SELECTED TOPICS LECTURE: INTRODUCTION TO NONLOCAL PDES

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The classical theory of second-order elliptic partial differential equations deals with differential operators like the Laplacian and models phenomena governed by purely local, continuous interactions, such as the diffusion of heat. However, many systems in nature or finance, such as gas dynamics or option pricing, demand models that account for long-range dependencies. Here, the dynamics at any given point are directly influenced by distant events. These phenomena are captured by nonlocal PDEs. This lecture provides an introduction to this topic, starting with the fundamental properties of the simplest nonlocal operator (the fractional Laplacian).

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1. The fractional Laplacian

Let us start by introducing the fractional Laplacian.

Definition 1.1. Let $0 < s < 1$ and $u \in C_c^\infty(\mathbb{R}^n)$. We define the fractional Laplacian as follows

$$(-\Delta)^s u(x) = c_{n,s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z, \tag{1.1}$$

where

$$c_{n,s} = \frac{2^{2s} s\,\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)} > 0, \tag{1.2}$$

and the principal value means

$$\mathrm{PV} \int_{\mathbb{R}^n} \cdots\,\mathrm{d}z := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \cdots\,\mathrm{d}z.$$

The constant (1.2) is chosen so that the Fourier multiplier is exactly $|\xi|^{2s}$ (we will see this later).

Sometimes it is convenient to write the integrand of the fractional Laplacian in terms of a second-order incremental quotient. The following lemma says that these defintions are equivalent.

Lemma 1.2. *For $u \in C_c^2(\mathbb{R}^n)$, the principal value in* (1.1) *exists and*

$$(-\Delta)^s u(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z. \tag{1.3}$$

The integral on the right converges absolutely.

Proof. Fix $x \in \mathbb{R}^n$. For $\delta > 0$ put

$$I_\delta(x) := \int_{|z|>\delta} \frac{u(x) - u(x+z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Note that $I_\delta(x)$ is finite, since u is bounded and the kernel is integrable away from the origin. Changing variables $z \mapsto -z$ gives

$$I_\delta(x) = \int_{|z|>\delta} \frac{u(x) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Summing up these two identities, we obtain

$$I_\delta(x) = \frac{1}{2} \int_{|z|>\delta} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} \,\mathrm{d}z.$$

Once we show that the limit as $\delta \to 0$ exists, the proof is complete. To do so, note that Taylor's formula yields, for $|z| < 1$,

$$\big|u(x+z) + u(x-z) - 2u(x)\big| = \left| \int_{-1}^{1} (1-|t|)\, z^T D^2 u(x+tz) z \,\mathrm{d}t \right| \leq \|D^2 u\|_{L^\infty(\mathbb{R}^n)} |z|^2.$$

since $\int_{-1}^{1} (1-|t|) \,\mathrm{d}t = 1$. Thus, writing $\omega_{n-1} = |\mathbb{S}^{n-1}|$ and using polar coordinates, as well as $s < 1$,

$$\begin{aligned} &\int_{|z|<1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} \,\mathrm{d}z \\ &\qquad \leq \|D^2 u\|_{L^\infty}\, \omega_{n-1} \int_0^1 r^{1-2s} \,\mathrm{d}r = \frac{\omega_{n-1}}{2(1-s)} \|D^2 u\|_{L^\infty} < \infty. \end{aligned}$$

On the other hand, for $|z| \geq 1$,

$$|2u(x) - u(x+z) - u(x-z)| \leq 4\|u\|_{L^\infty},$$

and hence

$$\begin{aligned} &\int_{|z|\geq 1} \frac{|2u(x) - u(x+z) - u(x-z)|}{|z|^{n+2s}} \,\mathrm{d}z \\ &\qquad \leq 4\|u\|_{L^\infty}\, \omega_{n-1} \int_1^\infty r^{-1-2s} \,\mathrm{d}r = \frac{2\omega_{n-1}}{s} \|u\|_{L^\infty} < \infty. \end{aligned}$$

Hence, $I_\delta(x)$ converges absolutely as $\delta \to 0$. This concludes the proof. □

Remark 1.3. Taylor's formula shows that, for $|z| < 1$,

$$|u(x) - u(x+z)| \leq \|\nabla u\|_{L^\infty} |z|.$$

Thus, when $0 < s < \frac{1}{2}$, the following integral is absolutely convergent near $z = 0$,

$$\int_{B_1} \frac{|u(x) - u(x+z)|}{|z|^{n+2s}} \,\mathrm{d}z \leq \|\nabla u\|_{L^\infty}\, \omega_{n-1} \int_0^1 r^{-2s} \,\mathrm{d}r = \frac{\omega_{n-1}}{1-2s} \|\nabla u\|_{L^\infty} < \infty.$$

Consequently, for $s < \frac{1}{2}$ the symbol PV may be omitted in (1.1).

Definition 1.4. For $0 < s < 1$, define

$$L^1_{2s}(\mathbb{R}^n) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{L^1_{2s}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(z)|}{1+|z|^{n+2s}} \,\mathrm{d}z < \infty \right\}.$$

We will sometimes call the integral in this definition the *tail integral*.

Note that $|x|^\alpha \in L^1_{2s}(\mathbb{R}^n)$ if and only if $\alpha < 2s$.

The following proposition establishes fine necessary conditions on u so that $(-\Delta)^s u$ is well-defined, meaning that (1.1) and (1.3) both converge. Moreover, we will show that $(-\Delta)^s u$ has $2s$ derivatives less than u. In this sence, $(-\Delta)^s$ is a *differential operator of order* $2s \in (0,2)$.

Proposition 1.5. *Let $0 < s < 1$ and $\varepsilon > 0$. If*

$$u \in C^{2s+\varepsilon}(B_1) \cap L^1_{2s}(\mathbb{R}^n),$$

then $(-\Delta)^s u$ is well-defined at every point of B_1 by (1.3), and

$$\|(-\Delta)^s u\|_{L^\infty(B_{1/2})} \le C\left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}\right).$$

Moreover, if $\varepsilon \notin \mathbb{N}$, *then $(-\Delta)^s u \in C^\varepsilon(B_{1/2})$ and*

$$\|(-\Delta)^s u\|_{C^\varepsilon(B_{1/2})} \le C\left(\|u\|_{C^{2s+\varepsilon}(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)}\right). \tag{1.4}$$

The constant C depends only on n, s, ε.

Proof. We first prove the estimate for $0 < \varepsilon < 1$ and put $\beta = 2s + \varepsilon \in (0,3)$.

Step 1. Fix $\eta \in C_c^\infty(B_{7/8})$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ in $B_{3/4}$, and write

$$u = v + w, \qquad v = \eta u, \qquad w = (1-\eta)u.$$

The idea is to estimate $(-\Delta)^s v$ and $(-\Delta)^s w$ separately. Note that by construction,

$$\|v\|_{C^\beta(\mathbb{R}^n)} \le C\|u\|_{C^\beta(B_1)}, \qquad \|w\|_{L^1_{2s}(\mathbb{R}^n)} \le \|u\|_{L^1_{2s}(\mathbb{R}^n)}.$$

Moreover, because $w = 0$ in $B_{3/4}$, for every $x \in B_{1/2}$ we have

$$(-\Delta)^s w(x) = -c_{n,s} \int_{\mathbb{R}^n \setminus B_{3/4}} \frac{w(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

Whenever $x \in B_{1/2}$ and $w(z) \neq 0$, we have $|x - z| \ge 1/4$. Consequently, for every multi-index γ,

$$\left|D_x^\gamma |x-z|^{-n-2s}\right| \le \frac{C_{n,s,\gamma}}{1+|z|^{n+2s}}.$$

We may differentiate under the integral to obtain

$$\|(-\Delta)^s w\|_{C^{m,1}(B_{1/2})} \le C\|w\|_{L^1_{2s}(\mathbb{R}^n)} \le C\|u\|_{L^1_{2s}(\mathbb{R}^n)} \quad (m = 0,1,2,\ldots).$$

Step 2. It remains to prove

$$\|(-\Delta)^s v\|_{C^\varepsilon(\mathbb{R}^n)} \le C\|v\|_{C^\beta(\mathbb{R}^n)}. \tag{1.5}$$

We define

$$\delta_z^2 v(x) = 2v(x) - v(x+z) - v(x-z).$$

The L^∞ bound follows by observing that

$$|\delta_z^2 v(x)| \le C\|v\|_{C^\beta}|z|^{\min\{\beta,2\}} \quad (|z| < 1), \qquad |\delta_z^2 v(x)| \le 4\|v\|_\infty \quad (|z| \ge 1)$$

in the same way as in the proof of Lemma 1.2, using that $\min\{\beta, 2\} > 2s$ when $0 < \varepsilon < 1$ and $\beta > 2s$.

To estimate the Hölder seminorm, take $x, x' \in \mathbb{R}^n$. We set $h = x - x'$ and assume that $|h| \in (0,1)$. Then, we have

$$|(-\Delta)^s v(x) - (-\Delta)^s v(x')| \le \frac{c_{n,s}}{2}\int_{|z|<|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}}\,\mathrm{d}z + \frac{c_{n,s}}{2}\int_{|z|\ge|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}}\,\mathrm{d}z$$

$$=: \frac{c_{n,s}}{2}\big(I_{\text{near}} + I_{\text{far}}\big).$$

It remains to show

$$I_{\text{near}} + I_{\text{far}} \leq C\|v\|_{C^\beta}|h|^\varepsilon. \tag{1.6}$$

We distinguish between two cases, depending on β.

Case A: $0 < \beta \leq 1$. Hölder continuity of v gives

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \leq \begin{cases} C\|v\|_{C^\beta}|z|^\beta, & |z| < |h|, \\ C\|v\|_{C^\beta}|h|^\beta, & |z| \geq |h|. \end{cases}$$

Indeed if $|z| < |h|$, we use that

$$|\delta_z^2 v(x)| = \big|\big(v(x) - v(x+z)\big) + \big(v(x) - v(x-z)\big)\big| \leq C\|v\|_{C^\beta}|z|^\beta,$$

and if $|z| \geq |h|$, we estimate

$$\begin{aligned} \big|\delta_z^2 v(x) - \delta_z^2 v(x')\big| &= \big|2\big(v(x) - v(x')\big) - \big(v(x+z) - v(x'+z)\big) - \big(v(x-z) - v(x'-z)\big)\big| \\ &\leq C\|v\|_{C^\beta}|h|^\beta \end{aligned}$$

Polar coordinates then yield

$$\begin{aligned} I_{\text{near}} &\leq C\|v\|_{C^\beta}\int_0^{|h|} \rho^{\beta-2s-1}\,\mathrm{d}\rho = \frac{C}{\varepsilon}\|v\|_{C^\beta}|h|^\varepsilon, \\ I_{\text{far}} &\leq C\|v\|_{C^\beta}|h|^\beta\int_{|h|}^\infty \rho^{-2s-1}\,\mathrm{d}\rho = \frac{C}{2s}\|v\|_{C^\beta}|h|^\varepsilon. \end{aligned}$$

Case B: $1 < \beta < 3$. We claim that by the fundamental theorem of calculus, for any $z \in \mathbb{R}^n$

$$\big|\delta_z^2 v(x) - \delta_z^2 v(x')\big| = \left|\int_0^1 \big(\delta_z^2 \nabla v\big)(x' + th)\cdot h\,\mathrm{d}t\right| \leq C\|v\|_{C^\beta}|h||z|^{\beta-1}. \tag{1.7}$$

For $1 < \beta \leq 2$, the proof of (1.7) uses that by $\nabla v \in C^{\beta-1}$, we have

$$|\nabla v(w \pm z) - \nabla v(w)| \leq [\nabla v]_{C^{\beta-1}}|z|^{\beta-1},$$

and hence we conclude

$$\begin{aligned} |\delta_z^2 \nabla v(w)| &= \big|\nabla v(w+z) + \nabla v(w-z) - 2\nabla v(w)\big| \\ &\leq |\nabla v(w+z) - \nabla v(w)| + |\nabla v(w-z) - \nabla v(w)| \\ &\leq 2[\nabla v]_{C^{\beta-1}}|z|^{\beta-1}, \end{aligned}$$

which implies (1.7).

For $2 < \beta < 3$, we prove (1.7) using $\nabla v \in C^{1,\beta-2}$ and $D^2 v \in C^{\beta-2}$. Indeed, by Taylor's formula,

$$\nabla v(w \pm z) = \nabla v(w) \pm D^2 v(w) z + R_\pm(w,z), \quad \text{where} \quad |R_\pm(w,z)| \leq C[D^2 v]_{C^\alpha}|z|^{\beta-1}.$$

Adding these two estimates and subtracting $2\nabla v(w)$, we obtain

$$\big|\delta_z^2 \nabla v(w)\big| = \big|2\nabla v(w) - \nabla v(w+z) - \nabla v(w-z)\big| \leq \big|R_+(w,z)\big| + \big|R_-(w,z)\big| \leq C[D^2 v]_{C^{\beta-2}}|z|^{\beta-1}.$$

This proves (1.7) also in this case and thus, we conclude

$$I_{\text{far}} = \int_{|z|\geq|h|} \frac{|\delta_z^2 v(x) - \delta_z^2 v(x')|}{|z|^{n+2s}}\,dz \leq C\|v\|_{C^\beta}|h|\int_{|z|\geq|h|} |z|^{\beta-1-n-2s}\,dz \leq C\|v\|_{C^\beta}|h|^\varepsilon.$$

To estimate I_{near}, note that for $1 < \beta \le 2$, we can use (1.7) to conclude since $|h| \le |z|$,

$$I_{\text{near}} \le C\|v\|_{C^\beta} \int_0^{|h|} |z|^{\beta-1-n-2s}\, d|z| \le C\|v\|_{C^\beta} |z|^\varepsilon.$$

For $2 < \beta < 3$, we use the integral representation of the second difference:

$$\delta_z^2 v(x) = -\int_{-1}^1 (1-|t|)\, z^T D^2 v(x+tz) z\, dt.$$

which implies that

$$|\delta_z^2 v(x) - \delta_z^2 v(x')| \le |z|^2 \int_{-1}^1 (1-|t|)|D^2 v(x+tz) - D^2 v(x'+tz)|\, dt \le C[D^2 v]_{C^{\beta-2}} |h|^{\beta-2} |z|^2.$$

Hence, it follows that

$$I_{\text{near}} \le C\|v\|_{C^\beta} |h|^{\beta-2} \int_0^{|h|} |z|^{2-n-2s}\, dz \le C\|v\|_{C^\beta} |h|^{\beta-2} |h|^{2-2s} = C\|v\|_{C^\beta} |h|^\varepsilon.$$

Hence, we have shown (1.6) also in this case.

When $|x - x'| \ge 1$, the same estimate follows from the L^∞ bound. This proves (1.5) and the proof is complete in case $\varepsilon \in (0,1)$.

Step 3. Now we consider the case $\varepsilon > 1$ with $\varepsilon \notin \mathbb{N}$. We only give a sketch of the argument. Write $\varepsilon = m + \alpha$, with $m \ge 1$ an integer and $0 < \alpha < 1$. As above, split $u = v + w$ with $v = \eta u$ and $w = (1-\eta)u$. Translation invariance of $(-\Delta)^s$ implies that

$$D^\gamma((-\Delta)^s v) = (-\Delta)^s (D^\gamma v), \qquad |\gamma| \le m.$$

Since $D^\gamma v \in C^{2s+\varepsilon-|\gamma|}(\mathbb{R}^n) \subset C^{2s+\alpha}(\mathbb{R}^n)$, the estimate just proved with $\alpha \in (0,1)$ applies to each $D^\gamma v$. From here, one can conclude the proof. □

Lemma 1.6. *For $u \in C_c^\infty(\mathbb{R}^n)$, the following identities hold:*

$$\begin{aligned}
(-\Delta)^s[u(\cdot + x_0)](x) &= ((-\Delta)^s u)(x + x_0), \\
(-\Delta)^s[u(Q\cdot)](x) &= ((-\Delta)^s u)(Qx) && \text{for } Q \in O(n), \\
(-\Delta)^s[u(\lambda\cdot)](x) &= \lambda^{2s}((-\Delta)^s u)(\lambda x) && \text{for } \lambda > 0.
\end{aligned}$$

Proof. In (1.1), use translation invariance, the change of variables $z \mapsto Qz$, and the substitution $w = \lambda z$, respectively. □

Next, we show that the fractional Laplacian arises as a Fourier multiplier operator.

Our Fourier-transform convention is

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, \mathrm{d}x.$$

Recall that $\mathcal{F}(-\Delta u)(\xi) = |\xi|^2 \widehat{u}(\xi)$.

Lemma 1.7. *For $u \in \mathcal{S}(\mathbb{R}^n)$ and $0 < s < 1$, it holds*

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi).$$

Proof. Taking the Fourier transform of (1.3) and using that

$$\mathcal{F}[u(x+z)](\xi) = e^{i\xi\cdot z}\widehat{u}(\xi), \quad e^{it} + e^{-it} = 2\cos t$$

we obtain

$$\widehat{(-\Delta)^s u}(\xi) = \frac{c_{n,s}}{2}\left(\int_{\mathbb{R}^n} \frac{2 - e^{i\xi\cdot z} - e^{-i\xi\cdot z}}{|z|^{n+2s}}\, dz\right)\widehat{u}(\xi) = c_{n,s}\left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z\right)\widehat{u}(\xi).$$

Here $1-\cos(\xi\cdot z) = O(|z|^2)$ near zero and is bounded at infinity, so the integral is finite. By rotation invariance and $w = |\xi|z$,

$$\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z = \int_{\mathbb{R}^n} \frac{1-\cos w_1}{(|w|/|\xi|)^{n+2s}}|\xi|^{-n}\, dw = |\xi|^{2s} J_{n,s}, \qquad J_{n,s} := \int_{\mathbb{R}^n} \frac{1-\cos w_1}{|w|^{n+2s}}\,\mathrm{d}w.$$

For completeness we evaluate the constant. The Gamma-function identity and the Gaussian integral

$$|w|^{-n-2s} = \frac{1}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{\frac{n}{2}+s-1}e^{-t|w|^2}\,\mathrm{d}t, \qquad \int_{\mathbb{R}^n}(1-\cos w_1)e^{-t|w|^2}\,dw = \left(\frac{\pi}{t}\right)^{n/2}(1-e^{-1/(4t)})$$

substituted into the definition of $J_{n,s}$ gives

$$\begin{aligned}
J_{n,s} &= \frac{1}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{\frac{n}{2}+s-1}\left[\int_{\mathbb{R}^n}(1-\cos w_1)e^{-t|w|^2}\,dw\right]dt \\
&= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+s)}\int_0^\infty t^{s-1}(1-e^{-1/(4t)})\,\mathrm{d}t \\
&= \frac{4^{-s}\pi^{n/2}}{\Gamma(\frac{n}{2}+s)}\int_0^\infty v^{-s-1}(1-e^{-v})\,\mathrm{d}v \\
&= \frac{4^{-s}\pi^{n/2}\Gamma(1-s)}{s\,\Gamma(\frac{n}{2}+s)}.
\end{aligned}$$

Formula (1.2) says precisely that $c_{n,s}J_{n,s} = 1$. This proves the desired result. □

Corollary 1.8. *If $u, v \in C_c^\infty(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} u\,(-\Delta)^s v\,\mathrm{d}x = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z.$$

In particular,

$$\int_{\mathbb{R}^n} u\,(-\Delta)^s u\,\mathrm{d}x = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{2s}|\widehat{u}(\xi)|^2\,\mathrm{d}\xi =: [u]_{H^s(\mathbb{R}^n)}^2 \geq 0$$

Proof. By the singular integral definition,

$$\begin{aligned}
\int_{\mathbb{R}^n} u(-\Delta)^s v\;\mathrm{d}x &= c_{n,s}\lim_{\varepsilon\to 0}\iint_{|x-z|>\varepsilon}\frac{u(x)(v(x)-v(z))}{|x-z|^{n+2s}}\;\mathrm{d}x\;\mathrm{d}z \\
&= \frac{c_{n,s}}{2}\lim_{\varepsilon\to 0}\iint_{|x-z|>\varepsilon}\frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}}\;\mathrm{d}x\;\mathrm{d}z,
\end{aligned}$$

where we interchanged the roles of x and z in order to get the last equality.

Since $u, v \in C_c^\infty(\mathbb{R}^n)$, the integrand is absolutely integrable. Thus we may let $\varepsilon \to 0$, obtaining the desired identity. The remaining claims follow immediately from Lemma 1.7. □

The following semigroup property follows immediately.

Corollary 1.9. *For $s,t>0$ with $s+t\le 1$, and $u\in\mathcal{S}(\mathbb{R}^n)$,*

$$(-\Delta)^s\big((-\Delta)^t u\big)=(-\Delta)^{s+t}u.$$

Proof. By Lemma 1.7, it holds $\widehat{v}(\xi)=|\xi|^{2t}\widehat{u}(\xi)$. Hence, by applying the inverse Fourier transform,

$$v(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2t}\widehat{u}(\xi)\,\mathrm{d}\xi.$$

Inserting this into the definition of the fractional Laplacian and using Fubini, we obtain

$$\begin{aligned}(-\Delta)^s v(x)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2t}\widehat{u}(\xi)\left(c_{n,s}\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,\mathrm{d}z\right)\mathrm{d}\xi\\&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2(s+t)}\widehat{u}(\xi)\,\mathrm{d}\xi\\&=(-\Delta)^{s+t}u(x).\end{aligned}$$

□

Our normalization makes $(-\Delta)^s$ interpolate between the identity and $-\Delta$.

Lemma 1.10. *Let $u\in C_c^\infty(\mathbb{R}^n)$. Then, it holds for every $x\in\mathbb{R}^n$,*

$$\lim_{s\downarrow 0}(-\Delta)^s u(x)=u(x),\qquad \lim_{s\uparrow 1}(-\Delta)^s u(x)=-\Delta u(x).$$

Proof. Let $\omega_{n-1}=|\mathbb{S}^{n-1}|=2\pi^{n/2}/\Gamma(n/2)$. Then

$$c_{n,s}\sim\frac{2s}{\omega_{n-1}}\quad(s\downarrow 0),\qquad c_{n,s}\sim\frac{4n(1-s)}{\omega_{n-1}}\quad(s\uparrow 1).$$

To see it, in (1.2) use $\Gamma(1-s)\to 1$ as $s\downarrow 0$ and $\Gamma(1-s)\sim(1-s)^{-1}$ as $s\uparrow 1$, together with $\Gamma(1+\frac{n}{2})=\frac{n}{2}\Gamma(\frac{n}{2})$.

Step 1: $s\downarrow 0$. Fix x and choose $R>1$ so large that $u(x+z)=0$ for $|z|>R$. We compute

$$(-\Delta)^s u(x)=c_{n,s}\ \mathrm{PV}\int_{B_R}\frac{u(x)-u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z+c_{n,s}u(x)\int_{\mathbb{R}^n\setminus B_R}|z|^{-n-2s}\,\mathrm{d}z=:I_{1,R}+I_{2,R}.$$

For $0<s\le\frac14$, the first integral is uniformly bounded since

$$|u(x)-u(x+z)|\le C\min\{|z|,1\}.$$

Since $c_{n,s}\to 0$, this implies $I_{1,R}\to 0$ as $s\to 0$. For $I_{2,R}$, we get

$$I_{2,R}=c_{n,s}u(x)\int_{\mathbb{R}^n\setminus B_R}|z|^{-n-2s}\,\mathrm{d}z=c_{n,s}\frac{\omega_{n-1}}{2s}R^{-2s}u(x)\longrightarrow u(x),$$

by the asymptotics of $c_{n,s}$.

Step 2: $s\uparrow 1$. Let us fix $\delta\in(0,1)$. Note that by Taylor's formula

$$2u(x)-u(x+z)-u(x-z)=-z^TD^2u(x)z+\rho_x(z)|z|^2,\qquad |\rho_x(z)|\le C|z|^2\quad\text{for}\quad|z|\le\delta.$$

We use (1.3) and split the integral at radius δ, namely

$$\begin{aligned}(-\Delta)^s u(x)=&\frac{c_{n,s}}{2}\int_{\mathbb{R}^n\setminus B_\delta}\frac{2u(x)-u(x+z)-u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z\\&-\frac{c_{n,s}}{2}\int_{B_\delta}\frac{z^TD^2u(x)z}{|z|^{n+2s}}\,\mathrm{d}z+\frac{c_{n,s}}{2}\int_{B_\delta}\rho_x(z)|z|^{-n+2-2s}\,\mathrm{d}z=:I_1+I_2+I_3.\end{aligned}$$

Since $c_{n,s} \to 0$, we have that $I_1 \to 0$ as $s \to 1$. Moreover, by symmetry on the sphere, we have

$$\int_{\mathbb{S}^{n-1}} \theta_i \theta_j \,\mathrm{d}\theta = \frac{\omega_{n-1}}{n} \delta_{ij}.$$

Therefore, as $s \to 1$,

$$\begin{aligned} I_2 &= -\frac{c_{n,s}}{2} \sum_{i,j} u_{ij}(x) \int_{B_\delta} \frac{z_i z_j}{|z|^{n+2s}} \,\mathrm{d}z \\ &= -\frac{c_{n,s}\omega_{n-1}}{2n} \Delta u(x) \int_0^\delta r^{1-2s} \,\mathrm{d}r = -\frac{c_{n,s}\omega_{n-1}}{4n(1-s)} \delta^{2-2s} \Delta u(x) \to -\Delta u(x). \end{aligned}$$

For I_3, we note that

$$|I_3| \le \frac{c_{n,s} C}{2} \int_{B_\delta} |z|^{-n+4-2s} \,\mathrm{d}z = \frac{c_{n,s} C \omega_{n-1}}{2(4-2s)} \delta^{4-2s} \to 0,$$

since $c_{n,s} \to 0$ as $s \to 1$. This concludes the proof. □

1.1. **The Caffarelli-Silvestre extension.** In this subsection, we prove that the nonlocal fractional Laplacian can be recast as a *local differential operator in one dimension higher*. This constitutes one of the most important properties of the fractional Laplacian and goes back to a famous paper by Caffarelli and Silvestre [CS07]. It is one of the most cited papers in partial differential equations of all times and it has many applications, as we will see below.

We first treat the case $s = \frac{1}{2}$.

Theorem 1.11. *Let $u \in C_c^\infty(\mathbb{R}^n)$, and let U be its bounded harmonic extension:*

$$\begin{cases} \Delta_{x,y} U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.8}$$

Then

$$(-\Delta)^{1/2} u(x) = -\partial_y U(x,0). \tag{1.9}$$

Proof. We apply Fourier transform in x to (1.8):

$$\partial_{yy} \widehat{U}(\xi, y) - |\xi|^2 \widehat{U}(\xi, y) = 0, \qquad \widehat{U}(\xi, 0) = \widehat{u}(\xi).$$

The bounded solution is $\widehat{U}(\xi, y) = e^{-|\xi| y} \widehat{u}(\xi)$. Thus $-\partial_y \widehat{U}(\xi, 0) = |\xi| \widehat{u}(\xi)$, which is exactly the Fourier transform of $(-\Delta)^{1/2} u$ by Theorem 1.7. Taking inverse Fourier transforms yields (1.9). □

Remark 1.12. The above theorem gives another justification for the notation $(-\Delta)^{1/2}$. Indeed, denote $Lu = -\partial_y U(x,0)$ and compute $L(Lu)$. Indeed, if U is the harmonic extension of u, then $-\partial_y U$ is the harmonic extension of Lu (since it coincides with Lu on $\{y = 0\}$ and it is harmonic). Thus,

$$L(Lu)(x) = -\left.\partial_y\right|_{y=0} (-\partial_y U) = \left.\partial_{yy}\right|_{y=0} U(x,y).$$

Now, since

$$\Delta_{x,y} U = \Delta_x U + \partial_{yy} U = 0,$$

we have that

$$L(Lu)(x) = -\Delta_x U(x,0) = -\Delta u(x).$$

That is, $L^2 u(x) = -\Delta u(x)$, which justifies the notation $L = \sqrt{-\Delta}$.

We will now generalize this result to general $s \in (0,1)$. This was achieved in the famous paper [CS07]. To do so, we set $a = 1-2s \in (-1,1)$. The *a-harmonic extension* of a function u is a solution U of

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} U\big) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) &= u(x) \quad \text{on } \mathbb{R}^n. \end{cases} \tag{1.10}$$

For $y > 0$, the equation is equivalently

$$\mathcal{L}_a U := \Delta_x U + \partial_{yy} U + \frac{a}{y}\partial_y U = 0.$$

Notice that $a = 0$ when $s = \frac{1}{2}$. For other values of s, y^a vanishes or becomes singular at $y = 0$; this is a degenerate or singular elliptic equation, rather than a uniformly elliptic one up to the boundary.

Lemma 1.13 (Poisson kernel for the weighted equation)**.** *Let*

$$P_s(z,y) = p_{n,s}\frac{y^{2s}}{(|z|^2+y^2)^{(n+2s)/2}}, \qquad p_{n,s} = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(s)}.$$

Then, for every $y > 0$,

$$P_s(\cdot, y) \geq 0, \qquad \int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = 1,$$

and P_s solves $\operatorname{div}_{x,y}(y^a \nabla_{x,y} P_s) = 0$ for $y > 0$.

Moreover, for $u \in C^{2s+\varepsilon}_{loc}(B_1)$ with $|u(x)| \leq C(1+|x|)^{2s-\varepsilon}$ for some $\varepsilon > 0$, the convolution

$$U(x,y) = \int_{\mathbb{R}^n} P_s(x-z,y)u(z)\,\mathrm{d}z \tag{1.11}$$

solves (1.10) with $U(\cdot,y) \to u$ locally uniformly in B_1 and a.e. in $\mathbb{R}^n \setminus B_1$ as $y \to 0$.

Moreover, (1.11) is the unique solution to (1.10) satisfying $|U(x,y)| \leq C(1+|(x,y)|)^{2s-\varepsilon}$.

Proof. The change of variables $z = yw$ gives

$$\int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = p_{n,s}\int_{\mathbb{R}^n}(1+|w|^2)^{-(n+2s)/2}\,\mathrm{d}w = p_{n,s}\frac{\pi^{n/2}\Gamma(s)}{\Gamma(\frac{n}{2}+s)} = 1.$$

In particular $P_s(z,y) = y^{-n}P_s(z/y,1)$, i.e. P_s is an approximate identity. To verify the PDE, set $r = (|z|^2+y^2)^{1/2}$, so that

$$P_s(z,y) = p_{n,s}y^{2s}r^{-(n+2s)}.$$

By Lemma 1.14 (see below), we have

$$\begin{aligned} &\left(\Delta_{z,y} + \frac{a}{y}\partial_y\right)(y^{2s}r^{-(n+2s)}) \\ &\quad = 2s(2s-1+a)y^{2s-2}r^{-(n+2s)} + (n+2s)(1-2s-a)y^{2s}r^{-(n+2s+2)} \\ &\quad = 0, \end{aligned}$$

since $a = 1-2s$. Thus,

$$\operatorname{div}(y^a \nabla P_s) = 0 \qquad \text{for } y > 0.$$

To prove that (1.11) satisfies the PDE, it suffices to observe that differentiation under the convolution is justified for $y > 0$. To prove that $U(\cdot,y) \to u$ locally uniformly in B_1 as $y \to 0$, we write

$$U(x,y) - u(x) = \int_{\mathbb{R}^n} P_s(z,y)\big(u(x-z) - u(x)\big)\,\mathrm{d}z.$$

Moreover, given $K \Subset B_1$, and $\delta > 0$ such that $\{x \in \mathbb{R}^n : \operatorname{dist}(x, K) < \delta\} \Subset B_1$, we define the modulus of continuity

$$\omega_u(\delta) = \sup_{\substack{x \in K \\ |z| \leq \delta}} |u(x - z) - u(x)|.$$

Then

$$\sup_{x \in K} |U(x, y) - u(x)| \leq \omega_u(\delta) + C \|(1 + |x|)^{\varepsilon - 2s} u(x)\|_{L^\infty(\mathbb{R}^n)} \int_{|z| \geq \delta} P_s(z, y)(1 + |z|)^{2s - \varepsilon} \, \mathrm{d}z.$$

Using the explicit expression for P_s,

$$\int_{|z| \geq \delta} P_s(z, y)(1 + |z|)^{2s - \beta} \, \mathrm{d}z \leq C y^{2s} \int_{|z| \geq \delta} \frac{(1 + |z|)^{2s - \varepsilon}}{|z|^{n + 2s}} \, \mathrm{d}z \leq C(\delta) y^{2s}.$$

Letting $y \to 0$ gives

$$\limsup_{y \downarrow 0} \|U(\cdot, y) - u\|_{L^\infty(K)} \leq \omega_u(\delta).$$

Since u is uniformly continuous in B_1, we have $\omega_u(\delta) \to 0$ as $\delta \downarrow 0$. Hence

$$\lim_{y \downarrow 0} \|U(\cdot, y) - u\|_{L^\infty(K)} = 0,$$

as desired.

The convergence $U(\cdot, y) \to u$ a.e. in $\mathbb{R}^n \setminus B_1$ follows from the Lebesgue differentiation theorem and since P_s is an approximate identity and $u \in L^1_{loc}(\mathbb{R}^n)$.

Finally, uniqueness of U can be proved by an argument based on the maximum principle applied to the operator $\operatorname{div}_{x,y}\big(y^a \nabla_{x,y}\big)$. Indeed, one can prove that the only solution W to

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} W\big) & = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ W(x, 0) & = 0 \quad \text{on } \mathbb{R}^n, \\ |W(x, y)| & \leq C(1 + |(x, y)|)^{2s - \varepsilon} \quad \text{in } \mathbb{R}^{n+1}_+ \end{cases}$$

is given by $W \equiv 0$.

To see it, we consider $\gamma \in (2s - \varepsilon, 2s)$ and consider

$$\Phi(x, y) = |(x, y)|^\gamma + \frac{n}{2s - \gamma} y^\gamma.$$

Since $a = 1 - 2s$, by Lemma 1.14 (see below), we get

$$\mathcal{L}_a(|(x, y)|^\gamma) = \gamma(\gamma + n - 2s)|(x, y)|^{\gamma - 2}, \qquad \mathcal{L}_a(y^\gamma) = \gamma(\gamma - 2s) y^{\gamma - 2}.$$

Consequently, using $y \leq |(x, y)|$ and $\gamma < 2$,

$$\mathcal{L}_a \Phi = \gamma(\gamma + n - 2s)|(x, y)|^{\gamma - 2} - n\gamma y^{\gamma - 2} \leq \gamma(\gamma - 2s)|(x, y)|^{\gamma - 2} < 0.$$

Let us now fix $R > 1$ sufficiently large and $\delta > 0$. Set $B_R^+ = B_R \cap \{y > 0\} \subset \mathbb{R}^{n+1}_+$. Then the previous observation implies

$$\mathcal{L}_a(W - \delta\Phi) = -\delta \mathcal{L}_a \Phi > 0 \quad \text{in } B_R^+.$$

Moreover, since $\gamma > 2s - \varepsilon$, for every sufficiently large R,

$$\delta R^\gamma > C(1 + R^\beta).$$

and therefore,

$$W - \delta\Phi \leq C(1 + R^{2s - \varepsilon}) - \delta R^\gamma \leq 0 \quad \text{on } \partial B_R^+ \setminus \{y = 0\}$$

$$W - \delta\Phi = -\delta|x|^\gamma \le 0 \quad \text{ on } \partial B_R^+ \cap \{y = 0\}$$

The maximum principle for $\mathcal{L}_a$ applied in B_R^+ therefore yields

$$W \le \delta\Phi \qquad \text{in } B_R^+.$$

Indeed, a positive interior maximum at $(x_0, y_0) \in \mathbb{R}^{n+1}_+$ would satisfy

$$\nabla(W - \delta\Phi)(x_0, y_0) = 0, \qquad \Delta(W - \delta\Phi)(x_0, y_0) \le 0,$$

and hence

$$\mathcal{L}_a(W - \delta\Phi)(x_0, y_0) \le 0,$$

a contradiction.

Letting $R \to \infty$ yields $W \le \delta\Phi$ in $\mathbb{R}^{n+1}_+$. Letting then $\delta \downarrow 0$ gives $W \le 0$ in $\mathbb{R}^{n+1}_+$. Applying the same argument to $-W$ yields $W \ge 0$, hence $W \equiv 0$. □

In the proof we used the following elementary lemma.

Lemma 1.14. *Let $a = 1 - 2s$ and*

$$\mathcal{L}_a = \Delta_{x,y} + \frac{a}{y}\partial_y.$$

For $z \in \mathbb{R}^n$, set $R = |(x, y) - (z, 0)|$, where $(x, y) \in \mathbb{R}^{n+1}_+$. Then, for every $\lambda, \mu \in \mathbb{R}$,

$$\mathcal{L}_a(y^\mu R^\lambda) = \mu(\mu + a - 1)y^{\mu-2}R^\lambda + \lambda(\lambda + n + a - 1 + 2\mu)y^\mu R^{\lambda-2}.$$

Proof. Since

$$\Delta_{x,y}R^\lambda = \lambda(\lambda + n - 1)R^{\lambda-2}, \qquad \partial_y R^\lambda = \lambda y R^{\lambda-2},$$

we obtain

$$\mathcal{L}_a(R^\lambda) = \lambda(\lambda + n + a - 1)R^{\lambda-2}.$$

Moreover,

$$\mathcal{L}_a(y^\mu) = \mu(\mu + a - 1)y^{\mu-2},$$

and

$$2\nabla_{x,y}(y^\mu) \cdot \nabla_{x,y}(R^\lambda) = 2\mu\lambda y^\mu R^{\lambda-2}.$$

The claim follows from the product rule

$$\mathcal{L}_a(fg) = f\mathcal{L}_a g + g\mathcal{L}_a f + 2\nabla_{x,y}f \cdot \nabla_{x,y}g.$$

□

Theorem 1.15 (Caffarelli–Silvestre extension)**.** *Let $0 < s < 1$, $a = 1 - 2s$, $u \in C^{2s+\varepsilon}_{loc}(B_1)$ with $|u(x)| \le C(1 + |x|)^{2s-\varepsilon}$ for some $\varepsilon > 0$, and let U be unique the solution to*

$$\begin{cases} \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} U\big) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x, 0) = u(x) & \text{on } \mathbb{R}^n \end{cases}$$

with $|U(x, y)| \le C(1 + |(x, y)|)^{2s-\varepsilon}$ given by the weighted Poisson extension (1.11)*. Then*

$$(-\Delta)^s u(x) = -d_s \lim_{y \downarrow 0} y^a \partial_y U(x, y), \qquad d_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1 - s)}. \tag{1.12}$$

In particular, note that for $s = \frac{1}{2}$, it holds $a = 0$ and $d_s = 1$, so we recover Theorem 1.11.

Proof. Use $\int_{\mathbb{R}^n} P_s(z,y)\,\mathrm{d}z = 1$ and evenness of $P_s(z,y)$ in z to write

$$U(x,y) - u(x) = p_{n,s} \int_{\mathbb{R}^n} \frac{y^{2s}\big(u(x-z) - u(x)\big)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z = \frac{p_{n,s}}{2} \int_{\mathbb{R}^n} \frac{y^{2s}\big(u(x+z) + u(x-z) - 2u(x)\big)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z.$$

Differentiate with respect to y and multiply by $y^a = y^{1-2s}$ to get

$$\begin{aligned} y^a \partial_y U(x,y) &= s p_{n,s} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{(|z|^2+y^2)^{(n+2s)/2}}\,\mathrm{d}z \\ &\quad - \frac{(n+2s)p_{n,s}}{2} y^2 \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{(|z|^2+y^2)^{(n+2s+2)/2}}\,\mathrm{d}z \\ &= I_1 + I_2. \end{aligned} \tag{1.13}$$

Fix $x \in B_1$ and choose $\delta > 0$ such that $B_{2\delta}(x) \subset B_1$. Since $u \in C^{2s+\varepsilon}_{\mathrm{loc}}(B_1)$, we may choose

$$2s < \alpha < \min\{2s+\varepsilon, 2\}$$

such that

$$|u(x+z) + u(x-z) - 2u(x)| \le C_x |z|^\alpha, \qquad |z| < \delta.$$

On the other hand, the growth assumption yields

$$|u(x+z) + u(x-z) - 2u(x)| \le C_x (1+|z|)^{2s-\varepsilon}, \qquad |z| \ge \delta.$$

Consequently,

$$\frac{|u(x+z) + u(x-z) - 2u(x)|}{(|z|^2+y^2)^{(n+2s)/2}} \le C_x \begin{cases} |z|^{\alpha-n-2s}, & |z| < \delta, \\ (1+|z|)^{2s-\varepsilon}|z|^{-n-2s}, & |z| \ge \delta. \end{cases}$$

The right-hand side belongs to $L^1(\mathbb{R}^n)$.

Hence, by dominated convergence, we get

$$\lim_{y \downarrow 0} I_1(y) = s p_{n,s} \int_{\mathbb{R}^n} \frac{u(x-z) + u(x+z) - 2u(x)}{|z|^{n+2s}}\,\mathrm{d}z = -2s p_{n,s}\,\mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z.$$

Similarly, for I_2, we observe that $\frac{y^2}{|z|^2+y^2} \le 1$, and therefore, dominated convergence implies that

$$\lim_{y \downarrow 0} I_2(y) = 0.$$

We therefore obtain from (1.13),

$$-\lim_{y \downarrow 0} y^a \partial_y U(x,y) = 2s p_{n,s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x-z)}{|z|^{n+2s}}\,\mathrm{d}z = \frac{2s p_{n,s}}{c_{n,s}} (-\Delta)^s u(x).$$

The constants satisfy

$$\frac{c_{n,s}}{2s p_{n,s}} = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} = d_s,$$

which is precisely (1.12). □

1.2. **The nonlocal Poisson kernel representation.** A major application of the Caffarelli-Silvestre extension is the following Poisson kernel representation of the fractional Laplacian.

Theorem 1.16 (Poisson kernel for the ball)**.** *Let $0 < s < 1$ and $r > 0$. The unique solution $u \in C^{2s+\varepsilon}_{loc}(B_1) \cap L^1_{2s}(\mathbb{R}^n) \cap C(\overline{B_1})$ of*

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } B_r, \\ u &= g \quad \text{in } \mathbb{R}^n \setminus B_r \end{cases}$$

for $g \in L^1_{2s}(\mathbb{R}^n) \cap C(\partial B_r)$, is given by

$$u(x) = \int_{\mathbb{R}^n \setminus B_r} \mathcal{P}_{r,s}(x,z) g(z) \, \mathrm{d}z, \qquad x \in B_r, \tag{1.14}$$

where

$$\mathcal{P}_{r,s}(x,z) = \kappa_{n,s} \frac{(r^2 - |x|^2)^s}{(|z|^2 - r^2)^s |x-z|^n}, \qquad \kappa_{n,s} = \frac{\Gamma(\frac{n}{2}) \sin(\pi s)}{\pi^{n/2+1}}.$$

Moreover,

$$\mathcal{P}_{r,s}(x,z) > 0, \qquad \int_{\mathbb{R}^n \setminus B_r} \mathcal{P}_{r,s}(x,z) \, \mathrm{d}z = 1.$$

Here, $g \in C(\partial B_r)$ is understood in the following sense:

$$\lim_{\substack{z \to x_0 \\ z \in \mathbb{R}^n \setminus B_r}} g(z) = g(x_0) \qquad \text{for every } x_0 \in \partial B_r.$$

Before we can prove this result, we need to give several auxilary lemmas. The first one is to find an explicit s-harmonic function in the half-space, which is a direct application of the Caffarelli-Silvestre extension.

Proposition 1.17. *Let $e \in \mathbb{S}^{n-1}$ and define*

$$u(x) = (x \cdot e)^s_+, \qquad t_+ = \max\{t, 0\}.$$

Then

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } \{x \cdot e > 0\}, \\ u &= 0 \quad \text{in } \{x \cdot e \le 0\}. \end{cases}$$

Proof. We use the Caffarelli–Silvestre extension. Write $t = x \cdot e$ and consider

$$U(x,y) = \left(\frac{\sqrt{t^2 + y^2} + t}{2} \right)^s.$$

First, we claim that

$$\widetilde{\mathcal{L}}_a U := \partial_{tt} U + \partial_{yy} U + \frac{1-2s}{y} \partial_y U = 0. \tag{1.15}$$

Indeed, set

$$R = \sqrt{t^2 + y^2}, \qquad v = R + t, \qquad U = 2^{-s} v^s.$$

Since $v = R + t$, and $\partial_t R = t R^{-1}$ we have

$$\partial_t v = \frac{t}{R} + 1, \qquad \partial_y v = \frac{y}{R}, \qquad |\nabla_{t,y} v|^2 = \frac{t^2}{R^2} + \frac{y^2}{R^2} + 1 + \frac{2t}{R} = \frac{2(R+t)}{R} = \frac{2v}{R}.$$

Moreover, by Lemma 1.14 (applied with $n = 1$), and using that $\widetilde{\mathcal{L}}_a t = 0$ since t is linear and independent of y,

$$\widetilde{\mathcal{L}}_a v = \widetilde{\mathcal{L}}_a R = (1+a)R^{-1}.$$

Hence, by the chain rule

$$\widetilde{\mathcal{L}}_a U = 2^{-s} s v^{s-1} \left(\widetilde{\mathcal{L}}_a v + (s-1)\frac{|\nabla_{t,y} v|^2}{v} \right) = \frac{2^{-s} s v^{s-1}}{R}(1 + a + 2s - 2) = 0.$$

This proves (1.15).

Since U depends on x only through $t = x \cdot e$ and $|e| = 1$, this is equivalent to

$$\mathcal{L}_a U = y^a \left(\Delta_x U + \partial_{yy} U + \frac{a}{y} \partial_y U \right) = y^a \left(\partial_{tt} U + \partial_{yy} U + \frac{1-2s}{y} \partial_y U \right) = 0 \quad \text{for } y > 0,$$

where we used that

$$\partial_{x_i} U = e_i \partial_t U, \qquad \partial_{x_i x_i} U = e_i^2 \partial_{tt} U, \qquad \Delta_x U = \sum_{i=1}^n e_i^2 \partial_{tt} U = |e|^2 \partial_{tt} U = \partial_{tt} U.$$

Moreover, taking $y \downarrow 0$,

$$U(x,0) = \left(\frac{|t| + t}{2} \right)^s = t_+^s = u(x).$$

Thus U is the weighted Poisson extension of u in the natural class of solutions with growth of order s.

We now compute the conormal derivative. First, we recall

$$\partial_y U = \frac{sy}{2R} \left(\frac{R+t}{2} \right)^{s-1}.$$

Therefore,

$$y^{1-2s} \partial_y U = \frac{s y^{2-2s}}{2r} \left(\frac{R+t}{2} \right)^{s-1}.$$

If $t > 0$, the factor $\left(\frac{R+t}{2}\right)^{s-1} / R$ remains bounded as $y \downarrow 0$. Since $s < 1$,

$$\lim_{y \downarrow 0} y^{1-2s} \partial_y U(x,y) = 0.$$

By the Caffarelli–Silvestre extension formula,

$$(-\Delta)^s u(x) = -d_s \lim_{y \downarrow 0} y^{1-2s} \partial_y U(x,y) = 0$$

whenever $x \cdot e > 0$. □

A key motivation to guess the right function in the proof comes from complex analysis, where it is a well-known fact that

$$U(t,y) = \operatorname{Re} \sqrt{t + iy} = \sqrt{\frac{\sqrt{t^2+y^2} + t}{2}}$$

is harmonic and has exactly the desired properties for $s = \frac{1}{2}$.

To be continued in the next lecture on Nov 10.

References

[CS07] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.