n,s} := \int_{\mathbb{R}^n} \frac{1-\cos w_1}{|w|^{n+2s}}\,\mathrm{d}w.$$

For completeness we evaluate the constant. The Gamma-function identity and the Gaussian integral

$$|w|^{-n-2s} = \frac{1}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{\frac{n}{2}+s-1} e^{-t|w|^2}\,\mathrm{d}t, \qquad \int_{\mathbb{R}^n}(1-\cos w_1)e^{-t|w|^2}\,dw = \left(\frac{\pi}{t}\right)^{n/2}(1-e^{-1/(4t)})$$

substituted into the definition of $J_{n,s}$ gives

$$\begin{aligned}
J_{n,s} &= \frac{1}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{\frac{n}{2}+s-1}\left[\int_{\mathbb{R}^n}(1-\cos w_1)e^{-t|w|^2}\,dw\right]dt \\
&= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{s-1}(1-e^{-1/(4t)})\,\mathrm{d}t \\
&= \frac{4^{-s}\pi^{n/2}}{\Gamma(\frac{n}{2}+s)}\int_0^\infty v^{-s-1}(1-e^{-v})\,\mathrm{d}v \\
&= \frac{4^{-s}\pi^{n/2}\Gamma(1-s)}{s\,\Gamma(\frac{n}{2}+s)}.
\end{aligned}$$

Formula (1.2) says precisely that $c_{n,s}J_{n,s} = 1$. This proves the desired result. □

Corollary 1.8. *If $u, v \in C_c^\infty(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} u\,(-\Delta)^s v\,\mathrm{d}x = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z.$$

In particular,

$$\int_{\mathbb{R}^n} u\,(-\Delta)^s u\,\mathrm{d}x = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{2s}|\widehat{u}(\xi)|^2\,\mathrm{d}\xi =: [u]^2_{H^s(\mathbb{R}^n)} \geq 0$$

Proof. By the singular integral definition,

$$\begin{aligned}
\int_{\mathbb{R}^n} u(-\Delta)^s v\ \mathrm{d}x &= c_{n,s}\lim_{\varepsilon\to 0}\iint_{|x-z|>\varepsilon}\frac{u(x)(v(x)-v(z))}{|x-z|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}z \\
&= \frac{c_{n,s}}{2}\lim_{\varepsilon\to 0}\iint_{|x-z|>\varepsilon}\frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}z,
\end{aligned}$$

where we interchanged the roles of x and z in order to get the last equality.

Since $u, v \in C_c^\infty(\mathbb{R}^n)$, the integrand is absolutely integrable. Thus we may let $\varepsilon \to 0$, obtaining the desired identity. The remaining claims follow immediately from Lemma 1.7. □

The following semigroup property follows immediately.

Corollary 1.9. *For $s, t > 0$ with $s + t \le 1$, and $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$(-\Delta)^s\big((-\Delta)^t u\big) = (-\Delta)^{s+t} u.$$

Proof. By Lemma 1.7, it holds $\widehat{v}(\xi) = |\xi|^{2t}\widehat{u}(\xi)$. Hence, by applying the inverse Fourier transform,

$$v(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2t}\widehat{u}(\xi)\,\mathrm{d}\xi.$$

Inserting this into the definition of the fractional Laplacian and using Fubini, we obtain

$$\begin{aligned}(-\Delta)^s v(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2t}\widehat{u}(\xi)\left(c_{n,s}\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z\right)\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2(s+t)}\widehat{u}(\xi)\,\mathrm{d}\xi\\ &= (-\Delta)^{s+t}u(x).\end{aligned}$$

□

Our normalization makes $(-\Delta)^s$ interpolate between the identity and $-\Delta$.

Lemma 1.10. *Let $u \in C_c^\infty(\mathbb{R}^n)$. Then, it holds for every $x \in \mathbb{R}^n$,*

$$\lim_{s\downarrow 0}(-\Delta)^s u(x) = u(x), \qquad \lim_{s\uparrow 1}(-\Delta)^s u(x) = -\Delta u(x).$$

Proof. Let $\omega_{n-1} = |\mathbb{S}^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. Then

$$c_{n,s} \sim \frac{2s}{\omega_{n-1}} \quad (s \downarrow 0), \qquad c_{n,s} \sim \frac{4n(1-s)}{\omega_{n-1}} \quad (s \uparrow 1).$$

To see it, in (1.2) use $\Gamma(1-s) \to 1$ as $s \downarrow 0$ and $\Gamma(1-s) \sim (1-s)^{-1}$ as $s \uparrow 1$, together with $\Gamma(1+\frac{n}{2}) = \frac{n}{2}\Gamma(\frac{n}{2})$.

Step 1: $s \downarrow 0$. Fix x and choose $R > 1$ so large that $u(x+z) = 0$ for $|z| > R$. We compute

$$(-\Delta)^s u(x) = c_{n,s}\ \mathrm{PV}\int_{B_R}\frac{u(x)-u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z + c_{n,s}u(x)\int_{\mathbb{R}^n\setminus B_R}|z|^{-n-2s}\,\mathrm{d}z =: I_{1,R} + I_{2,R}.$$

For $0 < s \le \frac{1}{4}$, the first integral is uniformly bounded since

$$|u(x) - u(x+z)| \le C\min\{|z|, 1\}.$$

Since $c_{n,s} \to 0$, this implies $I_{1,R} \to 0$ as $s \to 0$. For $I_{2,R}$, we get

$$I_{2,R} = c_{n,s}u(x)\int_{\mathbb{R}^n\setminus B_R}|z|^{-n-2s}\,\mathrm{d}z = c_{n,s}\frac{\omega_{n-1}}{2s}R^{-2s}u(x) \longrightarrow u(x),$$

by the asymptotics of $c_{n,s}$.

Step 2: $s \uparrow 1$. Let us fix $\delta \in (0,1)$. Note that by Taylor's formula

$$2u(x) - u(x+z) - u(x-z) = -z^T D^2u(x)z + \rho_x(z)|z|^2, \qquad |\rho_x(z)| \le C|z|^2 \quad \text{for} \quad |z| \le \delta.$$

We use (1.3) and split the integral at radius δ, namely

$$(-\Delta)^s u(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n \setminus B_\delta} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z \\ - \frac{c_{n,s}}{2} \int_{B_\delta} \frac{z^T D^2 u(x) z}{|z|^{n+2s}} \,\mathrm{d}z + \frac{c_{n,s}}{2} \int_{B_\delta} \rho_x(z) |z|^{-n+2-2s} \,\mathrm{d}z =: I_1 + I_2 + I_3.$$

Since $c_{n,s} \to 0$, we have that $I_1 \to 0$ as $s \to 1$. Moreover, by symmetry on the sphere, we have

$$\int_{\mathbb{S}^{n-1}} \theta_i \theta_j \,\mathrm{d}\theta = \frac{\omega_{n-1}}{n} \delta_{ij}.$$

Therefore, as $s \to 1$,

$$I_2 = -\frac{c_{n,s}}{2} \sum_{i,j} u_{ij}(x) \int_{B_\delta} \frac{z_i z_j}{|z|^{n+2s}} \,\mathrm{d}z \\ = -\frac{c_{n,s}\omega_{n-1}}{2n} \Delta u(x) \int_0^\delta r^{1-2s} \,\mathrm{d}r = -\frac{c_{n,s}\omega_{n-1}}{4n(1-s)} \delta^{2-2s} \Delta u(x) \to -\Delta u(x).$$

For I_3, we note that

$$|I_3| \le \frac{c_{n,s} C}{2} \int_{B_\delta} |z|^{-n+4-2s} \,\mathrm{d}z = \frac{c_{n,s} C \omega_{n-1}}{2(4-2s)} \delta^{4-2s} \to 0,$$

since $c_{n,s} \to 0$ as $s \to 1$. This concludes the proof. □

1.1. **The Caffarelli-Silvestre extension.** In this subsection, we prove that the nonlocal fractional Laplacian can be recast as a *local differential operator in one dimension higher*. This constitutes one of the most important properties of the fractional Laplacian and goes back to a famous paper by Caffarelli and Silvestre [CS07]. It is one of the most cited papers in partial differential equations of all times and it has many applications, as we will see below.

We first treat the case $s = \frac{1}{2}$.

Theorem 1.11. *Let $u \in C_c^\infty(\mathbb{R}^n)$, and let U be its bounded harmonic extension:*

$$\begin{cases} \Delta_{x,y} U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.8}$$

Then

$$(-\Delta)^{1/2} u(x) = -\partial_y U(x,0). \tag{1.9}$$

Proof. We apply Fourier transform in x to (1.8):

$$\partial_{yy} \widehat{U}(\xi, y) - |\xi|^2 \widehat{U}(\xi, y) = 0, \qquad \widehat{U}(\xi, 0) = \widehat{u}(\xi).$$

The bounded solution is $\widehat{U}(\xi, y) = e^{-|\xi| y} \widehat{u}(\xi)$. Thus $-\partial_y \widehat{U}(\xi, 0) = |\xi| \widehat{u}(\xi)$, which is exactly the Fourier transform of $(-\Delta)^{1/2} u$ by Theorem 1.7. Taking inverse Fourier transforms yields (1.9). □

Remark 1.12. The above theorem gives another justification for the notation $(-\Delta)^{1/2}$. Indeed, denote $Lu = -\partial_y U(x,0)$ and compute $L(Lu)$. Indeed, if U is the harmonic extension of u, then $-\partial_y U$ is the harmonic extension of Lu (since it coincides with Lu on $\{y = 0\}$ and it is harmonic). Thus,

$$L(Lu)(x) = -\left.\partial_y\right|_{y=0} (-\partial_y U) = \left.\partial_{yy}\right|_{y=0} U(x,y).$$

Now, since

$$\Delta_{x,y}U = \Delta_x U + \partial_{yy}U = 0,$$

we have that

$$L(Lu)(x) = -\Delta_x U(x,0) = -\Delta u(x).$$

That is, $L^2 u(x) = -\Delta u(x)$, which justifies the notation $L = \sqrt{-\Delta}$.

We will now generalize this result to general $s \in (0,1)$. This was achieved in the famous paper [CS07]. To do so, we set $a = 1-2s \in (-1,1)$. The *a-harmonic extension* of a function u is a solution U of

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} U\big) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) &= u(x) \quad \text{on } \mathbb{R}^n. \end{cases} \tag{1.10}$$

For $y > 0$, the equation is equivalently

$$\mathcal{L}_a U := \Delta_x U + \partial_{yy} U + \frac{a}{y}\partial_y U = 0.$$

Notice that $a = 0$ when $s = \frac{1}{2}$. For other values of s, y^a vanishes or becomes singular at $y = 0$; this is a degenerate or singular elliptic equation, rather than a uniformly elliptic one up to the boundary.

Lemma 1.13 (Poisson kernel for the weighted equation)**.** *Let*

$$P_s(z,y) = p_{n,s}\frac{y^{2s}}{(|z|^2+y^2)^{(n+2s)/2}}, \qquad p_{n,s} = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(s)}.$$

Then, for every $y > 0$,

$$P_s(\cdot, y) \geq 0, \qquad \int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = 1,$$

and P_s solves $\operatorname{div}_{x,y}(y^a \nabla_{x,y} P_s) = 0$ for $y > 0$.

Moreover, for $u \in C^{2s+\varepsilon}_{loc}(B_1)$ with $|u(x)| \leq C(1+|x|)^{2s-\varepsilon}$ for some $\varepsilon > 0$, the convolution

$$U(x,y) = \int_{\mathbb{R}^n} P_s(x-z,y)u(z)\,\mathrm{d}z \tag{1.11}$$

solves (1.10) with $U(\cdot,y) \to u$ locally uniformly in B_1 and a.e. in $\mathbb{R}^n \setminus B_1$ as $y \to 0$.

Moreover, (1.11) is the unique solution to (1.10) satisfying $|U(x,y)| \leq C(1+|(x,y)|)^{2s-\varepsilon}$.

Proof. The change of variables $z = yw$ gives

$$\int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = p_{n,s}\int_{\mathbb{R}^n}(1+|w|^2)^{-(n+2s)/2}\,\mathrm{d}w = p_{n,s}\frac{\pi^{n/2}\Gamma(s)}{\Gamma(\frac{n}{2}+s)} = 1.$$

In particular $P_s(z,y) = y^{-n}P_s(z/y,1)$, i.e. P_s is an approximate identity. To verify the PDE, set $r = (|z|^2+y^2)^{1/2}$, so that

$$P_s(z,y) = p_{n,s}y^{2s}r^{-(n+2s)}.$$

By Lemma 1.14 (see below), we have

$$\begin{aligned} &\left(\Delta_{z,y} + \frac{a}{y}\partial_y\right)(y^{2s}r^{-(n+2s)}) \\ &\quad = 2s(2s-1+a)y^{2s-2}r^{-(n+2s)} + (n+2s)(1-2s-a)y^{2s}r^{-(n+2s+2)} \\ &\quad = 0, \end{aligned}$$

since $a = 1 - 2s$. Thus,

$$\operatorname{div}(y^a \nabla P_s) = 0 \qquad \text{for } y > 0.$$

To prove that (1.11) satisfies the PDE, it suffices to observe that differentiation under the convolution is justified for $y > 0$. To prove that $U(\cdot, y) \to u$ locally uniformly in B_1 as $y \to 0$, we write

$$U(x,y) - u(x) = \int_{\mathbb{R}^n} P_s(z,y)\big(u(x-z) - u(x)\big)\,\mathrm{d}z.$$

Moreover, given $K \Subset B_1$, and $\delta > 0$ such that $\{x \in \mathbb{R}^n : \operatorname{dist}(x,K) < \delta\} \Subset B_1$, we define the modulus of continuity

$$\omega_u(\delta) = \sup_{\substack{x \in K \\ |z| \le \delta}} |u(x-z) - u(x)|.$$

Then

$$\sup_{x\in K} |U(x,y) - u(x)| \le \omega_u(\delta) + C\|(1+|x|)^{\varepsilon - 2s} u(x)\|_{L^\infty(\mathbb{R}^n)} \int_{|z|\ge\delta} P_s(z,y)(1+|z|)^{2s-\varepsilon}\,\mathrm{d}z.$$

Using the explicit expression for P_s,

$$\int_{|z|\ge\delta} P_s(z,y)(1+|z|)^{2s-\beta}\,\mathrm{d}z \le C y^{2s} \int_{|z|\ge\delta} \frac{(1+|z|)^{2s-\varepsilon}}{|z|^{n+2s}}\,\mathrm{d}z \le C(\delta) y^{2s}.$$

Letting $y \to 0$ gives

$$\limsup_{y \downarrow 0} \|U(\cdot,y) - u\|_{L^\infty(K)} \le \omega_u(\delta).$$

Since u is uniformly continuous in B_1, we have $\omega_u(\delta) \to 0$ as $\delta \downarrow 0$. Hence

$$\lim_{y\downarrow 0} \|U(\cdot,y) - u\|_{L^\infty(K)} = 0,$$

as desired.

The convergence $U(\cdot,y) \to u$ a.e. in $\mathbb{R}^n \setminus B_1$ follows from the Lebesgue differentiation theorem and since P_s is an approximate identity and $u \in L^1_{loc}(\mathbb{R}^n)$.

Finally, uniqueness of U can be proved by an argument based on the maximum principle applied to the operator $\operatorname{div}_{x,y}\big(y^a \nabla_{x,y}\big)$. Indeed, one can prove that the only solution W to

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} W\big) & = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ W(x,0) & = 0 \quad \text{on } \mathbb{R}^n, \\ |W(x,y)| & \le C(1+|(x,y)|)^{2s-\varepsilon} \quad \text{in } \mathbb{R}^{n+1}_+ \end{cases}$$

is given by $W \equiv 0$.

To see it, we consider $\gamma \in (2s - \varepsilon, 2s)$ and consider

$$\Phi(x,y) = |(x,y)|^\gamma + \frac{n}{2s-\gamma} y^\gamma.$$

Since $a = 1 - 2s$, by Lemma 1.14 (see below), we get

$$\mathcal{L}_a(|(x,y)|^\gamma) = \gamma(\gamma + n - 2s)|(x,y)|^{\gamma-2}, \qquad \mathcal{L}_a(y^\gamma) = \gamma(\gamma - 2s) y^{\gamma-2}.$$

Consequently, using $y \le |(x,y)|$ and $\gamma < 2$,

$$\mathcal{L}_a \Phi = \gamma(\gamma + n - 2s)|(x,y)|^{\gamma-2} - n\gamma y^{\gamma-2} \le \gamma(\gamma - 2s)|(x,y)|^{\gamma-2} < 0.$$

Let us now fix $R > 1$ sufficiently large and $\delta > 0$. Set $B_R^+ = B_R \cap \{y > 0\} \subset \mathbb{R}^{n+1}_+$. Then the previous observation implies

$$\mathcal{L}_a(W - \delta\Phi) = -\delta \mathcal{L}_a \Phi > 0 \quad \text{in } B_R^+.$$

Moreover, since $\gamma > 2s - \varepsilon$, for every sufficiently large R,

$$\delta R^\gamma > C(1 + R^\beta).$$

and therefore,

$$\begin{aligned} W - \delta\Phi \le C(1 + R^{2s-\varepsilon}) - \delta R^\gamma \le 0 \quad &\text{on } \partial B_R^+ \setminus \{y = 0\} \\ W - \delta\Phi = -\delta|x|^\gamma \le 0 \quad &\text{on } \partial B_R^+ \cap \{y = 0\} \end{aligned}$$

The maximum principle for $\mathcal{L}_a$ applied in B_R^+ therefore yields

$$W \le \delta\Phi \qquad \text{in } B_R^+.$$

Indeed, a positive interior maximum at $(x_0, y_0) \in \mathbb{R}^{n+1}_+$ would satisfy

$$\nabla(W - \delta\Phi)(x_0, y_0) = 0, \qquad \Delta(W - \delta\Phi)(x_0, y_0) \le 0,$$

and hence

$$\mathcal{L}_a(W - \delta\Phi)(x_0, y_0) \le 0,$$

a contradiction.

Letting $R \to \infty$ yields $W \le \delta\Phi$ in $\mathbb{R}^{n+1}_+$. Letting then $\delta \downarrow 0$ gives $W \le 0$ in $\mathbb{R}^{n+1}_+$. Applying the same argument to $-W$ yields $W \ge 0$, hence $W \equiv 0$. □

In the proof we used the following elementary lemma.

Lemma 1.14. *Let $a = 1 - 2s$ and*

$$\mathcal{L}_a = \Delta_{x,y} + \frac{a}{y}\partial_y.$$

For $z \in \mathbb{R}^n$, set $R = |(x, y) - (z, 0)|$, where $(x, y) \in \mathbb{R}^{n+1}_+$. Then, for every $\lambda, \mu \in \mathbb{R}$,

$$\mathcal{L}_a(y^\mu R^\lambda) = \mu(\mu + a - 1)y^{\mu-2}R^\lambda + \lambda(\lambda + n + a - 1 + 2\mu)y^\mu R^{\lambda-2}.$$

Proof. Since

$$\Delta_{x,y} R^\lambda = \lambda(\lambda + n - 1)R^{\lambda-2}, \qquad \partial_y R^\lambda = \lambda y R^{\lambda-2},$$

we obtain

$$\mathcal{L}_a(R^\lambda) = \lambda(\lambda + n + a - 1)R^{\lambda-2}.$$

Moreover,

$$\mathcal{L}_a(y^\mu) = \mu(\mu + a - 1)y^{\mu-2},$$

and

$$2\nabla_{x,y}(y^\mu) \cdot \nabla_{x,y}(R^\lambda) = 2\mu\lambda y^\mu R^{\lambda-2}.$$

The claim follows from the product rule

$$\mathcal{L}_a(fg) = f\mathcal{L}_a g + g\mathcal{L}_a f + 2\nabla_{x,y} f \cdot \nabla_{x,y} g.$$

□

Theorem 1.15 (Caffarelli–Silvestre extension)**.** *Let $0 < s < 1$, $a = 1 - 2s$, $u \in C^{2s+\varepsilon}_{loc}(B_1)$ with $|u(x)| \leq C(1+|x|)^{2s-\varepsilon}$ for some $\varepsilon > 0$, and let U be unique the solution to*

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} U\big) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x) & \text{on } \mathbb{R}^n \end{cases}$$

with $|U(x,y)| \leq C(1+|(x,y)|)^{2s-\varepsilon}$ given by the weighted Poisson extension (1.11)*. Then*

$$(-\Delta)^s u(x) = -d_s \lim_{y \downarrow 0} y^a \partial_y U(x,y), \qquad d_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}. \tag{1.12}$$

Moreover, (1.12) *remains true for any $u \in C^{2s+\varepsilon}_{loc}(B_1) \cap L^1_{2s}(\mathbb{R}^n)$ with U given by* (1.11)*.*

In particular, note that for $s = \frac{1}{2}$, it holds $a = 0$ and $d_s = 1$, so we recover Theorem 1.11.

Proof. Use $\int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = 1$ and evenness of $P_s(z,y)$ in z to write

$$U(x,y) - u(x) = p_{n,s} \int_{\mathbb{R}^n} \frac{y^{2s}\big(u(x-z) - u(x)\big)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z = \frac{p_{n,s}}{2} \int_{\mathbb{R}^n} \frac{y^{2s}\big(u(x+z) + u(x-z) - 2u(x)\big)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z.$$

Differentiate with respect to y and multiply by $y^a = y^{1-2s}$ to get

$$\begin{aligned} y^a \partial_y U(x,y) &= s p_{n,s} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z \\ &\quad - \frac{(n+2s)p_{n,s}}{2} y^2 \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{(|z|^2+y^2)^{(n+2s+2)/2}}\,\mathrm{d}z \\ &= I_1 + I_2. \end{aligned} \tag{1.13}$$

Fix $x \in B_1$ and choose $\delta > 0$ such that $B_{2\delta}(x) \subset B_1$. Since $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(B_1)$, we may choose

$$2s < \alpha < \min\{2s+\varepsilon, 2\}$$

such that

$$|u(x+z) + u(x-z) - 2u(x)| \leq C_x |z|^\alpha, \qquad |z| < \delta.$$

On the other hand, the growth assumption $|u(x)| \leq C(1+|x|)^{2s-\varepsilon}$ yields

$$|u(x+z) + u(x-z) - 2u(x)| \leq C_x (1+|z|)^{2s-\varepsilon}, \qquad |z| \geq \delta.$$

Consequently,

$$\frac{|u(x+z) + u(x-z) - 2u(x)|}{(|z|^2+y^2)^{(n+2s)/2}} \leq C_x \begin{cases} |z|^{\alpha-n-2s}, & |z| < \delta, \\ (1+|z|)^{2s-\varepsilon}|z|^{-n-2s}, & |z| \geq \delta. \end{cases}$$

The right-hand side belongs to $L^1(\mathbb{R}^n)$. Note that if $u \in L^1_{2s}(\mathbb{R}^n)$, we still have that $|u(x+z) + u(x-z) - 2u(x)||z|^{-n-2s} \in L^1(\mathbb{R}^n)$.

Hence, by dominated convergence, we get

$$\lim_{y \downarrow 0} I_1(y) = s p_{n,s} \int_{\mathbb{R}^n} \frac{u(x-z) + u(x+z) - 2u(x)}{|z|^{n+2s}}\,\mathrm{d}z = -2s p_{n,s}\,\mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z.$$

Similarly, for I_2, we observe that $\frac{y^2}{|z|^2+y^2} \leq 1$, and therefore, dominated convergence implies that

$$\lim_{y \downarrow 0} I_2(y) = 0.$$

We therefore obtain from (1.13),

$$-\lim_{y\downarrow 0} y^a \partial_y U(x,y) = 2sp_{n,s}\ \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z = \frac{2sp_{n,s}}{c_{n,s}}(-\Delta)^s u(x).$$

The constants satisfy

$$\frac{c_{n,s}}{2sp_{n,s}} = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)} = d_s,$$

which is precisely (1.12). □

Note that it is also possible to prove Theorem 1.15 via an adapation of the Fourier approach from Theorem 1.11. In that case, one has to solve a slightly more complicated ODE whose solutions are given as Bessel functions. Working out this alternative proof is a nice exercise.

1.2. **The nonlocal Poisson kernel representation.** A major application of the Caffarelli-Silvestre extension is the following Poisson kernel representation of the fractional Laplacian.

Theorem 1.16 (Poisson kernel for the ball)**.** *Let $0 < s < 1$ and $r > 0$. The unique solution $u \in C^{2s+\varepsilon}_{loc}(B_1) \cap L^1_{2s}(\mathbb{R}^n) \cap C(\overline{B_1})$ of*

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } B_r, \\ u &= g \quad \text{in } \mathbb{R}^n \setminus B_r \end{cases}$$

for $g \in L^1_{2s}(\mathbb{R}^n) \cap C(\partial B_r)$, is given by

$$u(x) = \int_{\mathbb{R}^n \setminus B_r} \mathcal{P}_{r,s}(x,z) g(z)\,\mathrm{d}z, \qquad x \in B_r, \tag{1.14}$$

where

$$\mathcal{P}_{r,s}(x,z) = \kappa_{n,s}\frac{(r^2-|x|^2)^s}{(|z|^2-r^2)^s|x-z|^n}, \qquad \kappa_{n,s} = \frac{\Gamma(\frac{n}{2})\sin(\pi s)}{\pi^{n/2+1}}.$$

Moreover,

$$\mathcal{P}_{r,s}(x,z) > 0, \qquad \int_{\mathbb{R}^n\setminus B_r} \mathcal{P}_{r,s}(x,z)\,\mathrm{d}z = 1.$$

Here, $g \in C(\partial B_r)$ is understood in the following sense:

$$\lim_{\substack{z\to x_0 \\ z\in\mathbb{R}^n\setminus B_r}} g(z) = g(x_0) \qquad \text{for every } x_0 \in \partial B_r.$$

Before we can prove this result, we need to give several auxilary lemmas. The first one is to find an explicit s-harmonic function in the half-space, which is a direct application of the Caffarelli-Silvestre extension.

Proposition 1.17. *Let $e \in \mathbb{S}^{n-1}$ and define*

$$u(x) = (x\cdot e)_+^s, \qquad t_+ = \max\{t, 0\}.$$

Then

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } \{x\cdot e > 0\}, \\ u &= 0 \quad \text{in } \{x\cdot e \le 0\}. \end{cases}$$

Proof. We use the Caffarelli–Silvestre extension. Write $t = x \cdot e$ and consider

$$U(x,y) = \left(\frac{\sqrt{t^2+y^2}+t}{2} \right)^s.$$

First, we claim that

$$\widetilde{\mathcal{L}}_a U := \partial_{tt} U + \partial_{yy} U + \frac{1-2s}{y} \partial_y U = 0. \tag{1.15}$$

Indeed, set

$$R = \sqrt{t^2+y^2}, \qquad v = R + t, \qquad U = 2^{-s} v^s.$$

Since $v = R + t$, and $\partial_t R = t R^{-1}$ we have

$$\partial_t v = \frac{t}{R} + 1, \qquad \partial_y v = \frac{y}{R}, \qquad |\nabla_{t,y} v|^2 = \frac{t^2}{R^2} + \frac{y^2}{R^2} + 1 + \frac{2t}{R} = \frac{2(R+t)}{R} = \frac{2v}{R}.$$

Moreover, by Lemma 1.14 (applied with $n = 1$), and using that $\widetilde{\mathcal{L}}_a t = 0$ since t is linear and independent of y,

$$\widetilde{\mathcal{L}}_a v = \widetilde{\mathcal{L}}_a R = (1+a) R^{-1}.$$

Hence, by the chain rule

$$\widetilde{\mathcal{L}}_a U = 2^{-s} s v^{s-1} \left(\widetilde{\mathcal{L}}_a v + (s-1) \frac{|\nabla_{t,y} v|^2}{v} \right) = \frac{2^{-s} s v^{s-1}}{R} (1 + a + 2s - 2) = 0.$$

This proves (1.15).

Since U depends on x only through $t = x \cdot e$ and $|e| = 1$, this is equivalent to

$$\mathcal{L}_a U = y^a \left(\Delta_x U + \partial_{yy} U + \frac{a}{y} \partial_y U \right) = y^a \left(\partial_{tt} U + \partial_{yy} U + \frac{1-2s}{y} \partial_y U \right) = 0 \quad \text{for } y > 0,$$

where we used that

$$\partial_{x_i} U = e_i \partial_t U, \qquad \partial_{x_i x_i} U = e_i^2 \partial_{tt} U, \qquad \Delta_x U = \sum_{i=1}^n e_i^2 \partial_{tt} U = |e|^2 \partial_{tt} U = \partial_{tt} U.$$

Moreover, taking $y \downarrow 0$,

$$U(x,0) = \left(\frac{|t|+t}{2} \right)^s = t_+^s = u(x).$$

Thus U is the weighted Poisson extension of u in the natural class of solutions with growth of order s.

We now compute the conormal derivative. First, we recall

$$\partial_y U = \frac{sy}{2R} \left(\frac{R+t}{2} \right)^{s-1}.$$

Therefore,

$$y^{1-2s} \partial_y U = \frac{s y^{2-2s}}{2r} \left(\frac{R+t}{2} \right)^{s-1}.$$

If $t > 0$, the factor $\left(\frac{R+t}{2}\right)^{s-1} / R$ remains bounded as $y \downarrow 0$. Since $s < 1$,

$$\lim_{y \downarrow 0} y^{1-2s} \partial_y U(x,y) = 0.$$

By the Caffarelli–Silvestre extension formula,

$$(-\Delta)^s u(x) = -d_s \lim_{y\downarrow 0} y^{1-2s}\partial_y U(x,y) = 0$$

whenever $x \cdot e > 0$. □

A key motivation to guess the right function in the proof comes from complex analysis, where it is a well-known fact that

$$U(t,y) = \operatorname{Re}\sqrt{t+iy} = \sqrt{\frac{\sqrt{t^2+y^2}+t}{2}}$$

is harmonic and has exactly the desired properties for $s = \frac{1}{2}$.

The following lemma establishes the action of the Kelvin transform on the fractional Laplacian. The Kelvin transform is a common tool in PDEs and it is also comonly used for the Laplacian. Via the Caffarelli-Silvestre extension we can lift it to the fractional Laplacian. The Kelvin transform turns the geometry of a space "inside out", while preserving the underlying PDE. It will allow us to construct a solution to a nonlocal equation in a ball from half-space solution in Proposition 1.17.

Lemma 1.18. *Let $0 < s < 1$, $a = 1-2s$, and $z \in \mathbb{R}^n$. Define*

$$I_z(X) = \frac{X-(z,0)}{|X-(z,0)|^2}, \qquad X = (x,y) \in \mathbb{R}^{n+1}_+.$$

If U satisfies $\operatorname{div}(y^a \nabla U) = 0$ in $\mathbb{R}^{n+1}_+$, then

$$\widetilde{U}(X) := |X-(z,0)|^{2s-n} U(I_z(X))$$

satisfies $\operatorname{div}(y^a \nabla \widetilde{U}) = 0$ in $\mathbb{R}^{n+1}_+$.

Moreover, let $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^n$. The fractional Kelvin transform

$$\mathcal{K}_z u(x) = |x-z|^{2s-n} u(T_z(x)) \in C^{2s+\varepsilon}_{loc}(T_z^{-1}(\Omega)\setminus\{z\}) \cap L^1_{2s}(\mathbb{R}^n), \qquad T_z(x) = \frac{x-z}{|x-z|^2},$$

satisfies

$$(-\Delta)^s(\mathcal{K}_z u)(x) = |x-z|^{-n-2s}(-\Delta)^s u(T_z(x)) \quad \text{in } T_z^{-1}(\Omega)\setminus\{z\}. \tag{1.16}$$

Proof. A direct computation gives

$$\mathcal{L}_a \widetilde{U}(X) = |X-(z,0)|^{2s-n-4}(\mathcal{L}_a U)(I_z(X)) = 0.$$

Indeed: Set

$$q = X-(z,0), \qquad R = |q|, \qquad Y = I_z(X) = \frac{q}{R^2}.$$

Then $\widetilde{U}(X) = R^{2s-n}U(Y)$. Moreover, note that

$$DI_z(X) = R^{-2}\left(I - 2\frac{q\otimes q}{R^2}\right).$$

Consequently,

$$\Delta(U\circ I_z) = R^{-4}\left[(\Delta U)(Y) - 2(n-1)q\cdot\nabla U(Y)\right].$$

Moreover, writing $\eta = y/R^2$ for the last coordinate of Y, we obtain

$$\frac{a}{y}\partial_y(U\circ I_z) = R^{-4}\left[\frac{a}{\eta}\partial_\eta U(Y) - 2a\, q\cdot\nabla U(Y)\right].$$

Combining these identities yields

$$\mathcal{L}_a(U \circ I_z) = R^{-4} \left[(\mathcal{L}_a U)(Y) + 2(2s-n)\, q \cdot \nabla U(Y) \right].$$

On the other hand, by Lemma 1.14,

$$\mathcal{L}_a(R^{2s-n}) = (2s-n)(2s-n+n+a-1)R^{2s-n-2} = 0,$$

and

$$\nabla(R^{2s-n}) \cdot \nabla(U \circ I_z) = -(2s-n)R^{2s-n-4} q \cdot \nabla U(Y).$$

Thus, by the product rule,

$$\mathcal{L}_a \widetilde{U} = R^{2s-n}\mathcal{L}_a(U \circ I_z) + 2\nabla(R^{2s-n}) \cdot \nabla(U \circ I_z) + (U \circ I_z)\mathcal{L}_a(R^{2s-n}) = R^{2s-n-4}(\mathcal{L}_a U)(Y).$$

This proves

$$\mathcal{L}_a \widetilde{U}(X) = R^{2s-n-4}(\mathcal{L}_a U)(I_z(X)) = 0.$$

Moreover, taking the weighted normal derivative,

$$\lim_{y \downarrow 0} y^a \partial_y \widetilde{U}(x,y) = |x-z|^{-n-2s} \lim_{y \downarrow 0} y^a \partial_y U(T_z(x), y).$$

Indeed, since

$$\partial_y Y = R^{-2} e_{n+1} - 2yR^{-4} q,$$

the chain rule gives

$$\partial_y \widetilde{U} = (2s-n) y R^{2s-n-2} U(Y) + R^{2s-n-2} \partial_\eta U(Y) - 2yR^{2s-n-4} q \cdot \nabla U(Y).$$

Multiplying by y^a and using $y^a = R^{2a}\eta^a$, we obtain

$$y^a \partial_y \widetilde{U} = R^{2s-n+2a-2} \eta^a \partial_\eta U(Y) + o(1) \qquad \text{as } y \downarrow 0,$$

for fixed $x \neq z$. Here the remaining terms vanish by the boundary regularity of the Poisson extension and the fact that $a > -1$. Since

$$Y \longrightarrow (T_z(x), 0), \qquad R \longrightarrow |x-z|,$$

we indeed conclude that

$$\lim_{y \downarrow 0} y^a \partial_y \widetilde{U}(x,y) = |x-z|^{-n-2s} \lim_{\eta \downarrow 0} \eta^a \partial_\eta U(T_z(x), \eta).$$

The identity (1.16) now follows from the fact that $\widetilde{U}(x,0) = \mathcal{K}_z u(x)$ and the Caffarelli–Silvestre extension (see Theorem 1.15), once we fix u as in the statement of the theorem, let U be defined via (1.11) from u and show that $\widetilde{U}$ is given via (1.11) from $\mathcal{K}_z u$.

(Note that $\mathcal{K}_z u \in L^1_{2s}(\mathbb{R}^n)$, as we compute below, but in general this function does not satisfy the pointwise growth condition under which we have uniqueness for the extended problem in Theorem 1.15).

To see it, we compute

$$\widetilde{U}(X) = R^{2s-n}U(Y) = p_{n,s} \frac{y^{2s}}{R^{n+2s}} \int_{\mathbb{R}^n} \frac{u(t)}{|Y-(t,0)|^{n+2s}}\, dt = p_{n,s} y^{2s} \int_{\mathbb{R}^n} \frac{|w-z|^{2s-n} u(T_z(w))}{|X-(w,0)|^{n+2s}}\, dw,$$

where we used the transformation $t = T_z(w)$ (note that $\mathrm{d}t = |w-z|^{-2n}\,\mathrm{d}w$ and $|Y-(t,0)| = \frac{|X-(w,0)|}{R|w-z|}$) and that U is given via (1.11) from u.

Note that it is easy to see that $\mathcal{K}_z u \in C^{2s+\varepsilon}_{loc}(T_z^{-1}(\Omega) \setminus \{z\})$ since $T_z, |\cdot - z|^{2s-n} \in C^\infty_{loc}(\mathbb{R}^n \setminus \{z\})$. Moreover, since

$$\int_{\mathbb{R}^n} \frac{|\mathcal{K}_z u(x)|}{1+|x-z|^{n+2s}}\, dx = \int_{\mathbb{R}^n} \frac{|x-z|^{2s-n}|u(T_z(x))|}{1+|x-z|^{n+2s}}\, dx = \int_{\mathbb{R}^n} \frac{|t|^{n-2s}|u(t)|\,|t|^{-2n}}{1+|t|^{-n-2s}}\, dt = \int_{\mathbb{R}^n} \frac{|u(t)|}{1+|t|^{n+2s}}\, dt,$$

where we used the transformation $t = T_z(x)$ (note that $\mathrm{d}x = |t|^{-2n}\,\mathrm{d}t$ and $|x-z| = |t|^{-1}$), and since $u \in L^1_{2s}(\mathbb{R}^n)$, we also have $\mathcal{K}_z u \in L^1_{2s}(\mathbb{R}^n)$. This concludes the proof. □

Remark 1.19. Note that $u \equiv 1$ satisfies $(-\Delta)^s u \equiv 0$. Note that $\mathcal{K}_0 u(x) = |x|^{2s-n}$. Hence, it holds $(-\Delta)^s(|x|^{2s-n}) = 0$ in $\mathbb{R}^n \setminus 0$. This function is the nonlocal analog of the fundamental solution for the Laplacian, given by $x \mapsto |x|^{2-n}$ for $n \geq 2$.

The following lemma constructs the torsion function for the fractional Laplacian.

Lemma 1.20. *Let $0 < s < 1$ and*

$$w(x) = (1-|x|^2)_+^s.$$

Then

$$(-\Delta)^s w(x) = Q_{n,s} \quad \text{in } B_1, \qquad \text{where} \qquad Q_{n,s} = 2^{2s}\frac{\Gamma(1+s)\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2})}. \tag{1.17}$$

Proof. Recall that $(-\Delta)^s(x\cdot e)_+^s = 0$ in $\{x\cdot e > 0\}$. Fix $\theta \in \mathbb{S}^{n-1}$ and apply Lemma 1.18 to

$$u(x) = \left(-\theta\cdot x - \frac{1}{2}\right)_+^s,$$

which satisfies

$$(-\Delta)^s u = 0 \quad \text{in } \left\{-\theta\cdot x > \frac{1}{2}\right\}$$

by Proposition 1.17. Since

$$-\theta\cdot T_\theta(x) - \frac{1}{2} = \frac{1-\theta\cdot x}{|x-\theta|^2} - \frac{1}{2} = \frac{1-|x|^2}{2|x-\theta|^2},$$

we have that $T_\theta(B_1) = \{-\theta\cdot x > \frac{1}{2}\}$ and conclude that

$$M_\theta(x) = \frac{(1-|x|^2)_+^s}{|x-\theta|^n}$$

satisfies $(-\Delta)^s M_\theta = 0$ in B_1.

On the other hand, applying the Poisson formula for the Laplacian (recall that the Poisson kernel in B_1 is given by $\frac{1-|x|^2}{|\mathbb{S}^{n-1}||x-\theta|^n}$) to the harmonic function $h(x) = x_i$, whose boundary values are $h(\theta) = \theta_i$, gives

$$x_i = \frac{1-|x|^2}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{\theta_i}{|x-\theta|^n}\,\mathrm{d}\sigma(\theta).$$

Consequently, recalling that $w(x) = (1-|x|^2)_+^s$, we obtain, for $x \in B_1$,

$$\partial_i w(x) = -2sx_i(1-|x|^2)^{s-1} = -\frac{2s(1-|x|^2)^s}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{\theta_i}{|x-\theta|^n}\,\mathrm{d}\sigma(\theta) = -\frac{2s}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\theta_i M_\theta(x)\,\mathrm{d}\sigma(\theta).$$

Since each M_θ satisfied $(-\Delta)^s M_\theta = 0$ in B_1 and differentiation commutes with $(-\Delta)^s$, this implies

$$\partial_i(-\Delta)^s w = (-\Delta)^s(\partial_i w) = -\frac{2s}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\theta_i(-\Delta)^s M_\theta(x)\,d\sigma(\theta) = 0 \qquad \text{in } B_1.$$

Thus $(-\Delta)^s w$ is constant in B_1.

Note that the first and second identity above can be justified by using dominated convergence, Fubini's theorem with the help of Proposition 1.5 and relying on $w, \partial_i w \in C^\infty_{loc}(B_1)\cap L^1_{2s}(\mathbb{R}^n)$, as well as $M_\theta \in C^\infty_{loc}(B_1)\cap L^1(\mathbb{R}^n)$ with $\sup_\theta \|M_\theta\|_{L^1(\mathbb{R}^n)} < \infty$. We skip the details.

To determine the constant, evaluate at the origin:

$$(-\Delta)^s w(0) = c_{n,s}|\mathbb{S}^{n-1}| \left(\int_0^1 \frac{1-(1-t^2)^s}{t^{1+2s}} \, \mathrm{d}t + \frac{1}{2s} \right) = \frac{c_{n,s}|\mathbb{S}^{n-1}|}{2} B(1-s,s) = 2^{2s} \frac{\Gamma(1+s)\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2})}.$$

Here we used integration by parts and the Gamma reflection formula. □

Now we are in a position to establish several elementary properties of the nonlocal Poisson kernel in the ball.

Lemma 1.21. *Let $0 < s < 1$ and $r > 0$. Extending $\mathcal{P}_{r,s}(\cdot, z)$ by zero outside B_r, we have*

$$(-\Delta)^s \mathcal{P}_{r,s}(\cdot, z)(x) = \frac{c_{n,s}}{|x-z|^{n+2s}}, \qquad x \in B_r. \tag{1.18}$$

Moreover,

$$\mathcal{P}_{r,s}(x,z) > 0, \qquad \int_{\mathbb{R}^n \setminus B_r} \mathcal{P}_{r,s}(x,z) \, \mathrm{d}z = 1.$$

Proof. By scaling, it suffices to consider $r = 1$. Fix $z \notin \overline{B_1}$ and set

$$D = |z|^2 - 1, \qquad R = D^{-1}, \qquad c = -z/D.$$

Note that T_z maps B_1 onto $B_R(c)$, and

$$R^2 - |T_z(x) - c|^2 = \frac{1-|x|^2}{D|x-z|^2}.$$

By Lemma 1.20 and scaling,

$$(-\Delta)^s (R^2 - |\xi - c|^2)_+^s = Q_{n,s} \qquad \text{in } B_R(c).$$

Applying the Kelvin transform yields

$$(-\Delta)^s \left(\frac{(1-|x|^2)_+^s}{|x-z|^n} \right) = Q_{n,s} \frac{D^s}{|x-z|^{n+2s}} \qquad \text{in } B_1.$$

Since

$$\frac{c_{n,s}}{Q_{n,s}} = \frac{\Gamma(\frac{n}{2}) \sin(\pi s)}{\pi^{n/2+1}} = \kappa_{n,s},$$

we obtain (1.18).

Positivity is immediate. To verify normalization, we use the classical identity

$$\int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}\sigma(\theta)}{|x-t\theta|^n} = \frac{|\mathbb{S}^{n-1}| t^{2-n}}{t^2 - |x|^2}, \qquad t > 1.$$

Writing $\rho = 1 - |x|^2$, polar coordinates give

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_1} \mathcal{P}_{1,s}(x,z) \, \mathrm{d}z &= \kappa_{n,s} |\mathbb{S}^{n-1}| \rho^s \int_1^\infty \frac{t \, \mathrm{d}t}{(t^2-1)^s (t^2 - |x|^2)} \\ &= \frac{\kappa_{n,s}|\mathbb{S}^{n-1}|}{2} B(1-s,s) = 1. \end{aligned}$$

The general case follows by scaling. □

Finally, we are in a position to prove Theorem 1.16.

Proof of Theorem 1.16. Define u by the formula (1.14) in B_r and set $u = g$ outside B_r. Writing

$$u_1 = u\mathbb{1}_{B_r}, \qquad u_2 = g\mathbb{1}_{\mathbb{R}^n\setminus B_r},$$

Lemma 1.21 gives, for $x \in B_r$,

$$(-\Delta)^s u_1(x) = \int_{\mathbb{R}^n\setminus B_r} g(z)(-\Delta)^s\big(\mathcal{P}_{r,s}(x,z)\mathbb{1}_{B_r}(x)\big)g(z)\,\mathrm{d}z = c_{n,s}\int_{\mathbb{R}^n\setminus B_r}\frac{g(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

On the other hand, since $u_2 \equiv 0$ in B_r,

$$(-\Delta)^s u_2(x) = -c_{n,s}\int_{\mathbb{R}^n\setminus B_r}\frac{u_2(z)}{|x-z|^{n+2s}}\,\mathrm{d}z = -c_{n,s}\int_{\mathbb{R}^n\setminus B_r}\frac{g(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

Hence

$$(-\Delta)^s u = (-\Delta)^s u_1 + (-\Delta)^s u_2 = 0 \qquad \text{in } B_r.$$

Next, we show that the assumed continuity of g implies

$$\lim_{\substack{x\to x_0\\ x\in B_r}} u(x) = g(x_0), \qquad x_0 \in \partial B_r.$$

To do so, we fix $x_0 \in \partial B_r$ and note first that for any $\varepsilon > 0$ there is $\delta > 0$ such that if $z \in B_r^c$ is such that $|x_0 - z| < \delta$, then $|g(z) - g(x_0)| < \varepsilon$.

Since $\mathcal{P}_{r,s}$ integrates to one,

$$\begin{aligned}|u(x) - g(x_0)| &\le \int_{B_r^c\cap\{|x_0-z|<\delta\}}\mathcal{P}_{r,s}(x,z)|g(z)-g(x_0)|\,\mathrm{d}z + \int_{B_r^c\cap\{|x_0-z|>\delta\}}\mathcal{P}_{r,s}(x,z)|g(z)-g(x_0)|\,\mathrm{d}z\\ &=: I_1 + I_2.\end{aligned}$$

Let us now fix $\varepsilon > 0$. Then, for I_1, by the continuity of g, it holds

$$I_1 \le \varepsilon\int_{\mathbb{R}^n\setminus B_r}\mathcal{P}_{r,s}(x,z)\,\mathrm{d}z = \varepsilon.$$

For I_2, we take $|x-x_0| < \frac{\delta}{2}$, which means that for any $|z-x_0| \ge \delta$, we have $|x-z| \ge |z-x_0| - |x-x_0| \ge \frac{\delta}{2}$. Moreover, we have $|x-z| \ge |z| - r$ and therefore,

$$1 + |z| = 1 + r + (|z| - r) \le 1 + r + |x-z| \le \left(1 + \frac{2(1+r)}{\delta}\right)|x-z|.$$

This implies

$$\begin{aligned}I_2(x) &\le C(r^2 - |x|^2)^s\int_{B_r^c\cap\{|z-x_0|\ge\delta\}}\frac{|g(z)-g(x_0)|}{(|z|^2-r^2)^s|x-z|^n}\,dz\\ &\le C(r^2-|x|^2)^s\int_{B_r^c}\frac{|g(z)-g(x_0)|}{(|z|^2-r^2)^s(1+|z|)^n}\,dz.\end{aligned}$$

Since $g \in C(\partial B_r)$, there is $\eta > 0$ such that $|g| \le M$ in $B_{r+\eta}\setminus B_r$ for some $M > 0$, and therefore,

$$\int_{B_{r+\eta}\setminus B_r}\frac{|g(z)-g(x_0)|}{(|z|^2-r^2)^s(1+|z|)^n}\,dz \le M\int_r^{r+\eta}\frac{\rho^{n-1}}{(\rho^2-r^2)^s(1+\rho)^n}\,\mathrm{d}\rho \le CM\int_r^{r+\eta}(\rho-r)^{-s}\,\mathrm{d}\rho < \infty.$$

Moreover, since $g \in L^1_{2s}(\mathbb{R}^n)$,

$$\int_{|z|\ge r+\eta}\frac{|g(z)-g(x_0)|}{(|z|^2-r^2)^s(1+|z|)^n}\,dz \le C\int_{\mathbb{R}^n}\frac{|g(z)|+|g(x_0)|}{1+|z|^{n+2s}}\,dz$$

$$\leq C \left(\|g\|_{L^1_{2s}(\mathbb{R}^n)} + |g(x_0)| \int_{\mathbb{R}^n} \frac{dz}{1+|z|^{n+2s}} \right) < \infty.$$

Thus, the integral in I_2 converges. Hence, we have $I_2(x) \to 0$ as $x \to x_0$. Since $\varepsilon > 0$ was arbitrary, we conclude that $u(x) \to g(x_0)$ as $x \to x_0$, as desired.

Uniqueness can be proved as follows. If there are two solutions $u, v \in C(\overline{B_1})$ then $w = u - v$ satisfies

$$(-\Delta)^s w = 0 \quad \text{in } B_r, \qquad w = 0 \quad \text{in } B_r^c.$$

If w attains a positive maximum in B_1, then

$$0 = (-\Delta)^s w(x_0) = c_{n,s} \operatorname{PV} \int_{\mathbb{R}^n} \frac{w(x_0) - w(z)}{|x_0 - z|^{n+2s}} \, \mathrm{d}z \geq c_{n,s} \int_{\mathbb{R}^n \setminus B_r} \frac{w(x_0)}{|x_0 - z|^{n+2s}} \, \mathrm{d}z > 0,$$

a contradiction. Similarly, we can rule out the existence of a negative minimum. Hence $w \equiv 0$, and therefore, $u \equiv v$, as desired. □

In the proof of uniqueness, we have used the following variant of the maximum principle for the fractional Laplacian.

Lemma 1.22. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap C(\overline{\Omega}) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Assume that*

$$\begin{cases} (-\Delta)^s u & \geq 0 \quad \text{in } \Omega, \\ u & \geq 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, $u \geq 0$ in $\mathbb{R}^n$. Moreover, either $u > 0$ in Ω or $u \equiv 0$ in $\mathbb{R}^n$.

Proof. Assume that $\inf_{\mathbb{R}^n} u = u(x_0) < 0$ for some $x_0 \in \Omega$ or that $\inf_{\mathbb{R}^n} u = u(x_0) \leq 0$ for some $x_0 \in \Omega$ and $u \not\equiv 0$. Then, it holds

$$(-\Delta)^s u(x_0) = c_{n,s} \operatorname{PV} \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+2s}} \, \mathrm{d}y < 0.$$

This is a contradiction. □

The Poisson kernel representation has several important applications, such as a nonlocal mean value property, a Harnack inequality, and interior regularity estimates. We will establish these results in the following subsections.

1.3. **Nonlocal Harnack inequalities.**

The goal of this subsection is to establish Harnack inequalities for s-harmonic functions.

Recall that classical harmonic functions satisfy the mean value property. The following is an analogous formula for s-harmonic functions, but the average is taken over the complement of a ball, rather than over its boundary.

Corollary 1.23 (Nonlocal mean value property)**.** *Let $u \in C^{2s+\varepsilon}_{loc}(B_r(x_0)) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$ be such that $(-\Delta)^s u = 0$ in $B_r(x_0)$. Then*

$$u(x_0) = \kappa_{n,s} r^{2s} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{u(z)}{(|z - x_0|^2 - r^2)^s |z - x_0|^n} \, \mathrm{d}z.$$

Moreover, if $(-\Delta)^s u \geq 0$ in $B_r(x_0)$ or $(-\Delta)^s u \leq 0$ in $B_r(x_0)$, then the identity for $u(x_0)$ holds with $\geq$ or $\leq$, respectively.

Proof. By translation, we may assume $x_0 = 0$. Evaluating the Poisson representation from Theorem 1.16 at $x = 0$ gives

$$u(0) = \int_{\mathbb{R}^n \setminus B_r} \mathcal{P}_{r,s}(0,z)u(z)\,\mathrm{d}z = \kappa_{n,s} \int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s}}{(|z|^2 - r^2)^s |z|^n} u(z)\,\mathrm{d}z,$$

in case $(-\Delta)^s u = 0$ in B_r as desired. If $(-\Delta)^s u \geq 0$ in B_r, we let v be the solution to

$$\begin{cases} (-\Delta)^s v &= 0 \quad \text{in } B_r, \\ v &= u \quad \text{in } B_r^c, \end{cases}$$

and note that by the above result, it holds

$$v(0) = \kappa_{n,s} \int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s}}{(|z|^2 - r^2)^s |z|^n} u(z)\,\mathrm{d}z.$$

Moreover, the maximum principle (see Lemma 1.22) applied to $u - v$ implies that $u \geq v$ in B_r and therefore, in particular, $u(0) \geq v(0)$. Together, this implies the desired result also in case $(-\Delta)^s u \geq 0$ in B_r. The proof in case $(-\Delta)^s u \leq 0$ in B_r goes analogously. □

Note that in contrast to the classical mean value property, in the nonlocal case, the value of u at the center depends on its values on the entire complement of the ball, not only on the sphere.

We also have the following variant of the mean value formula.

Corollary 1.24. *Let $u \in C^{2s+\varepsilon}_{loc}(B_r(x_0)) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$ be such that $(-\Delta)^s u = 0$ in $B_r(x_0)$. There exists a continuous, nonincreasing function $\omega_s : [0,\infty) \to (0,\infty)$, depending only on n and s, such that*

$$\int_{\mathbb{R}^n} \omega_s(|z|)\,dz = 1, \qquad \frac{C^{-1}}{1+t^{n+2s}} \leq \omega_s(t) \leq \frac{C}{1+t^{n+2s}} \quad \text{for all } t \geq 0,$$

and it holds

$$u(x_0) = R^{-n} \int_{\mathbb{R}^n} u(z)\,\omega_s\left(\frac{|z - x_0|}{R}\right) dz.$$

Moreover, if $(-\Delta)^s u \geq 0$ in $B_r(x_0)$ or $(-\Delta)^s u \leq 0$ in $B_r(x_0)$, then the identity for $u(x_0)$ holds with $\geq$ or $\leq$, respectively.

Proof. Define

$$\omega_s(t) := n\kappa_{n,s} \begin{cases} \displaystyle\int_0^1 \frac{\rho^{n+2s-1}}{(1-\rho^2)^s}\,d\rho, & 0 \leq t \leq 1, \\ \displaystyle\int_0^{1/t} \frac{\rho^{n+2s-1}}{(1-\rho^2)^s}\,d\rho, & t > 1. \end{cases}$$

Since $s < 1$, these integrals are finite. Thus ω_s is positive, continuous, and nonincreasing. For $t \geq 1$, substituting $\rho = \eta/t$ gives

$$\omega_s(t) = n\kappa_{n,s} t^{-n-2s} \int_0^1 \frac{\eta^{n+2s-1}}{(1-\eta^2/t^2)^s}\,d\eta,$$

and consequently, since also $1 \leq \frac{1}{(1-\eta^2/t^2)^s} \leq \frac{1}{(1-\eta^2)^s}$, we have

$$\frac{n\kappa_{n,s}}{n+2s}\, t^{-n-2s} \leq \omega_s(t) \leq \omega_s(0)\, t^{-n-2s}.$$

This proves the desired bounds on ω_s.

To prove the last claim, by translation and scaling, it suffices to consider $x_0 = 0$ and $R = 1$.

We first assume that $(-\Delta)^s u = 0$ in B_1. Applying Corollary 1.23 in $B_r \Subset B_1$, multiplying by nr^{n-1}, and integrating over $r \in (0,1)$, we obtain

$$\begin{aligned} u(0) &= u(0) \int_0^1 n r^{n-1} \,\mathrm{d}r \\ &= n\kappa_{n,s} \int_0^1 \int_{|z|>r} \frac{r^{n+2s-1} u(z)}{(|z|^2 - r^2)^s |z|^n} \,dz\, dr \\ &= \int_{\mathbb{R}^n} u(z) \left[\frac{n\kappa_{n,s}}{|z|^n} \int_0^{\min\{1,|z|\}} \frac{r^{n+2s-1}}{(|z|^2 - r^2)^s} \,dr \right] dz \\ &= \int_{\mathbb{R}^n} u(z) \omega_s(|z|) \,dz, \end{aligned}$$

where the last equality follows from substituting $r = |z|\rho$. Note that Fubini is justified by the bounds above and since $u \in L^1_{2s}(\mathbb{R}^n)$. Taking $u \equiv 1$ proves that $\int \omega_s = 1$.

The result in case $(-\Delta)^s u \geq 0$ in B_1 or $(-\Delta)^s u \leq 0$ in B_1 goes in the exact same way by applying the respective version of Corollary 1.23 in the computation above. □

In particular, we have the following nonlocal half-Harnack inequalities.

Lemma 1.25. *Let $u \in C^{2s+\varepsilon}_{loc}(B_1) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then, the following hold true.*

(i) *Let $(-\Delta)^s u \geq 0$ in B_1 and $u \geq 0$ in B_1. Then,*

$$\inf_{B_{1/2}} u \geq c_1 \int_{B_1} u(y) \,\mathrm{d}y - c_2 \int_{B_1^c} \frac{u_-(y)}{|y|^{n+2s}} \,\mathrm{d}y.$$

Let $(-\Delta)^s u \geq 0$ in B_1 and $u \geq 0$ in $\mathbb{R}^n$. Then,

$$\inf_{B_{1/2}} u \geq c_3 \int_{\mathbb{R}^n} \frac{u(y)}{1 + |y|^{n+2s}} \,\mathrm{d}y.$$

(ii) *Let $(-\Delta)^s u \leq 0$ in B_1. Then,*

$$\sup_{B_{1/2}} u \leq c_4 \int_{\mathbb{R}^n} \frac{u_+(y)}{1 + |y|^{n+2s}} \,\mathrm{d}y.$$

The constants $c_1, c_2, c_3, c_4 > 0$ depend only n, s.

Proof. This follows by combination of Corollary 1.24 with Lemma 1.22. □

As a direct corollary, we have the following nonlocal version of Harnack inequality.

Theorem 1.26 (Harnack inequality)**.** *Let $u \in C^{2s+\varepsilon}_{loc}(B_1) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Suppose that*

$$(-\Delta)^s u = 0 \qquad \text{in } B_1.$$

(i) *If $u \geq 0$ in B_1, then*

$$\sup_{B_{1/2}} u \leq C_1 \inf_{B_{1/2}} u + C_2 \int_{B_1^c} \frac{u_-(y)}{|y|^{n+2s}} \,\mathrm{d}y. \tag{1.19}$$

(ii) *If $u \geq 0$ in $\mathbb{R}^n$, then*

$$\sup_{B_{1/2}} u \leq C_2 \inf_{B_{1/2}} u.$$

Moreover, it holds

$$\int_{\mathbb{R}^n} \frac{u(y)}{1+|y|^{n+2s}}\,\mathrm{d}y \leq C_3 \sup_{B_{1/2}} u \leq C_4 \inf_{B_{1/2}} u \leq C_5 \int_{\mathbb{R}^n} \frac{u(y)}{1+|y|^{n+2s}}\,\mathrm{d}y.$$

All constants depends only on n and s.

Proof. Both claims follow immediately from the previous lemma. □

The last claim in Theorem 1.26 is particularly remarkable. It shows that the values of u in B_1^c, i.e. outside of the solution domain, can be controlled by the values of u in $B_{1/2}$.

Remark 1.27 (Harnack inequality via the extension)**.** There is another proof of the Harnack inequality for $(-\Delta)^s$ directly using the Caffarelli–Silvestre extension. We will give a brief sketch of the argument below (see also [CS07]):

Let U be the a-harmonic extension of u and define its even reflection

$$V(x,y) = U(x,|y|).$$

Since $(-\Delta)^s u = 0$ in B_r, we have

$$\lim_{y \downarrow 0} y^a \partial_y U(x,y) = 0, \qquad x \in B_r.$$

It follows that V is a weak solution of

$$\operatorname{div}(|y|^a \nabla V) = 0 \qquad \text{in } B_r^{n+1}.$$

Moreover, $V \geq 0$, since the Poisson kernel in Lemma 1.13 is nonnegative.

The weight $|y|^a$, with $a \in (-1,1)$, belongs to the so-called Muckenhoupt class A_2, i.e. it holds for $\omega(x,y) := |y|^a$

$$\sup_{B \subset \mathbb{R}^{n+1}} \left(\fint_B \omega(x,y)\,\mathrm{d}(x,y) \right) \left(\fint_B \omega(x,y)^{-1}\,\mathrm{d}(x,y) \right) < \infty.$$

The Harnack inequality for weighted elliptic equations with Muckenhoupt weights has been established in [FKS82] and applies to V. This gives the Harnack inequality for u.

Remark 1.28. In general, the Harnack inequality for $(-\Delta)^s$ fails if we do not assume $u \geq 0$ in $\mathbb{R}^n$ but only $u \geq 0$ in B_1. In that case, one can only show (1.19).

There is an explicit counterexample whose construction is due to Moritz Kassmann (see [Kas07]).

Let us now construct a function $u \in C^\infty_{loc}(B_1) \cap L^\infty(\mathbb{R}^n)$ satisfying

$$(-\Delta)^s u = 0 \quad \text{in } B_1, \qquad u \geq 0 \quad \text{in } B_1, \qquad \text{but} \qquad \inf_{B_{1/2}} u = 0 < \sup_{B_{1/2}} u,$$

showing that the Harnack inequality fails if we do not assume $u \geq 0$ in $\mathbb{R}^n$ but only $u \geq 0$ in B_1.

Set $R = 2/\sqrt{s}$ and $A = B_{2R} \setminus B_R$. Let v be the bounded solution of

$$\begin{cases} (-\Delta)^s v &= 0 \quad \text{in } B_1, \\ v &= \mathbb{1}_A \quad \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

By the Poisson representation formula (see Theorem 1.16),

$$v(x) = \kappa_{n,s}(1-|x|^2)^s \int_A \frac{dz}{(|z|^2-1)^s|x-z|^n}, \qquad x \in B_1.$$

In particular, $v \in C^\infty(B_1) \cap L^\infty(\mathbb{R}^n)$. Positivity and normalization of the Poisson kernel imply $0 < v < 1$ in B_1.

We claim that v is strictly decreasing as a function of $|x|$. Using polar coordinates and the identity

$$\int_{\mathbb{S}^{n-1}} \frac{d\sigma(\theta)}{|x-t\theta|^n} = \frac{|\mathbb{S}^{n-1}|\, t^{2-n}}{t^2-|x|^2}, \qquad t > 1, \quad |x| < 1,$$

which follows from normalization of the Poisson kernel for the Laplacian in the ball B_t, we obtain

$$v(x) = \kappa_{n,s}(1-|x|^2)^s \int_R^{2R} t^{n-1} \int_{\mathbb{S}^{n-1}} \frac{d\sigma(\theta)}{(|t|^2-1)^s|x-\theta t|^n}\,\mathrm{d}t = \kappa_{n,s}|\mathbb{S}^{n-1}| \int_R^{2R} \frac{t}{(t^2-1)^s} \frac{(1-|x|^2)^s}{t^2-|x|^2}\,dt.$$

In particular, v is a radial function. For $q \in [0,1)$ and $t \geq R$, we have

$$\frac{d}{dq}\left(\frac{(1-q)^s}{t^2-q}\right) = -\frac{s(1-q)^{s-1}}{t^2-q} + \frac{(1-q)^s}{(t^2-q)^2} = \frac{(1-q)^{s-1}}{(t^2-q)^2}\big(1-st^2-(1-s)q\big) < 0,$$

since $st^2 \geq sR^2 = 4$. This proves the claim.

Now set $M = v(0) \in (0,1)$ and define

$$u := M - v \qquad \text{in } \mathbb{R}^n.$$

Since constants are s-harmonic,

$$(-\Delta)^s u = 0 \qquad \text{in } B_1.$$

Moreover, the strict radial monotonicity of v gives

$$u(0) = 0, \qquad u(x) > 0 \quad \text{for } 0 < |x| < 1.$$

Therefore,

$$\inf_{B_{1/2}} u = 0 < \sup_{B_{1/2}} u,$$

so the nonlocal Harnack inequality (1.19) can, in general, not hold without the additional term $\int_{B_1^c} u_-(y)|y|^{-n-2s}\,\mathrm{d}y$.

1.4. **Interior regularity.** The Poisson representation also immediately gives interior smoothness of s-harmonic functions.

Corollary 1.29. *Let $s \in (0,1)$ and let $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(B_1) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$ satisfy*

$$(-\Delta)^s u = 0 \qquad \text{in } B_1.$$

Then $u \in C^\infty_{loc}(B_1)$. Moreover, for every $m \in \mathbb{N} \cup \{0\}$,

$$\|u\|_{C^m(B_{1/2})} \leq C\|u\|_{L^1_{2s}(\mathbb{R}^n)},$$

where C depends only on n, s, m.

Proof. For every $\rho \in [3/4, 7/8]$, the Poisson representation from Theorem 1.16 in $B_\rho \Subset B_1$ gives

$$u(x) = \int_{|z|>\rho} \mathcal{P}_{\rho,s}(x,z)u(z)\,dz, \qquad x \in B_{1/2},$$

where

$$\mathcal{P}_{\rho,s}(x,z) = \kappa_{n,s}\frac{(\rho^2-|x|^2)^s}{(|z|^2-\rho^2)^s|x-z|^n}.$$

We first record the derivative estimate with its scaling. Fix $\tau \in (0,1)$, let $\rho > 0$, and suppose that $x \in B_{\tau\rho}$ and $|z| > \rho$. For every multi-index β, scaling and the smoothness of $(1-|y|^2)^s$ on $\overline{B_\tau}$ give

$$\left|D_x^\beta(\rho^2-|x|^2)^s\right| = \rho^{2s-|\beta|}\left|D_y^\beta(1-|y|^2)^s\right|_{y=x/\rho} \le C_{\beta,s,\tau}\rho^{2s-|\beta|}.$$

Moreover, $|x-z| \ge (1-\tau)|z| \ge (1-\tau)\rho$, and hence, for every multi-index η,

$$\left|D_x^\eta|x-z|^{-n}\right| \le C_\eta|x-z|^{-n-|\eta|} \le C_{\eta,\tau}\rho^{-|\eta|}|z|^{-n}.$$

Fix a multi-index γ and write $k = |\gamma|$. The Leibniz rule therefore yields

$$\begin{aligned}
|D_x^\gamma\mathcal{P}_{\rho,s}(x,z)| &\le \frac{\kappa_{n,s}}{(|z|^2-\rho^2)^s}\sum_{\beta\le\gamma}\binom{\gamma}{\beta}\left|D_x^\beta(\rho^2-|x|^2)^s\right|\left|D_x^{\gamma-\beta}|x-z|^{-n}\right| \\
&\le C\kappa_{n,s}\frac{\rho^{2s-k}}{(|z|^2-\rho^2)^s|z|^n} \\
&= C\rho^{-k}\mathcal{P}_{\rho,s}(0,z),
\end{aligned}$$

where C depends only on n, s, k, and τ.

In particular, taking $\rho \in [3/4, 7/8]$ and $\tau = 2/3$, we have $B_{1/2} \subset B_{\tau\rho} = B_{2\rho/3}$ and obtain

$$|D_x^\gamma\mathcal{P}_{\rho,s}(x,z)| \le \frac{C_\gamma}{(|z|^2-\rho^2)^s|z|^n}, \qquad |x| \le \tfrac{1}{2}, \quad |z| > \rho, \tag{1.20}$$

uniformly in $\rho \in [3/4, 7/8]$.

To recover the $L^1_{2s}(\mathbb{R}^n)$ norm, we need to get rid of the singularity at $|z| = \rho$. We do so by averaging in ρ and therefore set

$$A_s(x,z) := 8\int_{3/4}^{7/8}\mathbb{1}_{\{|z|>\rho\}}\mathcal{P}_{\rho,s}(x,z)\,d\rho.$$

Note that

$$|D_x^\gamma A_s(x,z)| \le \frac{C_\gamma}{(1+|z|)^n}\int_{3/4}^{7/8}\mathbb{1}_{\{|z|>\rho\}}(|z|^2-\rho^2)^{-s}\,d\rho.$$

This implies

$$|D_x^\gamma A_s(x,z)| \le \frac{C_\gamma}{1+|z|^{n+2s}}, \qquad |x| \le \tfrac{1}{2}. \tag{1.21}$$

For $|z| \ge 2$, this follows from $|z|^2-\rho^2 \ge c|z|^2$, which yields

$$\int_{3/4}^{\min\{7/8,|z|\}}(|z|^2-\rho^2)^{-s}\,d\rho \le \int_{3/4}^{7/8}\mathrm{d}\rho|z|^{-2s} \le C|z|^{-2s}.$$

For $3/4 < |z| < 2$, it follows from

$$\int_{3/4}^{\min\{7/8,|z|\}}(|z|^2-\rho^2)^{-s}\,d\rho \le C\int_{3/4}^{\min\{7/8,|z|\}}(|z|-\rho)^{-s}\,d\rho \le C,$$

since $s < 1$. For $|z| \le 3/4$, there is nothing to prove. Hence, we have shown (1.21).

The estimate (1.21) applied with $\gamma = 0$ and the fact that $u \in L^1_{2s}(\mathbb{R}^n)$ justify the use of Fubini's theorem to derive the following formula:

$$u(x) = 8\int_{3/4}^{7/8} u(x)\,\mathrm{d}\rho = 8\int_{3/4}^{7/8}\int_{B_\rho^c} u(z)\mathcal{P}_{\rho,s}(x,z)\,\mathrm{d}z\,\mathrm{d}\rho = \int_{\mathbb{R}^n} A_s(x,z)u(z)\,dz.$$

All derivatives can now be taken under the integral sign using (1.21). Thus

$$|D^\gamma u(x)| \leq C_\gamma \|u\|_{L^1_{2s}(\mathbb{R}^n)}, \qquad x \in B_{1/2}.$$

The proof is complete. □

Remark 1.30. If u satisfies $(-\Delta)^s u = 0$ in $B_R(x_0)$, the estimate in Corollary 1.29 becomes

$$\begin{aligned}\|D^m u\|_{L^\infty(B_{R/2}(x_0))} &\leq CR^{2s-m}\int_{\mathbb{R}^n} \frac{|u(z)|}{R^{n+2s}+|z-x_0|^{n+2s}}\,dz \\ &\leq CR^{-m}\left(\fint_{B_R(x_0)} |u(z)|\,\mathrm{d}z + R^{2s}\int_{\mathbb{R}^n\setminus B_R(x_0)} \frac{|u(z)|}{|z-x_0|^{n+2s}}\,\mathrm{d}z\right).\end{aligned}$$

Indeed, this follows by applying Corollary 1.29 to $v(x) = u(x_0 + Rx)$.

The final goal of this subsection is to show that the interior regularity estimates of s-harmonic functions from Corollary 1.29 carry over to solutions of $(-\Delta)^s u = f$ for suitably regular f.

Theorem 1.31 (Interior Schauder estimate)**.** *Let $s \in (0,1)$, $k \in \mathbb{N}\cup\{0\}$, and $\alpha \in (0,1)$ satisfy $2s+\alpha \notin \mathbb{N}$. Suppose that $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(B_1) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$ satisfies*

$$(-\Delta)^s u = f \qquad \text{in } B_1,$$

where $f \in C^{k,\alpha}(B_1)$. Then $u \in C^{k+2s+\alpha}_{\mathrm{loc}}(B_1)$ and

$$\|u\|_{C^{k+2s+\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^1_{2s}(\mathbb{R}^n)} + \|f\|_{C^{k,\alpha}(B_1)}\right),$$

where C depends only on n, s, k, and α.

This result is a nonlocal analog of the classical Schauder theory for the Laplacian with two main differences:

- The solution u gains $2s$ derivatives compared to the source term f (corresponding to the order of the equation),
- The estimate requires information about u in the full space in terms of $\|u\|_{L^1_{2s}(\mathbb{R}^n)}$ (capturing the nonlocality of the equation).

Moreover, note that Theorem 1.31 is in some way a reverse version of Proposition 1.5. Instead of showing that the fractional Laplacian of a sufficiently regular function is regular, it establishes that a function is regular if its fractional Laplacian is regular. This is certainly a much deeper result.

Remark 1.32. By scaling, if u is a solution to $(-\Delta)^s u = f$ in $B_R(x_0)$, the estimate reads

$$\begin{aligned}&\|u\|_{L^\infty(B_{R/2}(x_0))} + R^{k+2s+\alpha}[u]_{C^{k+2s+\alpha}(B_{R/2}(x_0))} \\ &\qquad\leq C\Bigg(\fint_{B_R(x_0)} |u(z)|\,\mathrm{d}z + R^{2s}\int_{\mathbb{R}^n\setminus B_R(x_0)} \frac{|u(z)|}{|z-x_0|^{n+2s}}\,\mathrm{d}z \\ &\qquad\qquad + R^{2s}\|f\|_{L^\infty(B_R(x_0))} + R^{2s+k+\alpha}[f]_{C^{k,\alpha}(B_R(x_0))}\Bigg).\end{aligned}$$

The main technical tool to prove Theorem 1.31 is contained in the following lemma, which introduces the fundamental solution of the fractional Laplacian.

Lemma 1.33 (Fundamental solution)**.** *For $s \in (0,1)$, define*

$$\Phi_s(x) := \begin{cases} \dfrac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{n/2}\Gamma(s)}|x|^{2s-n}, & n \neq 2s, \\ -\dfrac{1}{\pi}\log|x|, & n = 2s = 1. \end{cases}$$

Let $k \in \mathbb{N} \cup \{0\}$, and $\alpha \in (0,1)$ satisfy $2s+\alpha \notin \mathbb{N}$. Let $f \in C_c^{k,\alpha}(B_2)$ and define

$$v(x) := (\Phi_s * f)(x) = \int_{\mathbb{R}^n} \Phi_s(x-z) f(z)\, dz.$$

Then, $v \in C^{2s+k+\alpha}_{loc}(\mathbb{R}^n) \cap L^1_{2s}(\mathbb{R}^n)$ and is a solution of

$$(-\Delta)^s v = f \qquad \text{in } \mathbb{R}^n.$$

Moreover, it holds

$$\|v\|_{C^{k+2s+\alpha}(B_1)} + \|v\|_{L^1_{2s}(\mathbb{R}^n)} \leq C\|f\|_{C^{k,\alpha}(B_2)},$$

where C depends only on n, s, k, α.

Proof. The proof is split into several steps:

Step 1: We verify the integral estimate. First, note that since $\Phi_s \in L^1_{loc}(\mathbb{R}^n)$ and since f has compact support in B_2, we have

$$|v(x)| \leq C\|f\|_{L^\infty(B_2)} \begin{cases} 1, & n > 2s, \\ 1+\log(2+|x|), & n = 2s, \\ 1+|x|^{2s-n}, & n < 2s. \end{cases} \tag{1.22}$$

Since each of these bounds is integrable against $(1+|x|^{n+2s})^{-1}$, this implies

$$\|v\|_{L^\infty(B_1)} + \|v\|_{L^1_{2s}(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(B_2)}.$$

We next establish the regularity estimate for $k = 0$. Write

$$2s+\alpha = m+\beta, \qquad m = \lfloor 2s+\alpha \rfloor \in \{0,1,2\}, \quad \beta \in (0,1).$$

We choose $\chi \in C_c^\infty(B_2)$ with $\chi = 1$ on B_1. Given $x, x' \in B_1$ with $h := |x-x'| \leq 1$, we decompose

$$\Phi_s = G_h + H_h, \qquad G_h(z) := \chi(z/h)\Phi_s(z), \quad H_h := \Phi_s - G_h.$$

For simplicity, we only explain the proof in case $n \neq 2s$. By the homogeneity of Φ_s, we have for $z \neq 0$

$$|D^j G_h(z)| \leq C|z|^{2s-n-j}\mathbb{1}_{\{|z|<2h\}}, \qquad |D^j H_h(z)| \leq C|z|^{2s-n-j}\mathbb{1}_{\{|z|>h\}}.$$

If $m \geq 1$, we get

$$D^m(G_h * f)(x) = \int_{\mathbb{R}^n} D^m G_h(z)\big(f(x-z) - f(x)\big)\, dz,$$

This can be justified by approximating G_h via a sequence of L^1 functions $G_{h,\varepsilon} = (1-\eta(z/\varepsilon))G_h(z)$, where η is a cutoff supported away from zero and using that by integration by parts, it holds $\int_{\mathbb{R}^n} D^m G_{h,\varepsilon} = 0$.

Consequently,

$$\|D^m(G_h * f)\|_{L^\infty} \leq C[f]_{C^\alpha(\mathbb{R}^n)} \int_0^{2h} r^{2s+\alpha-m-1}\,dr \leq C[f]_{C^\alpha(\mathbb{R}^n)} h^\beta.$$

If $m = 0$, we use $\|G_h\|_{L^1} \leq Ch^{2s}$ to obtain

$$|(G_h * f)(x) - (G_h * f)(x')| \leq \|G_h\|_{L^1}[f]_{C^\alpha(\mathbb{R}^n)} h^\alpha \leq C[f]_{C^\alpha(\mathbb{R}^n)} h^\beta.$$

Similar as for G_h, we obtain, using also that $m + 1 > 2s + \alpha$,

$$\|D^{m+1}(H_h * f)\|_{L^\infty} \leq C[f]_{C^\alpha(\mathbb{R}^n)} \int_h^\infty r^{2s+\alpha-m-2}\,dr \leq C[f]_{C^\alpha(\mathbb{R}^n)} h^{\beta-1}.$$

The mean value theorem therefore yields

$$|D^m(H_h * f)(x) - D^m(H_h * f)(x')| \leq Ch\|D^{m+1}(H_h * f)\|_{L^\infty} \leq C[f]_{C^\alpha(\mathbb{R}^n)} h^\beta.$$

Since $v = G_h * f + H_h * f$, combining the previous estimates gives

$$|D^m v(x) - D^m v(x')| \leq C[f]_{C^\alpha(\mathbb{R}^n)} |x - x'|^\beta.$$

Using also the L^∞ bound from above and Hölder interoplation, we conclude that

$$\|v\|_{C^{2s+\alpha}(B_1)} \leq C\|f\|_{C^\alpha(B_2)}.$$

For $k \geq 1$, we use that differentiation commutes with convolution:

$$D^\eta v = \Phi_s * D^\eta f, \qquad |\eta| \leq k.$$

Applying the estimate above to $D^\eta f$ gives

$$\|v\|_{C^{k+2s+\alpha}(B_1)} \leq C\|f\|_{C^{k,\alpha}(B_2)}.$$

This concludes the proof of the regularity estimate.

Step 2: It remains to show that v is a solution to $(-\Delta)^s v = f$ in $\mathbb{R}^n$. To prove it, we will employ Theorem 1.15. First, we define

$$\widetilde{\Phi}_s(x, y) := \begin{cases} \dfrac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{n/2}\Gamma(s)}(|x|^2 + y^2)^{s-n/2}, & n \neq 2s, \\ -\dfrac{1}{2\pi}\log(|x|^2 + y^2), & n = 2s = 1, \end{cases}$$

and consider

$$U(x, y) := \int_{\mathbb{R}^n} \widetilde{\Phi}_s(x - z, y) f(z)\,dz, \qquad y > 0.$$

We first identify U with the weighted Poisson extension of v. Taking $\mu = 0$ and $\lambda = 2s - n$ in Lemma 1.14, we obtain

$$\mathcal{L}_a \widetilde{\Phi}_s = 0,$$

since $\lambda + n + a - 1 = 0$. In the logarithmic case, $a = 0$, and the same conclusion follows from the harmonicity of $\log|(x, y)|$ away from the origin. Differentiating under the integral therefore gives

$$\operatorname{div}(y^a \nabla U) = 0 \qquad \text{in } \mathbb{R}^{n+1}_+.$$

Moreover, since $\widetilde{\Phi}_s \in L^1_{loc}(\mathbb{R}^n)$ and f has compact support, we get that U is continuous up to $y = 0$ and

$$U(x, 0) = v(x).$$

Choosing $0 < \varepsilon < \min\{\alpha, n, 2s\}$ and using the same argument as in (1.22), we get

$$|v(x)| \le C(1+|x|)^{2s-\varepsilon}, \qquad |U(x,y)| \le C(1+|(x,y)|)^{2s-\varepsilon}.$$

By uniqueness in Theorem 1.15, U is the weighted Poisson extension of v and it holds

$$(-\Delta)^s v(x) = -d_s \lim_{y\downarrow 0} y^{1-2s}\partial_y U(x,y).$$

We now compute the weighted normal derivative. Using

$$d_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}$$

and the identity $\Gamma(t+1) = t\Gamma(t)$, we compute

$$-d_s y^{1-2s}\partial_y \widetilde{\Phi}_s(z,y) = Q_y(z) := \frac{\Gamma(\frac{n}{2}+1-s)}{\pi^{n/2}\Gamma(1-s)}\frac{y^{2-2s}}{(|z|^2+y^2)^{n/2+1-s}}.$$

The same formula follows by direct calculation in the logarithmic case. Consequently,

$$-d_s y^{1-2s}\partial_y U(x,y) = (Q_y * f)(x).$$

The kernels Q_y are nonnegative and satisfy

$$Q_y(z) = y^{-n}Q_1(z/y), \qquad \int_{\mathbb{R}^n} Q_1(z)\,dz = 1.$$

The normalization follows from

$$\int_{\mathbb{R}^n}(1+|z|^2)^{-n/2-1+s}\,dz = \frac{\pi^{n/2}\Gamma(1-s)}{\Gamma(\frac{n}{2}+1-s)}.$$

Furthermore, for every $\delta > 0$,

$$\int_{|z|>\delta} Q_y(z)\,dz = \int_{|w|>\delta/y} Q_1(w)\,dw \longrightarrow 0 \qquad \text{as } y \downarrow 0.$$

Thus $(Q_y)_{y>0}$ is an approximation of the identity. Since f is bounded and continuous,

$$(Q_y * f)(x) \to f(x) \qquad \text{for every } x \in \mathbb{R}^n.$$

Hence, we conclude that for every $x \in \mathbb{R}^n$

$$(-\Delta)^s v(x) = -d_s \lim_{y\downarrow 0} y^{1-2s}\partial_y U(x,y) = \lim_{y\downarrow 0}(Q_y * f)(x) = f(x).$$

□

We are now ready to prove Theorem 1.31.

Proof of Theorem 1.31. Choose $\eta \in C_c^\infty(B_1)$ with $\eta = 1$ on $B_{3/4}$, and let $\widetilde{f} = \eta f$. Then $\widetilde{f} \in C_c^{k,\alpha}(B_2)$ and

$$\|\widetilde{f}\|_{C^{k,\alpha}(\mathbb{R}^n)} \le C\|f\|_{C^{k,\alpha}(B_1)}.$$

Set $w = \Phi_s * \widetilde{f}$. By Lemma 1.33,

$$(-\Delta)^s w = \widetilde{f} \quad \text{in } \mathbb{R}^n, \qquad \|w\|_{C^{k+2s+\alpha}(B_1)} + \|w\|_{L^1_{2s}(\mathbb{R}^n)} \le C\|f\|_{C^{k,\alpha}(B_1)}.$$

The remainder $h = u - w$ satisfies

$$(-\Delta)^s h = 0 \qquad \text{in } B_{3/4}, \qquad \|h\|_{L^1_{2s}(\mathbb{R}^n)} \le \|u\|_{L^1_{2s}(\mathbb{R}^n)} + C\|f\|_{C^{k,\alpha}(B_1)}.$$

Also, we have that $h \in C^{2s+\delta}_{\mathrm{loc}}(B_{3/4})$ for a sufficiently small $\delta \in (0, \min\{\varepsilon, \alpha\})$. Corollary 1.29 (applied in $B_{3/4}$) therefore yields, for every $N \in \mathbb{N} \cup \{0\}$,

$$\|h\|_{C^N(B_{1/2})} \leq C_N \left(\|u\|_{L^1_{2s}(\mathbb{R}^n)} + \|f\|_{C^{k,\alpha}(B_1)} \right).$$

Finally, we choose $N > k + 2s + \alpha$, note that then $\|h\|_{C^{k+2s+\alpha}(B_{1/2})} \leq C\|h\|_{C^N(B_{1/2})}$, and recall that $u = w + h$ to obtain the required estimate by combining the estimates for w and for h. □

Remark 1.34. The a priori assumption $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(B_1)$ can be removed if the equation is interpreted in a distributional sense. We will introduce this concept rigorously in the upcoming lectures.

1.5. **Boundary regularity.** We now turn to regularity of solutions to nonlocal equations up to the boundary.

Throughout this subsection, $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain within B_1. This means that there exist constants $r_0, N > 0$ such that for every $p \in \partial\Omega \cap B_1$, after a translation and rotation sending p to the origin, we can write

$$\Omega \cap B_{r_0} = \{x \in B_{r_0} : x_n > \varphi(x')\},$$

where $\varphi \in C^{1,1}(B'_{r_0})$ satisfies

$$\varphi(0) = 0, \qquad \nabla\varphi(0) = 0, \qquad \|\varphi\|_{C^{1,1}(B'_{r_0})} \leq N.$$

Throughout this section, we write

$$d_\Omega(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega.$$

First, we will use that the explicit torsion function in a ball from Lemma 1.20 provides a barrier for the Dirichlet problem in a general $C^{1,1}$ domain. This allows us to obtain the following decay estimate for solutions at the boundary.

Lemma 1.35. *Let $s \in (0,1)$ and suppose that $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(\Omega \cap B_1) \cap C(\overline{\Omega \cap B_1}) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Let $f \in L^\infty(\Omega \cap B_1)$ and assume that*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \cap B_1, \\ u = 0 & \text{in } B_1 \setminus \Omega, \end{cases}$$

Then, it holds for every $x \in \Omega \cap B_{3/4}$,

$$|u(x)| \leq C d_\Omega(x)^s \big(\|u\|_{L^\infty(B_1 \cap \Omega)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)} \|f\|_{L^\infty(\Omega)} \big),$$

where C depends only on n, s and Ω.

Proof. The proof is split into three steps. Set

$$M := \|f\|_{L^\infty(\Omega \cap B_1)} + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}.$$

Step 1: We first localize the equation. Choose $\eta \in C^\infty_c(B_{15/16})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_{7/8}$, and define $\widetilde{u} := \eta u$. In particular, $\widetilde{u} = 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$\|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(B_1)}.$$

For $x \in \Omega \cap B_{13/16}$, we have $\eta(x) = 1$, and therefore

$$(-\Delta)^s \widetilde{u}(x) = (-\Delta)^s u(x) + c_{n,s} \int_{\mathbb{R}^n} \frac{(1 - \eta(y))u(y)}{|x - y|^{n+2s}} \, dy =: f(x) + T(x).$$

Since $1-\eta$ vanishes on $B_{7/8}$, the term T is nonsingular for $x \in B_{13/16}$ and we can estimate

$$|T(x)| \le C \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}}\, dy \le C\|u\|_{L^1_{2s}(\mathbb{R}^n)}.$$

Consequently,

$$\|(-\Delta)^s \widetilde{u}\|_{L^\infty(\Omega \cap B_{13/16})} \le CM. \tag{1.23}$$

Step 2: We construct an exterior barrier as follows. Let $w(x) = (1-|x|^2)^s_+$ be the torsion function from Lemma 1.20. Its Kelvin transform is

$$\Psi(x) := \mathcal{K}_0 w(x) = \frac{(|x|^2-1)^s_+}{|x|^n}, \qquad x \neq 0,$$

with $\Psi(0) := 0$. Since $(-\Delta)^s w = Q_{n,s}$ in B_1, Lemma 1.18 yields

$$(-\Delta)^s \Psi(x) = |x|^{-n-2s}(-\Delta)^s w\left(\frac{x}{|x|^2}\right) = Q_{n,s}|x|^{-n-2s} > 0, \qquad |x|>1. \tag{1.24}$$

Notice that $\Psi \ge 0$, $\Psi = 0$ on $\overline{B_1}$, and $\Psi \in L^1_{2s}(\mathbb{R}^n)$.

Since Ω is $C^{1,1}$, there is $R > 0$, independent of $p \in \partial\Omega$, such that for every p an exterior ball $B_R(z_p) \subset \mathbb{R}^n \setminus \Omega$ is tangent to $\partial\Omega$ at p. We may assume that $R \le 1/96$.

Fix $p \in \partial\Omega \cap B_{25/32}$ and note that $\Omega \cap B_{3R}(p) \subset \Omega \cap B_{13/16}$ [Hence we can use (1.23)]. Define

$$\Psi_p(x) := \Psi\left(\frac{x-z_p}{R}\right).$$

By (1.24) and scaling,

$$(-\Delta)^s \Psi_p(x) = Q_{n,s}\frac{R^n}{|x-z_p|^{n+2s}}, \qquad x \in \Omega \cap B_{3R}(p).$$

In particular, on $\Omega \cap B_{3R}(p)$ we have $|x-z_p| \le 4R$, and therefore

$$(-\Delta)^s \Psi_p \ge Q_{n,s} 4^{-n-2s} R^{-2s} =: c_0 > 0 \qquad \text{in } \Omega \cap B_{3R}(p).$$

Moreover, for $x \in \Omega \cap \operatorname{supp}\eta \setminus B_{3R}(p)$,

$$2 \le \frac{|x-z_p|}{R} \le \frac{2}{R}.$$

Hence,

$$\Psi_p \ge c_1 > 0 \quad \text{in } \Omega \cap \operatorname{supp}\eta \setminus B_{3R}(p)$$

with c_1 independent of p.

By (1.23), we can choose $A \ge CM$ sufficiently large that

$$Ac_0 \ge \|(-\Delta)^s \widetilde{u}\|_{L^\infty(\Omega \cap B_{3R}(p))}, \qquad Ac_1 \ge \|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}.$$

Then the functions $V_\pm := A\Psi_p \pm \widetilde{u}$ satisfy

$$\begin{cases} (-\Delta)^s V_\pm \ge 0 & \text{in } \Omega \cap B_{3R}(p), \\ V_\pm \ge 0 & \text{in } \mathbb{R}^n \setminus (\Omega \cap B_{3R}(p)). \end{cases}$$

Indeed, outside Ω or outside $\operatorname{supp}\eta$, we have $\widetilde{u} = 0$. On the remaining part of the complement of $\Omega \cap B_{3R}(p)$, nonnegativity follows from $\Psi_p \ge c_1$.

Applying the maximum principle Lemma 1.22 in $\Omega \cap B_{3R}(p)$ gives $V_\pm \ge 0$, and hence we obtain

$$|\widetilde{u}(x)| \le A\Psi_p(x), \qquad x \in \Omega \cap B_{3R}(p). \tag{1.25}$$

Step 3: Since $\partial\Omega \in C^{1,1}(B_1)$, there is $\rho > 0$ such that, if $d_\Omega(x) < \rho$, then there is a unique $p \in \partial\Omega$ with $d_\Omega(x) = |x - p|$ and

$$x = p + d_\Omega(x)\nu(p),$$

where $\nu(p) \in \mathbb{S}^{n-1}$ is the inward unit normal. Moreover, note that $z_p = p - R\nu(p)$.

Now let $x \in \Omega \cap B_{3/4}$ and suppose

$$d_\Omega(x) < \delta_0 := \min\{\rho, R, 1/32\}.$$

Then its nearest boundary point satisfies $p \in B_{25/32}$ and $x \in \Omega \cap B_{3R}(p)$. Moreover,

$$\frac{|x - z_p|}{R} = 1 + \frac{d_\Omega(x)}{R}.$$

Writing $t = d_\Omega(x)/R$, and using that $(2 + t)^s \leq 2^s(1 + t)^s$, and $n > s$, we obtain

$$\Psi_p(x) = \frac{((1 + t)^2 - 1)^s}{(1 + t)^n} = \frac{(t(2 + t))^s}{(1 + t)^n} \leq 2^s t^s (1 + t)^{s-n} \leq 2^s R^{-s} d_\Omega(x)^s.$$

Since $\eta = 1$ on $B_{7/8}$, we have $\widetilde{u}(x) = u(x)$. Thus, (1.25) gives

$$|u(x)| \leq CMd_\Omega(x)^s.$$

Finally, if $x \in \Omega \cap B_{3/4}$ satisfies $d_\Omega(x) \geq \delta_0$, then

$$|u(x)| \leq \|u\|_{L^\infty(B_1)} \leq \delta_0^{-s} M d_\Omega(x)^s.$$

Combining the two cases proves the claim. □

The boundary decay estimate is the main ingredient of the global Hölder regularity theorem. To obtain the full C^s estimate, we combine it with an interior estimate for bounded right-hand sides.

Theorem 1.36 (Global C^s regularity)**.** *Let $s \in (0, 1)$ and suppose that $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(\Omega \cap B_1) \cap C(\overline{\Omega \cap B_1}) \cap L^1_{2s}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Let $f \in C^\alpha(\Omega \cap B_1)$ for some $\alpha > 0$ and assume that*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \cap B_1, \\ u = 0 & \text{in } B_1 \setminus \Omega. \end{cases}$$

Then, it holds $u \in C^s(B_{1/2})$ and

$$\|u\|_{C^s(B_{1/2})} \leq C \left(\|f\|_{C^{0,\alpha}(\Omega \cap B_1)} + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)} \right),$$

where C depends only on n, s, α and Ω.

Proof. Set

$$M := \|f\|_{C^{0,\alpha}(\Omega \cap B_1)} + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}.$$

By Lemma 1.35, we have

$$|u(x)| \leq CMd_\Omega(x)^s, \qquad x \in \Omega \cap B_{3/4}. \tag{1.26}$$

Choose $\alpha \in (0, \varepsilon)$ such that $2s + \alpha \notin \mathbb{N}$, and fix

$$s < \gamma < \min\{2s, 1\}.$$

The proof is split into two steps.

Step 1: We establish a scaled interior estimate. Fix $x_0 \in \Omega \cap B_{1/2}$ and set

$$r := \min\left\{ \frac{d_\Omega(x_0)}{2}, \frac{1}{8} \right\}.$$

Then $B_r(x_0) \subset \Omega \cap B_1$, and the rescaled function

$$v_r(z) := u(x_0 + rz)$$

satisfies

$$(-\Delta)^s v_r(z) = r^{2s} f(x_0 + rz) \qquad \text{in } B_1.$$

We first estimate the weighted norm of v_r. Suppose that $x_0 + rz \in B_{3/4}$. If $r = d_\Omega(x_0)/2$, then (1.26) gives

$$|v_r(z)| = |u(x_0 + rz)| \le CM|d_\Omega(x_0 + rz)|^s \le CM\big(d_\Omega(x_0) + r|z|\big)^s = CMr^s(2 + |z|)^s,$$

where we also used that $u = 0$ in $B_1 \setminus \Omega$ and that the distance function is 1-Lipschitz,

$$d_\Omega(x_0 + rz) \le d_\Omega(x_0) + |rz| = 2r + r|z| = r(2 + |z|).$$

If $r = 1/8$, the same estimate follows directly from the definition of M.

Consequently,

$$\int_{\{x_0 + rz \in B_{3/4}\}} \frac{|v_r(z)|}{1 + |z|^{n+2s}}\, dz \le CMr^s.$$

On the other hand, making the change of variables $y = x_0 + rz$, we obtain

$$\int_{\{x_0 + rz \notin B_{3/4}\}} \frac{|v_r(z)|}{1 + |z|^{n+2s}}\, dz = r^{2s} \int_{\mathbb{R}^n \setminus B_{3/4}} \frac{|u(y)|}{r^{n+2s} + |y - x_0|^{n+2s}}\, dy \le Cr^{2s} \|u\|_{L^1_{2s}(\mathbb{R}^n)},$$

where we used that $|x_0| < 1/2$ and hence $|y - x_0| \ge c(1 + |y|)$ for $|y| \ge 3/4$.

Combining the previous two estimates and using $r < 1$, we conclude that

$$\|v_r\|_{L^1_{2s}(\mathbb{R}^n)} \le CM(r^s + r^{2s}) \le CMr^s.$$

Decreasing α if necessary, we may assume that $2s + \alpha \notin \mathbb{N}$. Moreover, by scaling the Hölder norm,

$$\|r^{2s} f(x_0 + r\,\cdot)\|_{C^{0,\alpha}(B_1)} \le Cr^{2s} \|f\|_{C^{0,\alpha}(\Omega \cap B_1)}.$$

Thus, applying Theorem 1.31 to v_r, we obtain

$$[v_r]_{C^s(B_{1/2})} \le C\|v_r\|_{C^{2s+\alpha}(B_{1/2})} \le C\left(\|v_r\|_{L^1_{2s}(\mathbb{R}^n)} + r^{2s}\|f\|_{C^{0,\alpha}(\Omega \cap B_1)}\right) \le CMr^s.$$

Scaling back to u, we conclude that

$$[u]_{C^s(B_{r/2}(x_0))} = r^{-s}[v_r]_{C^s(B_{1/2})} \le CM. \tag{1.27}$$

Step 2: We conclude by comparing two points $x, y \in B_{1/2}$.

If $x, y \notin \Omega$, then $u(x) = u(y) = 0$.

If $x \in \Omega$ and $y \notin \Omega$, then $d_\Omega(x) \le |x - y|$, and (1.26) yields

$$|u(x) - u(y)| = |u(x)| \le CM|x - y|^s.$$

It remains to consider $x, y \in \Omega \cap B_{1/2}$. Set

$$t := |x - y|, \qquad r := \min\left\{\frac{d_\Omega(x)}{2}, \frac{1}{8}\right\}.$$

If $t < r/2$, then $y \in B_{r/2}(x)$, and (1.27) immediately gives

$$|u(x) - u(y)| \le CMt^s.$$

If $t \geq r/2$ and $r = d_\Omega(x)/2$, then

$$d_\Omega(x) \leq 4t, \qquad d_\Omega(y) \leq d_\Omega(x) + t \leq 5t.$$

Hence, by (1.26),

$$|u(x) - u(y)| \leq CM\big(d_\Omega(x)^s + d_\Omega(y)^s\big) \leq CMt^s.$$

Finally, if $t \geq r/2$ and $r = 1/8$, then $t \geq 1/16$, so

$$|u(x) - u(y)| \leq 2\|u\|_{L^\infty(B_1)} \leq CMt^s.$$

Combining all cases, we obtain $[u]_{C^s(B_{1/2})} \leq CM$, which proves the desired estimate.

The L^∞ bound follows from Lemma 1.35. This concludes the proof. □

Remark 1.37. The exponent s is optimal in general since already the torsion function $(1 - |x|^2)_+^s$ in B_1 given in Lemma 1.20 is not $C^{s+\delta}$ across the boundary for any $\delta > 0$. Note that this is in stark constrast to the local case, where solutions typically do not lose any interior regularity at the boundary, at least if $\partial\Omega$ is sufficiently regular.

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