ADVANCED TOPICS LECTURE: FREE BOUNDARY PROBLEMS

MARVIN WEIDNER

The goal of this lecture is to give an introduction to free boundary problems. These are partial differential equations which exhibit an a priori unknown interface. A prototype example is given by the melting of ice in water, but free boundary problems also exist in various other contexts such as, physics, material sciences, biology, finance, etc.

Typical questions:

- optimal regularity of solutions (across the free boundary)
- regularity of the free boundary
- singular free boundary points

(1) Basic properties of harmonic functions
  - mean value property, maximum principle
  - basic regularity results

(2) The obstacle problem [FRRO22, PSU12]
  - optimal regularity
  - Caffarelli's dichotomy: regular and singular points
  - $C^{1,\alpha}$ regularity of the free boundary near regular points
  - higher regularity of the free boundary
  - properties of singular points
  - outlook

(3) The Alt-Caffarelli problem [Vel23, CS05]
  - optimal regularity
  - improvement of flatness
  - higher regularity of the free boundary
  - singular points
  - outlook

(4) Further topics
  - thin obstacle problem and nonlocal operators
  - time-dependent free boundary problems
  - free boundary problems with multiple phases
  - ...

1. Basic properties of harmonic functions

The Dirichlet problem for the Laplace equation is given as follows

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where the boundary condition g and the source term f are given and $\Omega \subset \mathbb{R}^n$ is a bounded (Lipschitz) domain. There are different ways to make sense of solutions to this problem. Under suitable assumptions on f, g, there exists a unique solution.

From now on, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We recall several important facts and definitions.

- We have the following function space
$$H^1(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega) \text{ for } i \in \{1, \dots, n\}\},$$
where $\partial_i u$ are the weak partial derivatives of u and $\nabla u = (\partial_1 u, \dots, \partial_n u)$.
- When equipped with the following scalar product, $H^1(\Omega)$ is a Hilbert space
$$(u, v)_{H^1(\Omega)} = \int uv \,\mathrm{d}x + \int \nabla u \nabla v \,\mathrm{d}x, \qquad (u, u)_{H^1(\Omega)} = \|u\|^2_{H^1(\Omega)}.$$
- Recall the following integration by parts formula: if $u, v \in H^1(\Omega)$, then
$$\int_\Omega \partial_i uv \,\mathrm{d}x = -\int_\Omega u \partial_i v \,\mathrm{d}x + \int_{\partial\Omega} uv\nu_i \,\mathrm{d}x, \qquad i = 1, \dots, n,$$
where $\nu \in \mathbb{S}^{n-1}$ is the unit outward normal vector to $\partial\Omega$.
- There is a compact trace operator $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$, such that $\mathrm{Tr}\, u = u|_{\partial\Omega}$ whenever $u \in H^1(\Omega) \cap C(\overline{\Omega})$. We define
$$H^1_0(\Omega) := \overline{C^\infty_c(\Omega)}_{H^1(\Omega)}$$
as the closure of $C^\infty_c(\Omega)$ with respect to $\| \cdot \|_{H^1(\Omega)}$. It holds
$$H^1_0(\Omega) = \{u \in H^1(\Omega) : \mathrm{Tr}(u) = 0\}.$$
- Sobolev embedding
$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega), \quad \text{if } 2 < n,$$
Moreover, the embedding $H^1(\Omega) \Subset L^q(\Omega)$ is compact, whenever $q < \frac{2n}{n-2}$. In particular, $H^1(\Omega) \Subset L^2(\Omega)$.
- Poincaré inequality: for any $u \in H^1(\Omega)$ it holds
$$\int_\Omega |u - (u)_\Omega|^2 \,\mathrm{d}x \le C_1 \int_\Omega |\nabla u|^2 \,\mathrm{d}x,$$
$$\int_\Omega |u|^2 \,\mathrm{d}x \le C_2 \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |\mathrm{Tr}\, u|^2 \,\mathrm{d}x.$$
The constants C_1, C_2 only depend on n, Ω.
- Hölder spaces: Let $\alpha \in (0, 1]$. We define for $u \in C(\overline{\Omega})$
$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \qquad \|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\alpha}(\overline{\Omega})}.$$
Moreover, for $k \in \mathbb{N} \cup \{0\}$, we set
$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \|u\|_{C^k(\Omega)} + [D^k u]_{C^{0,\alpha}(\overline{\Omega})}, \qquad \|u\|_{C^k(\Omega)} = \sum_{j=1}^k \|D^j u\|_{L^\infty(\Omega)}.$$

Note that by Hölder interpolation, it holds

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \asymp \|u\|_{L^\infty(\overline{\Omega})} + [D^k u]_{C^{0,\alpha}(\overline{\Omega})}, \qquad \|u\|_{C^{k,1}(\overline{\Omega})} \asymp \|u\|_{L^\infty(\overline{\Omega})} + \|D^{k+1}u\|_{L^\infty(\Omega)}.$$

We define the spaces

$$C^{k,\alpha}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \|u\|_{C^{k,\alpha}(\overline{\Omega})} < \infty\}.$$

Sometimes, when $0 < k + \alpha = \beta \notin \mathbb{N}$, we define $C^\beta(\overline{\Omega}) := C^{k,\alpha}(\overline{\Omega})$. Note

$$C^\infty(\overline{\Omega}) \subset \cdots \subset C^{k,\alpha}(\overline{\Omega}) \subset C^{1,\alpha}(\overline{\Omega}) \subset C^1(\overline{\Omega}) \subset C^{0,1}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega}) \subset C(\overline{\Omega}).$$

- Arzelà-Ascoli's theorem: Given a sequence $(f_i)_i \subset C^{k,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$ and $k \in \mathbb{N} \cup \{0\}$ satisfying $\|f_i\|_{C^{k,\alpha}(\overline{\Omega})} \le C$ for some $C > 0$. Then, there exists a subsequence $(f_{i_j})_j \subset (f_i)_i$ which converges uniformly (if $k = 0$) and in $C^k(\overline{\Omega})$ (if $k \in \mathbb{N}$) to some $f \in C^{k,\alpha}(\overline{\Omega})$ and $\|f\|_{C^{k,\alpha}}(\overline{\Omega}) \le C$.

Literature recommendation: [Eva10]. Also recall functional analysis and PDE lecture.

Definition 1.1. Let $f \in L^2(\Omega)$. We say that u satisfies $-\Delta u = f$ in Ω in the weak sense whenever $u \in H^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \quad \text{for all } v \in H_0^1(\Omega). \tag{1.2}$$

Let $g \in L^2(\partial\Omega)$. We say that u is a weak solution of the Dirichlet problem (1.1) if $u \in H^1(\Omega)$ satisfies $\operatorname{Tr} u = g$, and (1.2).

We say that u is weakly superharmonic (resp. weakly subharmonic) in Ω, or satisfies $-\Delta u \ge 0$ in Ω in the weak sense (resp. $-\Delta u \le 0$ in the weak sense) if

$$\int_\Omega \nabla u \cdot \nabla v \, dx \ge 0 \quad \text{resp.} \quad \int_\Omega \nabla u \cdot \nabla v \, dx \le 0 \quad \text{for all } v \in H_0^1(\Omega), v \ge 0.$$

We say that $u \ge g$ on $\partial\Omega$ if $Tru \ge g$ on $\partial\Omega$.

Remark 1.2. If $u \in C^2(\overline{\Omega})$, then it holds $-\Delta u = f$ in Ω in the classical sense, if and only if it holds in the weak sense. Proof: integration by parts.

1.1. **Regularity of solutions and the maximum principle.** Throughout this section, whenever we say that $\Omega \subset \mathbb{R}^n$ is a domain, we mean that Ω is a connected, bounded, open set with $\partial\Omega \in C^{0,1}$. The latter assumption can usually be relaxed, but we assume it here for simplicity in order to have a well-defined trace operator.

Theorem 1.3 (Existence and uniqueness of weak solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in L^2(\Omega)$ and*

$$\{w \in H^1(\Omega) : \operatorname{Tr} w = g\} \neq \emptyset. \tag{1.3}$$

Then, there exists a unique weak solution to the Dirichlet problem (1.1).

Proof. Lax Milgram. (We expect this to be well-known.) □

Remark 1.4.
- A sufficient condition for (1.3) to hold true is if $g \in C^{0,1}(\partial\Omega)$.
- (1.3) holds true if and only if there exists $G \in H^1(\Omega)$ such that $\operatorname{Tr} G = g$. One can show that this is the case if and only if $g \in H^{1/2}(\partial\Omega)$.

The unique weak solution to the Dirichlet problem in a ball is explicit:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases} \implies u(x) = \omega_{n-1} \int_{\partial B_1} \frac{(1-|x|^2)g(y)}{|x-y|^n} \,\mathrm{d}y,$$

where $\omega_{n-1} = |\mathbb{S}^{n-1}|$.

By a rescaling argument, a similar formula holds in any ball $B_r(x_0) \subset \mathbb{R}^n$. Thus, we deduce that for any harmonic function $\Delta u = 0$ in Ω, with $B_r(x_0) \subset \Omega$, we have (Poisson kernel representation)

$$u(x) = \omega_{n-1} r^{-1} \int_{\partial B_r(x_0)} \frac{(r^2 - |x-x_0|^2)u(y)}{|x-y|^n} dy. \tag{1.4}$$

An immediate consequence of (1.4) is the following result.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^n$ be any open set, and $u \in H^1(\Omega)$ be any function satisfying $\Delta u = 0$ in Ω in the weak sense. Then, u is C^∞ inside Ω and u is a classical solution.*

Moreover, if u is bounded and $\Delta u = 0$ in B_1 in the weak sense, then we have the estimates

$$\|u\|_{C^k(B_{1/2})} \le C_k \|u\|_{L^\infty(B_1)}, \tag{1.5}$$

for all $k \in \mathbb{N}$, and for some constant C_k depending only on k and n.

Proof. For any ball $B_r(x_0) \subset \Omega$ it holds (1.4). By differentiating this formula it is immediate to see that $u \in C^\infty(B_{r/2}(x_0))$ and that (1.5) holds. Since this can be done for any ball $B_r(x_0) \subset \Omega$, we deduce that u is C^∞ inside Ω. □

Next, we prove the maximum principle for weak solutions.

Proposition 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in H^1(\Omega)$ satisfies, in the weak sense,*

$$\begin{cases} -\Delta u \ge 0 & \text{in } \Omega \\ u \ge 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $u \ge 0$ in Ω.

Proof. Notice that since $-\Delta u \ge 0$ in Ω we have

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \ge 0 \quad \text{for all } v \ge 0, \quad v \in H_0^1(\Omega). \tag{1.6}$$

Let us consider $u^- := \max\{-u, 0\}$ and $u^+ := \max\{u, 0\}$, so that $u = u^+ - u^-$. It is easy to check that $u^\pm \in H^1(\Omega)$ whenever $u \in H^1(\Omega)$, and that $u^- \in H_0^1(\Omega)$ since $\operatorname{Tr} u \ge 0$ on $\partial\Omega$. Hence we can choose $v = u^- \ge 0$ in (1.6). Then, using that $\nabla u = \nabla u^+ - \nabla u^-$ and $\nabla u^+ \cdot \nabla u^- = 0$, we get

$$0 \le \int_\Omega \nabla u \cdot \nabla u^- \,\mathrm{d}x = \int_\Omega \nabla u^+ \cdot \nabla u^- \,\mathrm{d}x - \int_\Omega |\nabla u^-|^2 \,\mathrm{d}x = -\int_\Omega |\nabla u^-|^2 \,\mathrm{d}x.$$

Hence, $\nabla u^- \equiv 0$ in Ω. Since $\operatorname{Tr} u^- \equiv 0$ this implies $u^- \equiv 0$ in Ω, that is, $u \ge 0$ in Ω. □

Remark 1.7.
- comparison principle: If $-\Delta u \ge -\Delta v$ in Ω and $u \ge v$ on $\partial\Omega$, then $u \ge v$ in Ω.
- in particular, superharmonic functions have their minimum on the boundary.
- Analogously, if $-\Delta u \le 0$ in Ω and $u \le 0$ on $\partial\Omega$, then $u \le 0$ in Ω.

A useful consequence of the maximum principle is the following.

Lemma 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let u be any weak solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)},$$

for a constant C depending only on the diameter of Ω.

Proof. Let us consider the function

$$\tilde{u}(x) := u(x)/(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}).$$

We want to prove that $|\tilde{u}| \leq C$ in Ω. Notice that $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } \Omega \\ \tilde{u} = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

with $|\tilde{g}| \leq 1$ and $|\tilde{f}| \leq 1$.

Let us choose R large enough so that $B_R \supset \Omega$; after a translation, we can take $R = \operatorname{diam}(\Omega)$. In B_R, let us consider the function

$$w(x) = \frac{R^2 - |x|^2}{2} + 1.$$

The function w satisfies

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega \\ w \geq 1 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by the comparison principle, we deduce that

$$\tilde{u} \leq w \quad \text{in } \Omega.$$

Since $w \leq C$ (with C depending only on R), we deduce that $\tilde{u} \leq C$ in Ω. Finally, repeating the same argument with $-\tilde{u}$ instead of $\tilde{u}$, we find that $|\tilde{u}| \leq C$ in Ω, and thus we are done. $\square$

The following result follows from the maximum principle and states how solutions to the Dirichlet problem behave near the boundary.

We say that Ω satisfies the *interior ball condition* whenever there exists $\rho_0 > 0$ such that every point on $\partial\Omega$ can be touched from inside with a ball of radius ρ_0 contained in Ω. That is, for any $x_0 \in \partial\Omega$ there exists $B_{\rho_0}(y_0) \subset \Omega$ with $x_0 \in \partial B_{\rho_0}(y_0)$.

It is not difficult to see that any C^2 domain satisfies such condition, and also any domain which is the complement of a convex set.

Lemma 1.9 (Hopf lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the interior ball condition and*

$$\bigcup_{x_0 \in \partial\Omega} B_{\rho_0}(y_0) \supset \{\operatorname{dist}(\cdot, \partial\Omega) \geq \rho_0/2\}.$$

Let $u \in C(\overline{\Omega})$ be a positive weakly superharmonic function in $\Omega \cap B_2$, with $u \geq 0$ on $\partial\Omega \cap B_2$. Then, $u \geq c_0 d$ in $\Omega \cap B_1$ for some $c_0 > 0$, where $d(x) := dist(x, \Omega^c)$.

Note that c_0 in general depends on u!

Proof. Since u is positive and continuous in $\Omega \cap B_2$, we have that

$$u \geq c_1 > 0 \quad \text{in } \{d \geq \rho_0/2\} \cap B_{3/2}$$

for some $c_1 > 0$. Let us consider the solution of

$$\begin{cases} -\Delta w &= 0 \quad \text{in } B_{\rho_0} \setminus B_{\rho_0/2}, \\ w &= 0 \quad \text{on } \partial B_{\rho_0}, \\ w &= 1 \quad \text{on } \partial B_{\rho_0/2}. \end{cases}$$

One can check

$$\begin{aligned} w(x) &= \frac{|x|^{2-n} - \rho_0^{2-n}}{(\rho_0/2)^{2-n} - \rho_0^{2-n}} \quad \text{if } n \geq 3, \\ w(x) &= \frac{\ln(\rho_0/|x|)}{\ln 2} \quad \text{if } n = 2, \\ w(x) &= \max\left\{1, \frac{2}{\rho_0}(\rho_0 - |x|)\right\} \quad \text{if } n = 1. \end{aligned}$$

In particular, it is immediate to check that $w \geq c_2(\rho_0 - |x|)$ in B_{ρ_0} for some $c_2 > 0$.

Let us take $x_0 \in \partial\Omega$, and apply the comparison principle to the functions u and $c_1 w(y_0 + x)$ in $\big(B_{\rho_0}(y_0) \setminus B_{\rho_0/2}(y_0)\big) \subset \Omega \cap B_{3/2}$, where y_0 is from the definition of the interior ball condition. (We are using that $u \in C(\overline{\Omega})$ to guarantee $u \geq 0$ on $\partial B_{\rho_0}(y_0)$). Hence, we deduce that

$$u(x) \geq c_1 w(y_0 + x) \geq c_1 c_2(\rho_0 - |x - y_0|) \geq c_1 c_2 d(x) \quad \text{in } B_{\rho_0}(y_0).$$

Setting $c_0 = c_1 c_2$ and using the previous inequality for $x_0 \in \partial\Omega$ and the corresponding ball $B_{\rho_0}(y_0) \subset \Omega \cap B_{3/2}$, the result follows. □

If Ω satisfies the *exterior ball condition*, i.e. there exists $\rho_0 > 0$ such that every point on $\partial\Omega$ can be touched from outside with a ball of radius ρ_0 contained in Ω, we also have the following result:

Lemma 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the exterior ball condition. Let $u \in C(\overline{\Omega})$ be a harmonic function in $\Omega \cap B_2$, with $u = 0$ on $\partial\Omega \cap B_2$. Then, $u \leq c_0 d$ in $\Omega \cap B_1$ for some $c_0 > 0$, where $d(x) := dist(x, \Omega^c)$.*

Proof. We employ a similar barrier argument as before. □

Remark 1.11. In particular, in nice domains (i.e. those satisfying the interior and exterior ball condition, e.g. if $\partial\Omega \in C^{1,1}$), harmonic functions with $u = 0$ on $\partial\Omega$ behave like linear functions near the boundary, i.e.

$$c_1 d \leq u \leq c_2 d \quad \text{close to } \partial\Omega$$

This property remains true in domains with $\partial\Omega \in C^{1,\alpha}$. However, it is dramatically different in bad domains. For instance,

$$\begin{aligned} &u_1(x) = x_1 x_2 \quad \text{solves } -\Delta u_1 = 0 \quad \text{in } \Omega_1 = \{x_1 x_2 > 0\} \quad \text{with } u_1 = 0 \quad \text{on } \partial\Omega_1, \\ &u_2(x) = r^{2/3}\sin(2\phi/3) \quad \text{solves } -\Delta u_2 = 0 \quad \text{in } \Omega_2 = \{x_1 < 0 \quad \text{or } x_2 < 0\} \text{ with } u_2 = 0 \quad \text{on } \partial\Omega_2. \end{aligned}$$

More generally, for any $\alpha > 0$, the function $u_\alpha(x) = r^\alpha \sin(\alpha\phi)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ and satisfies $u_\alpha = 0$ on $\partial\{(r\cos\phi, r\sin\phi) : \phi \in [0, \pi/\alpha]\}$.

Hence, in free boundary problems (where the boundary of the solution domain is unknown), it is a delicate question to analyze the behavior of the solution close to the boundary.

Remark 1.12. One can prove that solutions to the Dirichlet problem in Ω (1.1) always satisfy $u \in C(\overline{\Omega})$ if Ω satisfies the interior or exterior ball condition.

1.2. **The mean value property.**

Lemma 1.13. *Let $\Omega \subset \mathbb{R}^n$ be any open set. If $-\Delta u = 0$ in Ω, then*

$$u(x) = \fint_{\partial B_r(x)} u(y)\,\mathrm{d}y = \fint_{B_r(x)} u(y)\,\mathrm{d}y \quad \textit{for any ball } B_r(x) \subset \Omega. \tag{1.7}$$

Moreover, it holds for any weakly superharmonic (subharmonic) function $u \in H^1(\Omega)$,

$$r \mapsto \fint_{B_r(x)} u(y)dy \quad \textit{is monotone non-increasing (non-decreasing) for } r \in (0, \mathit{dist}(x, \partial\Omega)). \tag{1.8}$$

The property in (1.7) is called the *mean value property.*

Proof. If u is harmonic, the first equality in the mean value property follows by setting $x_0 = x$ in (1.4). The second equality follows by integrating the first one, namely

$$\fint_{B_r(x)} u(y)\,\mathrm{d}y = nr^{-n} \int_0^r \rho^{n-1} \fint_{B_\rho(x)} u(y)\,\mathrm{d}y\,\mathrm{d}\rho.$$

The claim for weakly subharmonic functions goes as follows. Fix $0 < \rho < r$ such that $B_r(x) \subset \Omega$. Let v be the solution to $-\Delta v = 0$ in $B_r(x)$ with $v = u$ on $\partial B_r(x)$. Then, by the maximum principle $u \le v$ in $B_r(x)$. Hence, by the mean value property

$$S(\rho) := \fint_{\partial B_\rho(x)} u(y)\,\mathrm{d}y \le \fint_{\partial B_\rho(x)} v(y)\,\mathrm{d}y = v(x) = \fint_{\partial B_r(x)} v(y)\,\mathrm{d}y = \fint_{\partial B_r(x)} u(y)\,\mathrm{d}y = S(r).$$

Then, by integrating over $(0, r)$,

$$A(r) := \fint_{B_r(x)} u(y)\,\mathrm{d}y = nr^{-n} \int_0^r \rho^{n-1} S(\rho)\,\mathrm{d}\rho \le S(r) nr^{-n} \int_0^r \rho^{n-1}\,\mathrm{d}\rho = S(r).$$

However, this yields

$$A'(r) = -n^2 r^{n-1} \int_0^r \rho^{n-1} S(\rho)\,\mathrm{d}\rho + nr^{-n} S(r) r^{n-1} = \frac{n}{r}(S(r) - A(r)) \ge 0,$$

as desired. $\square$

The following two lemmas yield the Harnack inequality for harmonic functions.

Lemma 1.14 (Weak Harnack inequality for weak supersolutions)**.** *Let $u \in C(B_1)$. Then,*

$$\begin{cases} -\Delta u \ge 0 & \textit{in } B_1 \\ u \ge 0 & \textit{in } B_1 \end{cases} \quad \Longrightarrow \quad \inf_{B_{1/2}} u \ge c\|u\|_{L^1(B_{1/2})},$$

for some $c > 0$ depending only on n.

Proof. By the Lebesgue differentiation theorem and (1.8), we have for any $x_0 \in B_{1/3}$

$$u(x_0) \ge \frac{1}{|B_{2/3}|} \int_{B_{2/3}(x_0)} u = c\|u\|_{L^1(B_{2/3}(x_0))} \ge c\|u\|_{L^1(B_{1/3})}$$

for some $c = c(n) > 0$, so that we have proved the property in a ball of radius $1/3$.

To prove it in $B_{1/2}$, consider $\bar{x}_0 \in \partial B_{1/3}$ and the ball $B_{1/6}(\bar{x}_0)$. We can repeat the previous steps to derive

$$\inf_{B_{1/6}(\bar{x}_0)} u \geq c\|u\|_{L^1(B_{1/6}(\bar{x}_0))}.$$

Moreover, if we denote $B := B_{1/3} \cap B_{1/6}(\bar{x}_0)$, then

$$\inf_{B_{1/6}(\bar{x}_0)} u \geq c\|u\|_{L^1(B_{1/6}(\bar{x}_0))} \geq c \int_B u \geq |B| \inf_B u \geq c \inf_{B_{1/3}} u.$$

This implies

$$\inf_{B_{1/2}} u \geq \inf_{B_{1/3}} u \wedge \inf_{x_0 \in \partial B_{1/3}} \inf_{B_{1/6}(\bar{x}_0)} u \geq c \inf_{B_{1/3}} u.$$

Similarly,

$$\|u\|_{L^1(B_{1/2})} \leq \|u\|_{L^1(B_{1/3})} + c \max_{x_0 \in \partial B_{1/3}} \|u\|_{L^1(B_{1/6}(\bar{x}_0))} \leq c\|u\|_{L^1(B_{1/3})}.$$

Altogether, from the first result in this proof, we can conclude

$$\inf_{B_{1/2}} u \geq c_1 \inf_{B_{1/3}} u \geq c_2\|u\|_{L^1(B_{1/3})} \geq c_3\|u\|_{L^1(B_{1/2})}$$

for some $c_3 = c_3(n) > 0$. In the last step we have used again (1.8). □

Lemma 1.15 (L^∞ bound for weak subsolutions)**.** *Let $u \in C(B_1)$. Then,*

$$-\Delta u \leq 0 \quad \text{in } B_1 \quad \Longrightarrow \quad \sup_{B_{1/2}} u \leq C\|u\|_{L^1(B_{3/4})},$$

for some C depending only on n.

We will see later that the L^1 norm in this estimate can be replaced by the L^ε norm for any $\varepsilon > 0$. This follows from Young's inequality and a covering argument.

Proof. The result follows from the the mean value property (1.8) in the same way as Lemma 1.14. □

Theorem 1.16 (Harnack inequality)**.** *Let $u \in C(B_1)$.*

$$\begin{cases} -\Delta u = 0 & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \end{cases} \quad \Longrightarrow \quad \sup_{B_{1/2}} u \leq c \inf_{B_{1/2}} u,$$

for some $c > 0$ depending only on n.

Proof. Combine Lemma 1.15 and Lemma 1.14. □

Remark 1.17. In particular, we have the following strict maximum principle: If $-\Delta u \geq 0$ in Ω with $u \geq 0$ in Ω and $u \not\equiv 0$, then $u > 0$ in Ω.

We end this subsection with three auxiliary lemmas that all follow from the mean value property and that will be used later in the lecture.

The first lemma says that the pointwise limit of a sequence of superharmonic uniformly bounded functions is superharmonic (in the sense that (1.8) holds).

Lemma 1.18. *Let $\Omega \subset \mathbb{R}^n$, and let $(w_k)_k$ be a sequence of uniformly bounded functions $w_k : \Omega \to \mathbb{R}$ satisfying* (1.8)*, converging pointwise to some $w : \Omega \to \mathbb{R}$. Then w satisfies* (1.8).

Proof. The proof is immediate. In fact, let $w_\infty := w$ and let us define for $k \in \mathbb{N} \cup \{\infty\}$, $\phi_{x,k}(r) := \fint_{B_r(x)} w_k$. Notice that $\phi_{x,k}(r)$ is non-increasing in r for all $k \in \mathbb{N}$. In particular, given $0 < r_1 < r_2 < R_x$, we have that $\phi_{x,k}(r_1) \geq \phi_{x,k}(r_2)$ for $k \in \mathbb{N}$. Now we let $k \to \infty$ and use that $w_k \to w$ pointwise to deduce, by the dominated convergence theorem (notice that w_k are uniformly bounded), that $\phi_{x,\infty}(r_1) \geq \phi_{x,\infty}(r_2)$. That is, $w_\infty = w$ satisfies (1.8). □

The second lemma shows that superharmonic functions are lower semicontinuous.

Lemma 1.19. *Let us assume that $w \in L^1_{loc}(\Omega)$ and satisfies (1.8) in $\Omega \subset \mathbb{R}^n$. Then, up to changing w in a set of measure 0, w is lower semicontinuous in Ω.*

Proof. We define $w_0(x) := \lim_{r \downarrow 0} \fint_{B_r(x)} w$ (which is well defined, since the average is monotone non-increasing). Then $w_0(x) = w(x)$ if x is a Lebesgue point, and thus $w_0 = w$ almost everywhere in Ω. Let us now consider $x_0 \in \Omega$, and let $x_k \to x_0$ as $k \to \infty$. Then, by the dominated convergence theorem,

$$\fint_{B_r(x_0)} w = \lim_{k \to \infty} \fint_{B_r(x_k)} w \leq \liminf_{k \to \infty} w_0(x_k) \tag{1.9}$$

for $0 < r < \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$. Now, by letting $r \downarrow 0$ on the left-hand side, we reach that

$$w_0(x_0) \leq \liminf_{k \to \infty} w_0(x_k), \tag{1.10}$$

that is, w_0 is lower semi-continuous at x_0. □

The next result yields a classification of global harmonic functions.

Theorem 1.20 (Liouville's theorem)**.** *Any bounded solution of $\Delta u = 0$ in $\mathbb{R}^n$ is constant.*

Proof. Let u be any global bounded solution of $\Delta u = 0$ in $\mathbb{R}^n$. Since u is smooth (by Corollary 1.5), each derivative $\partial_i u$ is well-defined and is harmonic. Thus, thanks to the mean-value property and the divergence theorem, for any $x \in \mathbb{R}^n$ and $R \geq 1$ we have

$$|\partial_i u(x)| = \left| \frac{c_n}{R^n} \int_{B_R(x)} \partial_i u \right| = \left| \frac{c_n}{R^n} \int_{\partial B_R(x)} u(y) \frac{y_i}{|y|} dy \right| \leq \frac{C}{R^n} \int_{\partial B_R(x)} |u|. \tag{1.11}$$

Thus, using that $|u| \leq M$ in $\mathbb{R}^n$, we find

$$|\partial_i u(x)| \leq \frac{c_n}{R^n} |\partial B_R(x)| M = \frac{c_n}{R^n} |\partial B_1| R^{n-1} M = \frac{c_n' M}{R} \to 0, \quad \text{as } R \to \infty. \tag{1.12}$$

Therefore, $\partial_i u(x) = 0$ for all $x \in \mathbb{R}^n$, and u is constant. □

2. The obstacle problem

In this chapter, we deal with our first free boundary problem: the obstacle problem.

There is a wide variety of problems in physics, industry, biology, finance, and other areas which can be described by PDEs that exhibit free boundaries. Many of such problems can be written as variational inequalities, for which the solution is obtained by minimizing a constrained energy functional. The obstacle problem is one of the most important and canonical examples.

Given smooth functions $\phi : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, the obstacle problem is the following:

$$\text{minimize } \frac{1}{2} \int_\Omega |\nabla v|^2 \, \mathrm{d}x \quad \text{among all functions } v \geq \phi \quad \text{in } \Omega \quad \text{with } v = g \quad \text{on } \partial\Omega.$$

- Interpretation: we look for the least energy function v, but the set of admissible functions consists only of functions that are above a certain "obstacle" ϕ.
- in 2D: Think of v as an elastic membrane that is constrained to be above ϕ
- We will see that the Euler-Lagrange equation is given as follows:

$$\begin{cases} v \geq \phi & \text{in } \Omega \\ -\Delta v \geq 0 & \text{in } \Omega \\ -\Delta v = 0 & \text{in the set } \{v > \phi\}, \end{cases}$$

  Intuition: Maybe you already know that the unconstrained problem leads to harmonic functions! Hence, if we denote $E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx$, then we will have $E(v + \varepsilon\eta) \geq E(v)$ for every $\varepsilon \geq 0$ and $\eta \geq 0$, $\eta \in C_c^\infty(\Omega)$, which yields $-\Delta v \geq 0$ in Ω. That is, we can perturb v with nonnegative functions ($\varepsilon\eta$) and we always get admissible functions ($v + \varepsilon\eta$). However, due to the constraint $v \geq \phi$, we cannot perturb v with negative functions in all of Ω, but only in the set $\{v > \phi\}$. This is why we get $-\Delta v \geq 0$ everywhere in Ω, but $-\Delta v = 0$ only in $\{v > \phi\}$. (We will show later that any minimizer v is continuous, so that $\{v > \phi\}$ is open.)
  Short form of the Euler-Lagrange equation:

$$\min\{-\Delta v, v - \phi\} = 0 \quad \text{in } \Omega.$$

- Consider $u := v - \phi$. Then, the obstacle problem is equivalent to

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ \Delta u \leq f & \text{in } \Omega \\ \Delta u = f & \text{in the set } \{u > 0\}, \end{cases}$$

  where $f := -\Delta\phi$. This way, we can assume without loss of generality that the obstacle is zero.
- The previous problem is the Euler-Lagrange equation associated to the following minimization problem:

$$\text{minimize} \int_\Omega \frac{1}{2}|\nabla u|^2 + fu\, dx \quad \text{among all functions } u \geq 0 \quad \text{with } u = g - \phi \quad \text{on } \partial\Omega.$$

- A key feature of the obstacle problem is that it has two unknowns:

$$\text{the solution } u, \text{ and the contact set } \{u = 0\}.$$

  In other words, there are two regions in Ω, characterized by the minimization problem:

$$\text{one in which } u = 0, \text{ and one in which } -\Delta u = f.$$

  Moreover, we denote the **free boundary** by

$$\Gamma := \partial\{u > 0\} \cap \Omega,$$

- We will see that since u is a nonnegative supersolution, it will hold $\nabla u = 0$ on Γ, that is, we will have that $u \geq 0$ solves

$$\begin{cases} \Delta u = f & \text{in } \{u > 0\} \\ u = 0 & \text{on } \Gamma \\ \nabla u = 0 & \text{on } \Gamma. \end{cases}$$

  This is yet another way to write the Euler Lagrange equation (this time explicitly including the interface Γ).

- We see that we have both Dirichlet and Neumann conditions on Γ. This would usually be an over-determined problem (too many boundary conditions on Γ, recall Lax-Milgram), but since Γ is also free, it turns out that the problem has a unique solution (where Γ is part of the solution).

Some applications of the obstacle problem

- Dam problem,
- Stefan problem,
- Hele-Shaw flow,
- optimal stopping, finance,
- interacting particle systems,
- elasticity

2.1. **Well-posedness and the Euler Lagrange equation.** Existence and uniqueness of solutions follows easily from the fact that the functional $\int_\Omega |\nabla v|^2 dx$ is convex, and that we want to minimize it in the closed convex set $\{v \in H^1(\Omega) : v \geq \phi\}$. The following proof is standard in the calculus of variations

Proposition 2.1 (Existence and uniqueness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ and $\phi \in H^1(\Omega)$ be such that*

$$\mathcal{C} = \{w \in H^1(\Omega) : w \geq \phi \text{ in } \Omega, \operatorname{Tr} w = g\} \neq \emptyset.$$

Then, there exists a unique minimizer of

$$E(v) := \int_\Omega |\nabla v|^2 \, \mathrm{d}x \quad \text{among all } v \in \mathcal{C}. \tag{2.1}$$

Proof. Let us define

$$\theta_0 := \inf\left\{E(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 \, \mathrm{d}x : w \in K\right\},$$

that is, the infimum value of $E(w)$ among all admissible functions $w \in \mathcal{C}$. Let us take a sequence of functions $\{v_k\}$ such that

(i) $v_k \in H^1(\Omega)$,
(ii) $\operatorname{Tr} v_k = g$ and $v_k \geq \phi$ in Ω,
(iii) $E(v_k) \to \theta_0$ as $k \to \infty$.

By (i), $\|v_k\|_{L^2(\Omega)}$ is uniformly bounded, and by the Poincaré inequality,

$$\|v_k\|_{L^2(\Omega)} \leq C\|\nabla v_k\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)},$$

i.e., the sequence $\{v_k\}$ is uniformly bounded in $H^1(\Omega)$. Therefore, a subsequence $\{v_{k_j}\}$ will converge to a certain function v strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$.

Moreover, by compactness of the trace operator $\operatorname{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$, we will have $\operatorname{Tr} v_{k_j} \to \operatorname{Tr} v$ in $L^2(\partial\Omega)$, so that $\operatorname{Tr} v = g$.

Furthermore, v satisfies (weak lower semi-continuity of $\|\cdot\|_{H^1(\Omega)}$ and compactness of $H^1(\Omega) \subset L^2(\Omega)$)

$$\|v\|_{H^1(\Omega)} \leq \liminf_{j\to\infty} \|v_j\|_{H^1(\Omega)}, \qquad \|v\|_{L^2(\Omega)} = \lim_{j\to\infty} \|v_j\|_{L^2(\Omega)},$$

and therefore,

$$E(v) = \frac{1}{2}[v]_{H^1(\Omega)} \le \frac{1}{2}\liminf_{j\to\infty}[v_j]_{H^1(\Omega)} = \liminf_{j\to\infty} E(v_{k_j}).$$

Hence, v is a minimizer of the energy functional. Since $v_{k_j} \ge \phi$ in Ω and $v_{k_j} \to v$ in $L^2(\Omega)$, we have $v \ge \phi$ in Ω. Thus, we have proved the existence of a minimizer v.

The uniqueness of the minimizer follows from the strict convexity of the functional $E(v)$, as follows:

First, observe that the set $\mathcal{C}$ is convex, i.e. if $u, v \in \mathcal{C}$ are both minimizers, then for $t \in (0,1)$, we have

$$w_t := tu + (1-t)v \in \mathcal{C}.$$

By minimality of u and v,

$$E(u) = E(v) \le E(w_t). \tag{2.2}$$

On the other hand, for the gradients we have the identity

$$\begin{aligned}|\nabla w_t|^2 &= t^2|\nabla u|^2 + (1-t)^2|\nabla v|^2 + 2t(1-t)\nabla u \nabla v \\ &= t^2|\nabla u|^2 + (1-t)^2|\nabla v|^2 - t(1-t)\big(|\nabla u - \nabla v|^2 - |\nabla u|^2 - |\nabla v|^2\big) \\ &= t|\nabla u|^2 + (1-t)|\nabla v|^2 - t(1-t)|\nabla u - \nabla v|^2.\end{aligned}$$

Integrating over Ω yields

$$E(w_t) = tE(u) + (1-t)E(v) - \frac{1}{2}t(1-t)\int_\Omega |\nabla u - \nabla v|^2\,dx.$$

Since $E(u) = E(v)$, this simplifies to

$$E(w_t) = E(u) - \frac{1}{2}t(1-t)\int_\Omega |\nabla u - \nabla v|^2\,dx \le E(u). \tag{2.3}$$

Combining (2.2) and (2.3) gives equality, and therefore it must be,

$$\int_\Omega |\nabla u - \nabla v|^2\,dx = 0. \tag{2.4}$$

Therefore $\nabla u = \nabla v$ a.e. in Ω, so $u - v$ is constant a.e. Since $u - v = 0$ on $\partial\Omega$, the constant must be zero. Hence $u = v$. □

From now on, we will always assume that $\phi \in C^\infty(\overline{\Omega})$ for simplicity. One gets analogous results under much weaker regularity assumptions on ϕ, but the proofs might be more technical.

Our goal is to derive the Euler-Lagrange equation for minimizers v of (2.1).

We start with the following lemma.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in C^\infty(\overline{\Omega})$, and $v \in H^1(\Omega)$ be any minimizer of (2.1). Then, $-\Delta v \ge 0$ in Ω.*

Proof. Since v minimizes E among all functions above the obstacle ϕ (and with fixed boundary conditions on $\partial\Omega$), we have that

$$E(v + \varepsilon\eta) \ge E(v) \quad \text{for every } \varepsilon \ge 0 \text{ and } \eta \ge 0, \eta \in C_c^\infty(\Omega).$$

This yields

$$\varepsilon \int_\Omega \nabla v \cdot \nabla \eta + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\eta|^2 dx \ge 0 \quad \text{for every } \varepsilon \ge 0 \text{ and } \eta \ge 0, \eta \in C_c^\infty(\Omega),$$

and thus

$$\int_\Omega \nabla v \cdot \nabla \eta \geq 0 \quad \text{for every } \eta \geq 0, \eta \in C_c^\infty(\Omega).$$

This means that $-\Delta v \geq 0$ in Ω in the weak sense, as desired. □

From here, by showing first that $\{v > \phi\}$ is open, we obtain the Euler-Lagrange equations for the functional:

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in C^\infty(\overline{\Omega})$, and $v \in H^1(\Omega)$ be any minimizer of* (2.1)*. Then, $v \in C_{loc}(\Omega)$ and it holds*

$$\begin{cases} v & \geq \phi \quad \text{in } \Omega \\ -\Delta v & \geq 0 \quad \text{in } \Omega \\ \Delta v & = 0 \quad \text{in } \{v > \phi\} \cap \Omega. \end{cases} \tag{2.5}$$

Proof. By construction, we already know that $v \geq \phi$ in Ω and, thanks to Lemma 2.2, $-\Delta v \geq 0$ in Ω, i.e, v is (weakly) superharmonic. Up to replacing v in a set of measure zero, we may also assume that v is lower semi-continuous (by Lemma 1.19). Thus, we only need to prove that $\Delta v = 0$ in $\{v > \phi\} \cap \Omega$ and that v is continuous.

First, we show that $\{v > \phi\} \cap \Omega$ is open. Let $x_0 \in \{v > \phi\} \cap \Omega$ be such that $v(x_0) - \phi(x_0) > \varepsilon_0 > 0$. Since v is lower semi-continuous and ϕ is continuous, there exists some $\delta > 0$ such that

$$v(x) - \phi(x) > \varepsilon_0/2 \quad \forall x \in B_\delta(x_0).$$

Hence $B_\delta(x_0) \subset \{v > \phi\}$. Since x_0 was arbitrary, this means that $\{v > \phi\}$ is open.

This implies, also, that $\Delta v = 0$ weakly in $\{v > \phi\} \cap \Omega$. Indeed, for any $x_0 \in \{v > \phi\}$ and $\eta \in C_c^\infty(B_\delta(x_0))$ with $|\eta| \leq 1$, we have $v \pm \varepsilon\eta \geq \phi$ in Ω for all $|\varepsilon| < \varepsilon_0/2$, and therefore it is an admissible competitor. Thus, we have

$$E(v + \varepsilon\eta) \geq E(v) \quad \forall |\varepsilon| < \varepsilon_0.$$

In particular, the map $\varepsilon \to E(v + \varepsilon\eta)$ has a critical point at $\varepsilon = 0$, i.e.

$$\frac{d}{d\varepsilon} E(v + \varepsilon\eta)|_{\varepsilon=0} = 0.$$

Equivalently,

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla(v + \varepsilon\eta)|^2 \, dx \\ &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla v|^2 + \varepsilon^2 |\nabla \eta|^2 + 2\varepsilon \nabla v \nabla \eta \, dx \\ &= 2 \int_\Omega \nabla v \nabla \eta \, dx, \end{aligned}$$

i.e. v is weakly harmonic in $B_\delta(x_0)$. Hence, we deduce that v is harmonic in $\{v > \phi\} \cap \Omega$.

Finally, let us show that v is continuous. We already know, by the regularity of harmonic functions (see Corollary 1.5), that v is continuous in $\{v > \phi\} \cap \Omega$. Let us now show that v is continuous in $\{v = \phi\} \cap \Omega$, as well.

Let $y_0 \in \{v = \phi\} \cap \Omega$, and let us argue by contradiction. Since v is lower semi-continuous, it suffices to assume that there is a sequence $y_k \to y_0$ such that

$$v(y_k) \to v(y_0) + \varepsilon_0 = \phi(y_0) + \varepsilon_0$$

for some $\varepsilon_0 > 0$.

Since ϕ is continuous, we may assume also that $y_k \in \{v > \phi\}$. Let us denote by z_k the projection of y_k towards $\{v = \phi\}$, so that $\delta_k := |z_k - y_k| \le |y_0 - y_k| \downarrow 0$ and

$$v(z_k) \to v(y_0) = \phi(y_0). \tag{2.6}$$

Now, since v is superharmonic, by (1.8),

$$v(z_k) \ge \fint_{B_{2\delta_k}(z_k)} v = (1 - 2^{-n}) \fint_{B_{2\delta_k}(z_k) \setminus B_{\delta_k}(y_k)} v + 2^{-n} \fint_{B_{\delta_k}(y_k)} v = I_1 + I_2.$$

For the first equality, we used that $B_{\delta_k}(y_k) \subset B_{2\delta_k}(z_k)$. Observe that, for I_1, since v is lower semi-continuous and $\delta_k \downarrow 0$, we can assume that, for k large enough, $v \ge \phi(y_0) - 2^{-n}\varepsilon_0$ in $B_{2\delta_k}(z_k)$, so that

$$I_1 \ge (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0].$$

On the other hand, since v is harmonic in $B_{\delta_k}(y_k)$, we have by the mean-value property that

$$I_2 = 2^{-n} v(y_k).$$

Combining everything, we get

$$v(z_k) \ge (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0] + 2^{-n} v(y_k) \to \phi(y_0) + 2^{-2n}\varepsilon_0,$$

which contradicts (2.6). Hence, v is continuous in Ω. □

Remark 2.4. As in the case of harmonic functions, it is easy to show that if a function v satisfies

$$\begin{cases} v \ge \phi & \text{in } \Omega, \\ \Delta v \le 0 & \text{in } \Omega, \\ \Delta v = 0 & \text{in the set } \{v > \phi\}, \end{cases}$$

then it must actually be a minimizer of (2.1).

We next prove the following result, which says that v can be characterized as the least supersolution above the obstacle.

Proposition 2.5 (Least supersolution)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in H^1(\Omega)$, and $v \in H^1(\Omega)$ be any minimizer of (2.1). Then, for any function w satisfying $-\Delta w \ge 0$ in Ω, $w \ge \phi$ in Ω, and $\operatorname{Tr} w \ge \operatorname{Tr} v$, we have $w \ge v$ in Ω. In other words, if w is any supersolution above the obstacle ϕ, then $w \ge v$.*

Proof. If w is any function satisfying $-\Delta w \ge 0$ in Ω, $w \ge \phi$ in Ω, and $\operatorname{Tr} w \ge \operatorname{Tr} v$, it simply follows from the maximum principle that $w \ge v$. Indeed, we have $-\Delta w \ge -\Delta v$ in $\Omega \cap \{v > \phi\}$, and on the boundary of Ω we have $\operatorname{Tr} w \ge \operatorname{Tr} v$ and $w \ge \phi = v$ on $\{v = \phi\}$. □

2.2. **Optimal regularity of solutions.** Thanks to Proposition 2.3, we know that any minimizer of (2.1) is continuous and solves (2.5).

From now on, we will restrict our study to solutions of the Euler Lagrange equation without any boundary conditions on $\partial\Omega$. This means, we localize the problem and study it in a ball:

For $\phi \in C^\infty(B_1)$, we consider

$$\begin{cases} v & \ge \phi \quad \text{in } B_1, \\ -\Delta v & \ge 0 \quad \text{in } B_1, \\ -\Delta v & = 0 \quad \text{in } \{v > \phi\} \cap B_1. \end{cases} \tag{2.7}$$

Our next goal is to answer the following question:

Question: What is the optimal regularity of solutions?

Remark 2.6. Notice that in the set $\{v > \phi\}$ we have $\Delta v = 0$, while in the interior of the set $\{v = \phi\}$ we have $\Delta v = \Delta\phi$ (since $v = \phi$ there). Thus, since $\Delta\phi$ is in general not zero, Δv is discontinuous across the free boundary $\partial\{v > \phi\}$ in general. In particular, $v \notin C^2$.
Example: in 1D, consider $v(x) = -x_+^2$, which solves (2.7) in $(-1, 1)$ with $\phi = -x^2$.

We will now prove that any minimizer of (2.1) is actually $C^{1,1}$, which by the previous remark is the optimal regularity.

Theorem 2.7 (Optimal regularity)**.** *Let $\phi \in C^\infty(B_1)$, and v be any solution to* (2.7)*. Then, v is $C^{1,1}(B_{1/2})$, with the estimate*

$$\|v\|_{C^{1,1}(B_{1/2})} \le C\|v\|_{L^\infty(B_{3/4})} + \|\phi\|_{C^{1,1}(B_{3/4})}.$$

The constant C depends only on n.

To prove this, the main step is the following lemma, which establishes that solutions detach at most quadratically from the free boundary.

Lemma 2.8. *Let $\phi \in C^\infty(B_1)$, and v be any solution to* (2.7)*. Let $x_0 \in B_{1/2}$ be any point on $\{v = \phi\}$. Then, for any $r \in (0, 1/4)$ we have*

$$0 \le \sup_{B_r(x_0)} (v - \phi) \le C\|\phi\|_{C^{1,1}(B_{3/4})} r^2,$$

with C depending only on n.

In particular, Lemma 2.8 implies that $v \in L^\infty(B_{3/4})$.

Proof. After dividing v by a constant if necessary, we may assume that $\|\phi\|_{C^{1,1}(B_1)} \le 1$. Let

$$\ell(x) := \phi(x_0) + \nabla\phi(x_0) \cdot (x - x_0)$$

be the linear part of ϕ at x_0. Let $r \in (0, 1/4)$. Then, by the $C^{1,1}$ regularity of ϕ, in $B_r(x_0)$ we have

$$\ell(x) - r^2 \le \phi(x) \le v(x). \tag{2.8}$$

Next, we consider

$$w(x) := v(x) - \ell(x) + r^2.$$

Our goal is to show that in the ball $B_r(x_0)$, we have

$$w \le Cr^2.$$

This function w satisfies $w \ge 0$ in $B_r(x_0)$ by (2.8), and $-\Delta w = -\Delta v \ge 0$ in $B_r(x_0)$. Let us split w into $w = w_1 + w_2$, with

$$\begin{cases} -\Delta w_1 = 0 & \text{in } B_r(x_0) \\ w_1 = w & \text{on } \partial B_r(x_0) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_2 \ge 0 & \text{in } B_r(x_0) \\ w_2 = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Notice that by the maximum principle, $0 \le w_1 \le w$ and $0 \le w_2$, and hence $0 \le w_2 \le w$.

Moreover, note that

$$w_1(x_0) \le w(x_0) = v(x_0) - \ell(x_0) + r^2 = r^2,$$

and thus by the Harnack inequality (see Theorem 1.16),

$$\|w_1\|_{L^\infty(B_{r/2}(x_0))} \le Cr^2.$$

For w_2, notice that $-\Delta w_2 = -\Delta v$, and in particular $-\Delta w_2 = 0$ in $\{v > \phi\}$. This means that w_2 attains its maximum on $\{v = \phi\}$. But in the set $\{v = \phi\}$ we have

$$w_2 \le w = \phi - \ell + r^2 \le Cr^2,$$

and therefore we deduce that

$$\|w_2\|_{L^\infty(B_r(x_0))} \le Cr^2.$$

Combining the bounds for w_1 and w_2, we get

$$\|w\|_{L^\infty(B_r(x_0))} \le Cr^2,$$

as desired. Recalling the definition of w, and using that $\|\phi\|_{C^{1,1}(B_1)} \le 1$, we find by (2.8),

$$v - \phi = w + \ell - \phi + r^2 \le Cr^2 \quad \text{in } B_{r/2}(x_0),$$

as desired. □

As shown next, the previous lemma easily implies the $C^{1,1}$ regularity.

Proof of Theorem 2.7. Dividing v by a constant if necessary, we may assume that

$$\|v\|_{L^\infty(B_{3/4})} + \|\phi\|_{C^{1,1}(B_{3/4})} \le 1.$$

We already know that $v \in C^\infty_{\text{loc}}(\{v > \phi\})$, since v is harmonic there. Moreover, v is $C^\infty(\{v = \phi\})$, since $\phi \in C^\infty$. Hence, it remains to show smoothness of v across the interface $\Gamma = \partial\{v > \phi\}$. For this, we will use Lemma 2.8.

Let $x_1 \in \{v > \phi\} \cap B_{1/2}$, and let $x_0 \in \Gamma$ be the closest free boundary point. Denote $\rho = |x_1 - x_0|$. Then, we have $-\Delta v = 0$ in $B_\rho(x_1)$, and thus we have also $-\Delta(v - \ell) = 0$ in $B_\rho(x_1)$, where ℓ is the linear part of ϕ at x_0. By estimates for harmonic functions (see Corollary 1.5), the quadratic growth from Lemma 2.8, and since $\phi \in C^{1,1}$ (arguing as in (2.8)), we find

$$\begin{aligned} \|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} = \|D^2(v - \ell)\|_{L^\infty(B_{\rho/2}(x_1))} &\le \frac{C}{\rho^2}\|v - \ell\|_{L^\infty(B_\rho(x_1))} \\ &\le \frac{C}{\rho^2}\|v - \phi\|_{L^\infty(B_\rho(x_1))} + \frac{C\rho^2}{\rho^2} \le \frac{C\rho^2}{\rho^2} = C. \end{aligned}$$

[The factor ρ^{-2} in the second step comes from rescaling Corollary 1.5, i.e. applying it to $v_\rho(x) := v(\rho x)$ and using that $\|D^2 v\|_{L^\infty(B_{\rho/2})} = \rho^{-2}\|D^2 v_\rho\|_{L^\infty(B_{1/2})}$].

In particular, $|D^2 v(x_1)| \le C$. We can do this for all $x_1 \in \{v > \phi\} \cap B_{1/2}$. Moreover, for $x_1 \in \partial\{v > \phi\}$, we deduce $|D^2 v(x_1)| \le C$ from Lemma 2.8. Altogether, it follows $\|v\|_{C^{1,1}(B_{1/2})} \le C$, as desired. □

2.3. **Nondegeneracy.** Next, we want to prove that, at all free boundary points, v separates from ϕ at least quadratically (we already know at most quadratically). That is, we want

$$0 < cr^2 \le \sup_{B_r(x_0)} (v - \phi) \le Cr^2 \tag{2.9}$$

for all free boundary points $x_0 \in \partial\{v > \phi\}$. This property is essential in order to study the free boundary later.

We will prove it under an additional assumption:

Assumption: The obstacle ϕ satisfies

$$-\Delta\phi \geq c_0 > 0 \quad \text{in } B_1. \tag{2.10}$$

Remark 2.9. The assumption (2.10) is quite mild.

- Since $-\Delta v \geq 0$ everywhere, it is clear that if $x_0 \in \partial\{v > \phi\}$, then $-\Delta\phi(x_0) \geq 0$.
  In fact, if $-\Delta\phi(x_0) < 0$, then, since v touches ϕ from above at x_0, the function $v - \phi$ has a global minimum there, i.e. $(-\Delta)(v - \phi) \leq 0$, i.e.$-\Delta v(x_0) < 0$, a contradiction).
- It can be proved that, in fact, if $\Delta\phi$ and $\nabla\Delta\phi$ do not vanish simultaneously, then $-\Delta\phi > 0$ near all free boundary points [Caf98].
- The assumption (2.10) is somewhat necessary. Without it, the lower bound in (2.9) actually fails and one can construct counterexamples in which the free boundary is a fractal set with infinite perimeter (see [Caf98]).
  Idea: Just choose $u = 0$ and note that given any fractal set, we can find ϕ such that $\{\phi = 0\}$ is this set. Then, $u = 0$ solves the obstacle problem with obstacle ϕ.

Proposition 2.10 (Nondegeneracy)**.** *Let $\phi \in C^\infty(B_1)$, and v be any solution to* (2.7)*. Assume that ϕ satisfies $-\Delta\phi \geq c_0 > 0$ in B_1. Then, for every free boundary point $x_0 \in \partial\{v > \phi\} \cap B_{1/2}$, we have*

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \quad \textit{for all } r \in (0, 1/4),$$

with a constant $c > 0$ depending only on n and c_0.

Proof. Let $x_1 \in \{v > \phi\}$ be any point close to x_0 (we will let $x_1 \to x_0$ at the end of the proof). Consider the function [we will see that the r^2 essentially comes from the fact that $\Delta(|x - x_1|^2) = 2n$.]

$$w(x) := v(x) - \phi(x) - \frac{c_0}{2n}|x - x_1|^2.$$

Then, in $\{v > \phi\} \cap B_r(x_1)$, we have

$$-\Delta w = -\Delta v + \Delta\phi + c_0 = \Delta\phi + c_0 \leq 0,$$

Moreover, $w(x_1) > 0$. Hence, by the maximum principle, w attains a positive maximum on $\partial(\{v > \phi\} \cap B_r(x_1))$. But on the free boundary $\partial\{v > \phi\}$ we clearly have $w < 0$. Therefore, there is a point on $\partial B_r(x_1)$ at which $w > 0$. In other words,

$$0 < \sup_{\partial B_r(x_1)} w = \sup_{\partial B_r(x_1)} (v - \phi) - \frac{c_0}{2n} r^2.$$

Letting now $x_1 \to x_0$, we find $\sup_{\partial B_r(x_0)}(v - \phi) \geq cr^2 > 0$, as desired. □

Remark 2.11. Note that we have used the fact that $-\Delta v \geq 0$ in B_1 only for continuity of v in the proof of the nondegeneracy!

This ends the study of basic properties of the obstacle problem. Before we continue, let us quickly summarize:

Summary of basic properties. Let $\phi \in C^\infty(B_1)$ and v be any solution to the obstacle problem

$$\begin{cases} v \geq \phi & \text{in } B_1 \\ -\Delta v \geq 0 & \text{in } B_1 \\ \Delta v = 0 & \text{in } \{v > \phi\} \cap B_1. \end{cases}$$

Then, we have:

- **Optimal regularity:** $\|v\|_{C^{1,1}(B_{1/2})} \leq C(\|v\|_{L^\infty(B_1)} + \|\phi\|_{C^{1,1}(B_1)})$.

- **Quadratic growth:** If $-\Delta\phi \geq c_0 > 0$, then
$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \quad \text{for all } r \in (0, 1/2)$$
at all free boundary points $x_0 \in \partial\{v > \phi\} \cap B_{1/2}$.

2.4. **An alternative way to formulate the obstacle problem.** Recall the obstacle problem (2.7) problem
$$\begin{cases} v \geq \phi & \text{in } B_1, \\ \Delta v \leq 0 & \text{in } B_1, \\ \Delta v = 0 & \text{in } \{v > \phi\} \cap B_1 \end{cases}$$
for some $\phi \in C^\infty(B_1)$ with $-\Delta\phi \geq c_0 > 0$. Clearly, this problem is equivalent to
$$\begin{cases} u \geq 0 & \text{in } B_1, \\ \Delta u \leq f & \text{in } B_1, \\ \Delta u = f & \text{in } \{u > 0\} \cap B_1, \end{cases} \tag{2.11}$$
where $f = -\Delta\phi \geq c_0 > 0$.

Let us quickly explain that this problem arises as the Euler-Lagrange equation of an alternative energy functional, without going into too much detail.

Proposition 2.12 (An alternative energy functional)**.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ be such that*
$$\mathcal{C} = \{u \in H^1(\Omega) : u \geq 0 \text{ in } \Omega, u|_{\partial\Omega} = g\} \neq \emptyset.$$
Then, for any $f \in L^2(\Omega)$ with $f \geq 0$ there exists a unique minimizer of
$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega fu \tag{2.12}$$
among all functions $u \in \mathcal{C}$.

Moreover, the following are equivalent.

(i) *u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu$ among all functions satisfying $u \geq 0$ in Ω and $\operatorname{Tr} u = g$.*
(ii) *u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+$ among all functions satisfying $\operatorname{Tr} u = g$.*

Proof. We skip the proof of the existence and uniqueness. The equivalence of (i) and (ii) follows once we show that minimizers to (ii) are nonnegative. (Note that $\mathcal{C} \neq \emptyset$ implies that $g \geq 0$ on $\partial\Omega$.)
To show this, recall that $|\nabla u|^2 = |\nabla u^+|^2 + |\nabla u^-|^2$, and therefore, since $f \geq 0$ in Ω,
$$\frac{1}{2}\int_\Omega |\nabla u^+|^2 + \int_\Omega fu^+ \leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+,$$
with strict inequality unless $u = u^+$. Hence, any minimizer u of the functional in (ii) must be nonnegative. □

The equivalence of (i) and (ii) will help us understand the connection between the obstacle problem and the Alt-Caffarelli free boundary problem later.

The Euler-Lagrange equation associated to (2.12) is given as follows:

Proposition 2.13. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $f \in C^\infty(\Omega)$, and $u \in H^1(\Omega)$ be any minimizer of* (2.12) *subject to the boundary conditions* $\operatorname{Tr} u = g$. *Then, u solves*

$$\begin{cases} \Delta u & = f\chi_{\{u>0\}} \quad \text{in } \Omega, \\ u & \geq 0 \quad \text{in } \Omega \end{cases}$$

in the weak sense.

Proof. Notice that, by Proposition 2.12, u is actually a minimizer of

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega f u^+$$

subject to the boundary conditions $\operatorname{Tr} u = g$. Hence, for any $\eta \in H^1_0(\Omega)$ and $\varepsilon > 0$ we have

$$E(u + \varepsilon\eta) \geq E(u).$$

In particular, we obtain

$$0 \leq \lim_{\varepsilon \downarrow 0} \frac{E(u+\varepsilon\eta) - E(u)}{\varepsilon} = \int_\Omega \nabla u \cdot \nabla \eta + \lim_{\varepsilon \downarrow 0} \int_\Omega f \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon}.$$

Notice that

$$\lim_{\varepsilon \downarrow 0} \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in } \{u > 0\}, \\ \eta^+ & \text{in } \{u = 0\}, \end{cases}$$

so that we have

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta\chi_{\{u>0\}} + \int_\Omega f\eta^+\chi_{\{u=0\}} \geq 0 \quad \text{for all } \eta \in H^1_0(\Omega).$$

Assume first that $\eta \geq 0$, so that

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta \geq 0 \quad \text{for all } \eta \in H^1_0(\Omega), \eta \geq 0,$$

which implies that $\Delta u \leq f$ in the weak sense. On the other hand, if $\eta \leq 0$, then

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta\chi_{\{u>0\}} \geq 0 \quad \text{for all } \eta \in H^1_0(\Omega), \eta \leq 0,$$

which implies that $\Delta u \geq f\chi_{\{u>0\}}$ in the weak sense. Hence, (recall that $f \geq 0$),

$$f\chi_{\{u>0\}} \leq \Delta u \leq f \quad \text{in } \Omega.$$

In particular, notice that $\Delta u = f$ in $\{u > 0\}$.

Now, since f is smooth, this implies that $\Delta u \in L^\infty_{\text{loc}}(\Omega)$. One can show (elliptic regularity theory and Calderón-Zygmund estimates) that this implies $u \in C^{1,1-\varepsilon}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$. Thus, $\Delta u = 0$ almost everywhere in the level set $\{u = 0\}$ and we have

$$\Delta u = f\chi_{\{u>0\}} \quad \text{a.e. in } \Omega.$$

From here, one can easily deduce that $\Delta u = f\chi_{\{u>0\}}$ in Ω in the weak sense. □

As we mentioned before, the formulation of the obstacle problem (2.12) is equivalent to the one from (2.1). One can also deduce the $C^{1,1}$ regularity and nondegeneracy from the Euler-Lagrange equation in Proposition 2.13. This is a little shorter, however, more complicated tools like Schauder theory and the Harnack inequality for equations of the form $-\Delta u = f$ have to be used. For more details see [FRRO22].

Summary of basic properties. Let $f \in C^\infty(B_1)$ and u be any solution to the obstacle problem

$$\begin{cases} u \geq 0 & \text{in } B_1, \\ \Delta u = f\chi_{\{u>0\}} & \text{in } B_1. \end{cases}$$

Then, we have:

- **Optimal regularity:** $\|u\|_{C^{1,1}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,1}(B_1)})$.
- **Quadratic growth:** If $f \geq c_0 > 0$, then

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \quad \text{for all } r \in (0, 1/2)$$

at all free boundary points $x_0 \in \partial\{u > 0\} \cap B_{1/2}$.

2.5. **Regularity of free boundaries: an overview.** The next goal of this chapter is to understand properties of the free boundary in the obstacle problem.

We will from now on consider solutions to

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \text{ in } B_1, \\ \Delta u = f \text{ in } \{u > 0\} \cap B_1, \end{cases} \tag{2.13}$$

with

$$f \geq c_0 > 0 \quad \text{and} \quad f \in C^\infty(B_1).$$

Note that all of these properties are in particular satisfied by solutions to the obstacle problem, as we have seen before.

Remark 2.14. Several remarks are in order:

- Note that on the interface

$$\Gamma = \partial\{u > 0\} \cap B_1,$$

since $u \in C^{1,1}$ and $u \geq 0$, we have that

$$u = 0 \text{ on } \Gamma, \qquad \nabla u = 0 \text{ on } \Gamma.$$

(if $\nabla u \neq 0$ on Γ, there would be a sign change).
- Due to Remark 2.11, the nondegeneracy from Proposition 2.10 still holds true. Hence, under (2.13), we still have for some $0 < c < C$ (now with C depending on $\|u\|_{C^{1,1}(B_1)}$),

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \ \ \forall x_0 \in \partial\{u > 0\}. \tag{2.14}$$

- Since $u \in C^{1,1}$, we have that $\Delta u \in L^\infty$, i.e. it holds $\Delta u = f$ a.e. in $\{u > 0\} \cap B_1$. Moreover, since $u \in C^{1,1}$, we have that $\nabla u \in H^1$, it holds that $\Delta u = 0$ a.e. on $\{\nabla u = 0\} \supset \{u = 0\}$ (It is a general fact that derivatives of an H^1 function v vanish a.e. on $\{v = 0\}$, and it follows from the fact that $\nabla v = \nabla v_+ - \nabla v^-$ a.e.). From here, we can deduce that for any $\eta \in C_c^\infty(B)$ and $B \Subset B_1$,

$$\int_B \nabla u \nabla \eta = -\int_B \Delta u \eta + \int_{\partial B} \partial_\nu u \eta = -\int_B f\chi_{\{u>0\}}\eta \,\mathrm{d}x,$$

i.e. u solves in the weak sense

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } B_1.$$

For simplicity, we will assume from now on that

$$f \equiv 1,$$

i.e. we will consider solutions u to

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \text{ in } B_1, \\ \Delta u = 1 \text{ in } \{u > 0\} \cap B_1, \end{cases} \tag{2.15}$$

It is also possible to study the problem with a general $f \in C^\infty$, but it is more technically involved.

The central mathematical challenge in the obstacle problem is to understand the geometry/regularity of the free boundary Γ. Clearly, despite knowing that $u \in C^{1,1}$, Γ could still be a very irregular object, even a fractal set with infinite perimeter.

Our goal will be to prove Caffarelli's dichotomy, which splits the free boundary Γ into a set of **regular points** and a set of **singular points**. We will show that

(i) Γ is C^∞ near regular points
(ii) Characterize the set of singular points and prove that they are contained in an $(n-1)$-dimensional C^1 manifold.

These are the main and most important result in the obstacle problem. (i) was proved by Caffarelli in 1977 (see [Caf77]), and it is one of the major results for which he received the Wolf Prize in 2012, the Shaw Prize in 2018, and the Abel Prize in 2023.

Definition 2.15 (blow-up)**.** We say that u_0 is a blow-up of u (satisfying (2.15)) at $x_0 \in \partial\{u > 0\} \cap B_1$, if there is a sequence $r_k \searrow 0$ such that

$$u_{r_k,x_0}(x) := \frac{u(x_0 + r_k x)}{r^2}$$

satisfies

$$u_{r_k} \to u_0 \quad \text{in } C^1_{loc}(\mathbb{R}^n).$$

If $x_0 = 0$, we denote $u_{r_k,x_0} = u_{r_k}$.

Clearly, blow-ups always exist by Arzelà-Ascoli's theorem and the $C^{1,1}$ regularity of u. Moreover, it is not difficult to see that they are global solutions to the obstacle problem (2.15).

Overview of the strategy.

- Given any free boundary point x_0, one considers the rescalings u_{r_k,x_0} ("zooming in" at a free boundary point).
- By $C^{1,1}$ estimates, a subsequence of $u_{r_k} \to u_0$ (blow-up) in $C^1_{loc}(\mathbb{R}^n)$ as $r_k \to 0$.
- Main issue: **classify blow-ups**:
  - either $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ (regular points)
  - or $u_0(x) = \frac{1}{2}x^T Ax$ (singular points).

  Here, $e \in \mathbb{S}^{n-1}$ and $A \geq 0$ is a positive semi-definite matrix satisfying $\mathrm{tr}A = 1$.
- transfer information from u_0 to u:

– free boundary is $C^{1,\alpha}$ near regular points (for some small $\alpha > 0$).
– $C^{1,\alpha}$ implies C^∞ (reminiscent of Hilbert's XIX problem).

2.6. **Classification of blow-ups.** The aim of this section is to classify all possible blow-ups u_0. For this, we proceed in three steps:

- prove that blow-ups are 2-homogeneous, i.e. $u_0(\lambda x) = \lambda^2 u_0(x)$ for all $\lambda \geq 0$.
- prove that blow-ups are convex, i.e. $D^2 u_0 \geq 0$.
- complete classification of blow-ups

Proposition 2.16 (Homogeneity of blow-ups)**.** *Let u be any solution to* (2.15) *with $0 \in \partial\{u > 0\}$. Then, any blow-up of u at 0 is 2-homogeneous.*

Remark 2.17. Note that not all global solutions to the obstacle problem in $\mathbb{R}^n$ are homogeneous. There exist global solutions u_0 that are convex, $C^{1,1}$, and whose contact set $\{u_0 = 0\}$ is an ellipsoid. In fact, it was shown recently in [EFW25] (it was a conjecture for more than 90 years) that the coincidence set of a global solution with non-empty interior has to be either a half-space, an ellipsoid, a paraboloid, or a cylinder with an ellipsoid or paraboloid as base.
The result Proposition 2.16 says that such non-homogeneous solutions cannot appear as blow-ups.

Our proof uses a very important tool in the theory of free boundaries, namely a monotonicity formula.

Theorem 2.18 (Weiss' monotonicity formula)**.** *Let u be any solution to* (2.15) *with $0 \in \partial\{u > 0\}$. Then, the quantity*

$$W_u(r) := \frac{1}{r^{n+2}} \int_{B_r} \left(\frac{1}{2}|\nabla u|^2 + u \right) - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 \tag{2.16}$$

is monotone in r, i.e.

$$\frac{d}{dr} W_u(r) = \frac{1}{r^{n+4}} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2 dx \geq 0 \quad \forall r \in (0,1).$$

Proof. Let $u_r(x) = r^{-2}u(rx)$, and observe that by scaling

$$W_u(r) = \int_{B_1} \left(\frac{1}{2}|\nabla u_r|^2 + u_r \right) - \int_{\partial B_1} u_r^2. \tag{2.17}$$

Using this, together with $\frac{d}{dr}(\nabla u_r) = \nabla \frac{d}{dr} u_r$, we find

$$\frac{d}{dr} W_u(r) = \int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r.$$

Now, integrating by parts we get

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = - \int_{B_1} \Delta u_r \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r.$$

Now, note that

$$\frac{d}{dr} u_r = -2r^{-3}u(rx) + r^{-2} x \cdot \nabla u(rx) = \frac{1}{r}\{x \cdot \nabla u_r - 2u_r\}. \tag{2.18}$$

Thus, $\frac{d}{dr} u_r = 0$ in $\{u_r = 0\}$ (recall that $\nabla u_r = u_r = 0$ on $\{u_r = 0\}$ by Remark 2.14). Moreover, since $\Delta u_r = 1$ in $\{u_r > 0\}$, we have

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = - \int_{B_1} \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r.$$

Thus, we deduce, using also that $\partial_\nu = x\cdot\nabla$ on ∂B_1 together with (2.18)

$$\begin{aligned}\frac{d}{dr}W_u(r) &= \int_{\partial B_1}\partial_\nu(u_r)\frac{d}{dr}u_r - 2\int_{\partial B_1}u_r\frac{d}{dr}u_r\\ &= \int_{\partial B_1}x\cdot\nabla u_r r^{-1}\{x\cdot\nabla u_r - 2u_r\} - 2\int_{\partial B_1}u_r r^{-1}\{x\cdot\nabla u_r - 2u_r\}\\ &= \frac{1}{r}\int_{\partial B_1}(x\cdot\nabla u_r - 2u_r)^2,\end{aligned}$$

which gives the desired result after scaling back from u_r to u. □

Proof of Proposition 2.16. Let $u_r(x) = r^{-2}u(rx)$, and notice that we have the scaling property

$$W_{u_r}(\rho) = W_u(\rho r),$$

for any $r,\rho > 0$. Indeed,

$$\begin{aligned}W_{u_r}(\rho) &= \rho^{-n-2}\int_{B_\rho}\left(\frac{1}{2}|\nabla u_r|^2 + u_r\right) - \rho^{-n-3}\int_{\partial B_\rho}u_r^2\\ &= \rho^{-n-2}r^{-2}\int_{B_\rho}\left(\frac{1}{2}|\nabla u|^2 + u\right) - \rho^{-n-3}r^{-4}\int_{\partial B_\rho}u^2\\ &= (r\rho)^{-n-2}\int_{B_{r\rho}}\left(\frac{1}{2}|\nabla u|^2 + u\right) - (r\rho)^{-n-3}\int_{\partial B_{r\rho}}u^2 = W_u(r\rho).\end{aligned}$$

If u_0 is any blow-up of u at 0 then there is a sequence $r_j \to 0$ satisfying $u_{r_j} \to u_0$ in $C^1_{\rm loc}(\mathbb{R}^n)$. Thus, for any $\rho > 0$ we have

$$W_{u_0}(\rho) = \lim_{r_j\to 0}W_{u_{r_j}}(\rho) = \lim_{r_j\to 0}W_u(\rho r_j) = W_u(0+). \tag{2.19}$$

Notice that the limit $W_u(0+) := \lim_{r\to 0}W_u(r)$ exists by monotonicity of W and since $u \in C^{1,1}$ implies $W_u(r) \geq -C$ for all $r \geq 0$. Moreover, the second equality follows by scaling (see (2.17)).

Hence, the function $W_{u_0}(\rho)$ is constant in ρ. However, by Theorem 2.18 this yields that

$$x\cdot\nabla u_0 - 2u_0 = 0 \quad \text{in } \mathbb{R}^n,$$

and therefore u_0 is 2-homogeneous. (Note that u_0 is a global solution to (2.15), and therefore we can take any $r > 0$ in Theorem 2.18.) Indeed, this property implies that

$$\psi(\lambda) = \lambda^{-2}u_0(\lambda x)$$

satisfies

$$\psi'(\lambda) = \lambda^{-3}(-2u_0(\lambda x) + (\lambda x)\cdot\nabla u_0(\lambda x)) = 0 \;\; \forall\lambda \geq 0,$$

which implies that

$$\lambda^{-2}u_0(\lambda x) = \psi(\lambda) = \psi(1) = u_0(x).$$

□

Using the 2-homogeneity of blow-ups, we can now show that they are also convex. We actually prove a slightly more general result:

Proposition 2.19. *Let $u_0 \in C^{1,1}$ be any 2-homogeneous global solution to*

$$\begin{cases} u_0 \geq 0 & \text{in } \mathbb{R}^n \\ \Delta u_0 = 1 & \text{in } \{u_0 > 0\} \end{cases}$$

such that $0 \in \partial\{u > 0\}$. Then, u_0 is convex.

[Heuristic idea of the proof: $D^2 u_0$ is harmonic in $\{u_0 > 0\}$ and $D^2 u_0 \geq 0$ on $\partial\{u_0 > 0\}$ (since $u_0 \geq 0$, it is convex at the free boundary). Since $D^2 u_0$ is also 0-homogeneous, by the maximum principle, $D^2 u_0 \geq 0$ everywhere.]

We need the following auxiliary lemma.

Lemma 2.20. *Let $\Lambda \subset B_1$ be closed. Let $w \in H^1(B_1) \cap C(B_1)$ be such that $w \geq 0$ on Λ and such that w is superharmonic in the weak sense in $B_1 \setminus \Lambda$. Then $\min\{w, 0\} = -w^-$ is superharmonic in the weak sense in B_1.*

Proof. It is a well-known fact that if $-\Delta v \geq 0$ in Ω the weak sense, then $-\Delta \min\{v, 0\} \geq 0$ in Ω in the weak sense. To see it, note that if $F \in C^\infty(\mathbb{R})$ is non-decreasing and concave, then $F(v) \in H^1(B_1)$, and moreover, for any $\eta \in H_0^1(B_1)$ with $\eta \geq 0$,

$$\begin{aligned} \int_{B_1} \nabla F(v) \nabla \eta \, \mathrm{d}x &= F'(v) \nabla v \nabla \eta \, \mathrm{d}x \\ &= \int_{B_1} \nabla(F'(v)\eta) \nabla v \, \mathrm{d}x - \int_{B_1} \eta F''(v) |\nabla v|^2 \, \mathrm{d}x. \end{aligned}$$

Since $F'(v) \geq 0$ and $0 \leq F'(v)\eta \in H_0^1(B_1)$ is an admissible test-function, and thus, the first term is non-negative. Moreover, we have $F''(v) \leq 0$ by concavity, and therefore

$$\int_{B_1} \nabla F(v) \nabla \eta \, \mathrm{d}x \geq 0,$$

i.e. $-\Delta(F(v)) \geq 0$. Then, the fact follows by taking a sequence $F_k(t) \to -t_-$ as $k \to \infty$ uniformly, and taking limits.

We define $w_\varepsilon = \min\{w, -\varepsilon\} \in H^1(B_1)$. By continuity, we know that in a neighborhood of $\{w = -\varepsilon\}$, it holds $-\Delta w \geq 0$. By application of the previous fact to $v := w + \varepsilon$, we have that

$$0 \leq -\Delta \min\{w + \varepsilon, 0\} = -\Delta(\min\{w + \varepsilon, 0\} - \varepsilon) = -\Delta w_\varepsilon$$

in B_1 in the weak sense.

Since the functions $(w_\varepsilon)_\varepsilon$ are uniformly bounded in $H^1(B_1)$, up to subsequences they converge weakly to $\min\{w, 0\}$. Since the weak limit of weakly superharmonic functions is superharmonic, we deduce the desired result. □

[It is possible to remove the continuity assumption on $w \in H^1(B_1)$.]

[Recall Lemma 1.18 and Lemma 1.19.]

Proof of Proposition 2.19. Let $e \in \mathbb{S}^{n-1}$ and consider the second derivatives $\partial_{ee} u_0$. We define

$$w_0 := \min\{\partial_{ee} u_0, 0\}$$

and we claim that w_0 is superharmonic in $\mathbb{R}^n$, in the sense (1.8), i.e. such that

$$r \mapsto \fint_{B_r(x)} w_0(y) \, \mathrm{d}y \quad \text{is monotone non-increasing.} \tag{2.20}$$

Indeed, let $\delta_t^2 u_0(x)$ for $t > 0$ be defined by

$$\delta_t^2 u_0(x) := \frac{u_0(x+te) + u_0(x-te) - 2u_0(x)}{t^2}.$$

Now, since $\Delta u_0 = \chi_{\{u_0>0\}}$ by Remark 2.14, we have that in the weak sense,

$$\Delta \delta_t^2 u_0 = \frac{1}{t^2}(\chi_{\{u_0(\cdot+te)>0\}} + \chi_{\{u_0(\cdot-te)>0\}} - 2) \leq 0 \quad \text{in } \{u_0 > 0\}$$

Moreover, it holds $\delta_t^2 u_0 \geq 0$ in $\{u_0 = 0\}$ and $\delta_t^2 u_0 \in C^{1,1}$.

Thus, by Lemma 2.20, $w_t := \min\{\delta_t^2 u_0, 0\}$ is weakly superharmonic, and hence w_t satisfies (2.20).

Since $u_0 \in C^{1,1}$, we have that $\delta_t^2 u_0(x)$ is uniformly bounded independently of t, and therefore w_t is uniformly bounded in t and converges pointwise to w_0 as $t \downarrow 0$. In particular, by Lemma 1.18 we have that w_0 satisfies (2.20), as claimed.

Up to changing it in a set of measure 0, w_0 is lower semi-continuous by Lemma 1.18. In particular, since w_0 is 0-homogeneous by assumption, it must attain its minimum at a point $y_0 \in B_1$. Here, we used that lower semi-continuous functions attain their minimum in compact sets. But for now, w_0 is defined in $\mathbb{R}^n$. 0-homogeneity allows us to restrict the search for the minimum to $\mathbb{S}^{n-1}$.)

But since $⨍_{B_r(y_0)} w_0$ is non-increasing for $r > 0$, we must have that w_0 is constant.

Since w_0 vanishes on the free boundary due to (2.14), we have $w_0 \equiv 0$.

That is, for any $e \in \mathbb{S}^{n-1}$ we have that $\partial_{ee} u_0 \geq 0$ and therefore u_0 is convex. □

Remark 2.21. The original proof by Caffarelli yields a quantitative estimate on the convexity without using the homogeneity assumption. More precisely, for any solution u to (2.15) with $0 \in \partial\{u > 0\}$,

$$\partial_{ee} u(x) \geq -C|\log|x||^{-\varepsilon} \quad \text{for all } e \in \mathbb{S}^{n-1}, x \in B_{1/2},$$

for some $\varepsilon > 0$.

[Since $C|\log|x||^{-\varepsilon} \to 0$ as $x \to 0$, it says that u becomes closer and closer to being convex as we approach to the free boundary. Rescaling this result to B_R, and letting $R \to \infty$, this implies that any global solution is convex.]

Let us summarize our findings in the following proposition.

Proposition 2.22. *Let u be any solution to (2.15) with $0 \in \partial\{u > 0\}$, and let $u_r(x) := u(rx)/r^2$. Then, for any sequence $r_k \to 0$ there is a subsequence $r_{k_j} \to 0$ such that*

$$u_{r_{k_j}} \to u_0 \quad \text{in } C^1_{loc}(\mathbb{R}^n)$$

as $k_j \to \infty$, for some function u_0 satisfying

$$\begin{cases} u_0 \in C^{1,1}_{loc}(\mathbb{R}^n), \\ u_0 \geq 0 \text{ in } \mathbb{R}^n, \\ \Delta u_0 = 1 \text{ in } \{u_0 > 0\}, \\ 0 \in \partial\{u_0 > 0\}, \\ u_0 \text{ is convex}, \\ u_0 \text{ is homogeneous of degree 2.} \end{cases}$$

Proof. Recall that by the $C^{1,1}$ regularity of u, and by nondegeneracy, we have that (see (2.14))

$$\frac{1}{C} \le \sup_{B_1} u_r \le C$$

for some $C > 0$. Moreover, again by $C^{1,1}$ regularity of u, we have

$$\|D^2 u_r\|_{L^\infty(B_{1/(2r)})} = \|D^2 u\|_{L^\infty(B_{1/2})} \le C.$$

Since the sequence $\{u_{r_k}\}$, for $r_k \to 0$, is uniformly bounded in $C^{1,1}(K)$ for each compact set $K \subset \mathbb{R}^n$, by Arzelà-Ascoli's theorem there is a subsequence $r_{k_j} \to 0$ such that

$$u_{r_{k_j}} \to u_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n)$$

for some $u_0 \in C^{1,1}(K)$. Moreover, u_0 satisfies

$$\|D^2 u_0\|_{L^\infty(K)} \le C$$

with C independent of K, and $u_0 \ge 0$ in K.

Next, we prove that $\Delta u_0 = 1$ in $\{u_0 > 0\} \cap K$: For any $\eta \in C_c^\infty(\{u_0 > 0\} \cap K)$ we have that, for k_j large enough, $u_{r_{k_j}} > 0$ in the support of η, and thus

$$\int_{\mathbb{R}^n} \nabla u_{r_{k_j}} \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} \eta \, dx.$$

Since $u_{r_{k_j}} \to u_0$ in $C^1(K)$, we can take the limit $k_j \to \infty$ to get

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} \eta \, dx.$$

Since $\eta \in C_c^\infty(\{u > 0\} \cap K)$, and $K \subset \mathbb{R}^n$ were arbitrary, it follows that $\Delta u_0 = 1$ in $\{u_0 > 0\}$.

The fact that $0 \in \partial\{u_0 > 0\}$ follows by taking limits to $u_{r_{k_j}}(0) = 0$ and $\|u_{r_{k_j}}\|_{L^\infty(B_\rho)} \approx \rho^2$ for all $\rho \in (0, 1)$. Finally, the homogeneity and convexity of u_0 follow from Proposition 2.16 and Proposition 2.19. $\square$

Our next goal is to prove the following.

Theorem 2.23 (Classification of blow-ups)**.** *Let u be any solution to* (2.15) *with $0 \in \partial\{u > 0\}$, and let u_0 be any blow-up of u at 0. Then,*

(a) *either*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2$$

for some $e \in \mathbb{S}^{n-1}$.

(b) *or*

$$u_0(x) = \frac{1}{2} x^T A x$$

for some matrix $A \ge 0$ with tr$A = 1$.

Important comment: At this point, blow-ups are not unique, i.e. different subsequences could lead to different blow-ups u_0.

Before we can classify blow-ups, we need three additional elementary lemmas.

Lemma 2.24. *Let $\Sigma \subset \mathbb{R}^n$ be a closed convex cone with nonempty interior with vertex at the origin. Let $w \in C(\mathbb{R}^n)$ be a function satisfying*

$$\Delta w = 0 \text{ in } \Sigma^c, \quad w > 0 \text{ in } \Sigma^c, \quad \text{and} \quad w = 0 \text{ in } \Sigma.$$

Assume in addition that w is homogeneous of degree 1. Then, Σ must be a half-space.

Proof. By convexity of Σ, there exists a half-space $H = \{x \cdot e > 0\}$, with $e \in \mathbb{S}^{n-1}$, such that $H \subset \Sigma^c$. Let $v(x) = (x \cdot e)_+$. v is harmonic and positive in H, and vanishes in H^c.

By the Hopf Lemma (see Lemma 1.9; Σ^c satisfies the interior ball condition by convexity of Σ), we have that

$$w \geq c_0 d_\Sigma \quad \text{in } \Sigma^c \cap B_1,$$

where $d_\Sigma(x) = \mathrm{dist}(x, \Sigma)$ and c_0 is a small positive constant.

In particular, since both w and d_Σ are homogeneous of degree 1, we deduce that

$$w \geq c_0 d_\Sigma \quad \text{in } \Sigma^c.$$

Thus, since $d_\Sigma \geq d_{H^c} = v$, we deduce that

$$w \geq c_0 v$$

for some $c_0 > 0$.

[The idea is now to consider the functions w and cv, and let $c > 0$ increase until the two functions touch at one point, which will give us a contradiction, since two harmonic functions cannot touch at an interior point.]

Define

$$c^* := \sup\{c > 0 : w \geq cv \text{ in } \Sigma^c\}.$$

Notice that $c^* \geq c_0 > 0$. Then, we consider the function $w - c^* v \geq 0$.

Assume that $w - c^* v$ is not identically zero. Since this function is harmonic in H, by the strict maximum principle, $w - c^* v > 0$ in H.

Then, using the Hopf Lemma in H (see Lemma 1.9) and repeating the arguments from before, we deduce that

$$w - c^* v \geq c_0 d_{H^c} = c_0 v,$$

since $v = d_{H^c}$. This implies

$$w - (c^* + c_0) v \geq 0,$$

a contradiction with the definition of c^*. Therefore, it must be $w - c^* v \equiv 0$. This means that w is a multiple of v, and therefore $\Sigma = H^c$, a half-space. □

[An alternative way to argue in the previous lemma is by harmonic functions on the sphere (compare with Remark 1.11). Any function w which is harmonic in a cone Σ^c and homogeneous of degree α can be written as a function on the sphere, satisfying $\Delta_{\mathbb{S}^{n-1}} w = \mu w$ on $\mathbb{S}^{n-1} \cap \Sigma^c$ with $\mu = \alpha(n + \alpha - 2)$ – in our case $\alpha = 1$. (Here, $\Delta_{\mathbb{S}^{n-1}}$ denotes the spherical Laplacian, i.e. the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$.) In other words, homogeneous harmonic functions solve an eigenvalue problem on the sphere. Nnotice that $w > 0$ in Σ^c and $w = 0$ in Σ imply that w is the first eigenfunction of $\mathbb{S}^{n-1} \cap \Sigma^c$. The first eigenvalue is $\mu = n - 1$. But, on the other hand, the same happens for the domain $H = \{x \cdot e > 0\}$, since $v(x) = (x \cdot e)_+$ is a positive harmonic function in H. This means that both domains $\mathbb{S}^{n-1} \cap \Sigma^c$

and $\mathbb{S}^{n-1} \cap H$ have the same first eigenvalue μ. But then, by strict monotonicity of the first eigenvalue with respect to domain inclusions, we deduce that $H \subset \Sigma^c$ implies $H = \Sigma^c$, as desired.]

Lemma 2.25. *Assume that $\Delta u = 1$ in $\mathbb{R}^n \setminus \partial H$, where ∂H is a hyperplane. If $u \in C^1(\mathbb{R}^n)$, then $\Delta u = 1$ in $\mathbb{R}^n$.*

Proof. Assume $\partial H = \{x_1 = 0\}$. For any ball $B_R \subset \mathbb{R}^n$, we consider the solution to

$$\begin{cases} \Delta w &= 1 \quad \text{in } B_R, \\ w &= u \quad \text{on } \partial B_R, \end{cases}$$

and define $v = u - w$. Then, we have

$$\begin{cases} \Delta v &= 0 \quad \text{in } B_R \setminus \partial H, \\ v &= 0 \quad \text{on } \partial B_R. \end{cases}$$

We want to show that u coincides with w, that is, $v \equiv 0$ in B_R.

For this, notice that since v is bounded in B_R, for $\kappa > 0$ large enough we have by the maximum principle (applied in both halfs of $B_R \setminus \partial H$ separately)

$$v(x) \le \kappa(2R - |x_1|) \quad \text{in } B_R,$$

since $2R - |x_1|$ is positive in B_R and harmonic in $B_R \setminus \{x_1 = 0\}$. Thus, we may consider

$$\kappa^* := \inf\{\kappa \ge 0 : v(x) \le \kappa(2R - |x_1|) \text{ in } B_R\}.$$

Assume $\kappa^* > 0$. Since v and $2R - |x_1|$ are continuous in $\overline{B_R}$, and $v = 0$ on ∂B_R, we must have a point $p \in B_R$ at which

$$v(p) = \kappa^*(2R - |p_1|).$$

Moreover, since v is C^1, and the function $2R - |x_1|$ has a wedge on $\partial H = \{x_1 = 0\}$, we must have $p \in B_R \setminus \partial H$.

This is not possible, as two harmonic functions cannot touch tangentially at an interior point p.

This means that $\kappa^* = 0$, and hence $v \le 0$ in B_R.

Repeating the same argument with $-v$ instead of v, we deduce that $v \equiv 0$ in B_R, and thus the lemma is proved. □

Finally, we will use the following basic property of convex functions.

Lemma 2.26. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that the set $\{u = 0\}$ contains the straight line $\{te_0 : t \in \mathbb{R}\}$, $e_0 \in \mathbb{S}^{n-1}$. Then, $u(x + te_0) = u(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.*

Proof. After a rotation, assume $e_0 = e_n$. Then, writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have that $u(0, x_n) = 0$ for all $x_n \in \mathbb{R}$, and we want to prove that

$$u(x', x_n) = u(x', 0) \;\; \forall x' \in \mathbb{R}^{n-1}, \quad x_n \in \mathbb{R}.$$

By convexity, given x' and x_n, for every $\varepsilon > 0$ and $M \in \mathbb{R}$ we have

$$(1 - \varepsilon)u(x', x_n) + \varepsilon u(0, x_n + M) \ge u((1 - \varepsilon)x', x_n + \varepsilon M).$$

Since $u(0, x_n + M) = 0$, choosing $M = \lambda/\varepsilon$ and letting $\varepsilon \to 0$ we deduce that

$$u(x', x_n) \ge u(x', x_n + \lambda).$$

Since this can be done for any $\lambda \in \mathbb{R}$ and $x_n \in \mathbb{R}$, the result follows. □

We finally establish the classification of blow-ups at regular points.

Proof of Theorem 2.23. Let u_0 be any blow-up of u at 0. We already proved that u_0 is convex and homogeneous of degree 2. We divide the proof into two cases.

Case 1. Assume that $\{u_0 = 0\}$ has nonempty interior. Then, by convexity and homogeneity of u_0, we have $\{u_0 = 0\} = \Sigma$, a closed convex cone with nonempty interior.

For any direction $\tau \in \mathbb{S}^{n-1}$ such that $-\tau \in \mathring{\Sigma}$, we claim that

$$w := \partial_\tau u_0 \geq 0 \quad \text{in } \mathbb{R}^n.$$

Indeed, for every $x \in \mathbb{R}^n$ we have that $u_0(x + \tau t)$ is zero for $t \ll -1$, and therefore by convexity of u_0 we get that $\partial_t u_0(x + \tau t)$ is monotone non-decreasing in t, and zero for $t \ll -1$. This means that $\partial_t u_0(x + \tau t) \geq 0$, and thus $\partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$, as claimed.

Note that, at least for some $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \mathring{\Sigma}$, the function w is not identically zero (otherwise, we would get a contradiction with the nondegeneracy (2.14)). Moreover, since it is harmonic in Σ^c (recall that $\Delta u_0 = 1$ in Σ^c), it holds $w > 0$ in Σ^c.

But then, since w is homogeneous of degree 1, we can apply Lemma 2.24 to deduce that Σ is a half-space.

By convexity of u_0 and Lemma 2.26, this means that u_0 is a one-dimensional function, i.e.

$$u_0(x) = U(x \cdot e)$$

for some $U : \mathbb{R} \to \mathbb{R}$ and some $e \in \mathbb{S}^{n-1}$.

Thus, we have that $U \in C^{1,1}$ solves

$$U''(t) = 1 \quad \text{for } t > 0, \qquad U(t) = 0 \quad \text{for } t \leq 0.$$

From ODE theory, we deduce that $U(t) = \frac{1}{2}t_+^2$, and therefore

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2.$$

Case 2. Assume now that $\{u_0 = 0\}$ has empty interior. Then, by convexity, $\{u_0 = 0\}$ is contained in a hyperplane ∂H.

Hence, $\Delta u_0 = 1$ in $\mathbb{R}^n \setminus \partial H$, with ∂H being a hyperplane, and $u_0 \in C^{1,1}$. Lemma 2.25 yields that

$$\Delta u_0 = 1 \quad \text{in } \mathbb{R}^n.$$

Then all second derivatives of u_0 are harmonic and globally bounded (due to their 0-homogeneity) in $\mathbb{R}^n$, so by the Liouville theorem (see Theorem 1.20) they must be constant. Hence, u_0 is a quadratic polynomial. Finally, since $u_0(0) = 0$, $\nabla u_0(0) = 0$, and $u_0 \geq 0$, we deduce

$$u_0(x) = \frac{1}{2}x^T A x$$

for some $A \geq 0$, and since $\Delta u_0 = 1$, we have $\mathrm{tr}A = 1$. □

2.7. Lipschitz regularity of the free boundary near regular points.

Definition 2.27. Let u be any solution to (2.15) satisfying for some $x_0 \in B_{1/2} \cap \partial\{u > 0\}$

$$\limsup_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0 \tag{2.21}$$

(i.e., the contact set has positive density at x_0). Then, x_0 is called a *regular free boundary point*.

Our goal is to show that the free boundary $\partial\{u>0\}$ is C^∞ in a neighborhood of regular points x_0. This is usually done in three steps:

(1) Lipschitz regularity of the free boundary near regular points,
(2) Lipschitz implies $C^{1,\alpha}$,
(3) $C^{1,\alpha}$ implies C^∞.

To prove the first step, we transfer the local information on u into a blow-up u_0. More precisely, we will show that

$$x_0 \text{ is a regular point } \implies \text{The contact set of a blow-up } u_0 \text{ has nonempty interior.}$$

Lemma 2.28. *Let u be any solution to (2.15) and assume that $0 \in \partial\{u>0\}$ is a regular point. Then, there is at least one blow-up u_0 of u at 0 such that the contact set $\{u_0 = 0\}$ has nonempty interior.*

Proof. Let $r_k \to 0$ be a sequence along which

$$\lim_{r_k \to 0} \frac{|\{u=0\} \cap B_{r_k}|}{|B_{r_k}|} \geq \theta > 0.$$

Such a sequence exists (with $\theta > 0$ small enough) by assumption. Thanks to Proposition 2.22, there exists a subsequence $r_{k_j} \downarrow 0$ along which $u_{r_{k_j}} \to u_0$ uniformly on compact sets of $\mathbb{R}^n$, where $u_r(x) = r^{-2}u(rx)$ and u_0 is convex.

Assume by contradiction that $\{u_0 = 0\}$ has empty interior. Then, by convexity, we have that $\{u_0 = 0\}$ is contained in a hyperplane, say $\{u_0 = 0\} \subset \{x_1 = 0\}$. Since $u_0 > 0$ in $\{x_1 \neq 0\}$ and u_0 is continuous, we have that for each $\delta > 0$ there is some $\varepsilon > 0$ such that

$$u_0 \geq \varepsilon > 0 \quad \text{in } \{|x_1| > \delta\} \cap B_1.$$

Therefore, by uniform convergence of $u_{r_{k_j}} \to u_0$ in B_1, there is $r_{k_j} > 0$ small enough such that

$$u_{r_{k_j}} \geq \frac{\varepsilon}{2} > 0 \quad \text{in } \{|x_1| > \delta\} \cap B_1.$$

In particular, the contact set of $u_{r_{k_j}}$ is contained in $\{|x_1| \leq \delta\} \cap B_1$, i.e.

$$\frac{|\{u_{r_{k_j}} = 0\} \cap B_1|}{|B_1|} \leq \frac{|\{|x_1| \leq \delta\} \cap B_1|}{|B_1|} \leq C\delta.$$

Rescaling back to u, we find

$$\frac{|\{u=0\} \cap B_{r_{k_j}}|}{|B_{r_{k_j}}|} = \frac{|\{u_{r_{k_j}} = 0\} \cap B_1|}{|B_1|} < C\delta.$$

Since we can do this for every $\delta > 0$, we find that

$$\lim_{r_{k_j} \to 0} \frac{|\{u=0\} \cap B_{r_{k_j}}|}{|B_{r_{k_j}}|} = 0,$$

a contradiction. Thus, the lemma is proved. $\square$

Combining the previous lemma with the classification of blow-ups (see Theorem 2.23), we deduce:

Corollary 2.29. *Let u be any solution to (2.15), and assume that $0 \in \partial\{u>0\}$ is a regular point. Then, there is at least one blow-up of u at 0 of the form*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2, \quad e \in \mathbb{S}^{n-1}. \tag{2.22}$$

Next, we use this information to show that the free boundary must be smooth in a neighborhood of any regular point. Our first goal is to establish Lipschitz regularity of the free boundary.

Proposition 2.30. *Let u be any solution to* (2.15)*, and assume that $0 \in \partial\{u > 0\}$ is a regular point. Let $\varepsilon > 0$. Then, there exist $e \in \mathbb{S}^{n-1}$ and $r_0 > 0$ such that*

$$|u_{r_0}(x) - \frac{1}{2}(x \cdot e)_+^2| \le \varepsilon \quad \text{in } B_1,$$

and

$$|\partial_\tau u_{r_0}(x) - (x \cdot e)_+(\tau \cdot e)| \le \varepsilon \quad \text{in } B_1$$

for all $\tau \in \mathbb{S}^{n-1}$.

Proof. By Corollary 2.29 and Proposition 2.22 there are a subsequence $r_j \to 0$ and $e \in \mathbb{S}^{n-1}$ for which

$$u_{r_j} \to \frac{1}{2}(x \cdot e)_+^2 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n).$$

In particular, for every $\tau \in \mathbb{S}^{n-1}$ we have

$$u_{r_j} \to \frac{1}{2}(x \cdot e)_+^2, \qquad \partial_\tau u_{r_j} \to \partial_\tau \frac{1}{2}(x \cdot e)_+^2 = (x \cdot e)_+(\tau \cdot e) \qquad \text{uniformly in } B_1.$$

Hence, given $\varepsilon > 0$, there exists j_0 such that

$$|u_{r_{j_0}}(x) - \frac{1}{2}(x \cdot e)_+^2| \le \varepsilon \quad \text{in } B_1,$$

and

$$|\partial_\tau u_{r_{j_0}}(x) - (x \cdot e)_+(\tau \cdot e)| \le \varepsilon \quad \text{in } B_1.$$

$\square$

Note that if $(\tau \cdot e) > 0$, then the derivatives $\partial_\tau u_0 = (x \cdot e)_+(\tau \cdot e)$ are nonnegative, and strictly positive in $\{x \cdot e > 0\}$.

We want to transfer this information to u_{r_0}, and prove that $\partial_\tau u_{r_0} \ge 0$ in B_1 for all $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \ge 1/2$. For this, we need the following auxiliary lemma.

Lemma 2.31. *Let u be any solution to* (2.15)*, and consider $u_{r_0}(x) = r_0^{-2}u(r_0x)$ and $\Omega = \{u_{r_0} > 0\}$. Assume that a function $w \in C(B_1)$ satisfies:*

(a) *w is bounded and harmonic in $\Omega \cap B_1$.*
(b) *$w = 0$ on $\partial\Omega \cap B_1$.*
(c) *Denoting $N_\delta := \{x \in B_1 : \operatorname{dist}(x, \partial\Omega) < \delta\}$, we have $w \ge -c_1$ in N_δ and $w \ge C_2 > 0$ in $\Omega \setminus N_\delta$.*

If c_1/C_2 is small enough, and $\delta > 0$ is small enough, then $w \ge 0$ in $B_{1/2} \cap \Omega$.

Proof. Notice that in $\Omega \setminus N_\delta$ we already know that $w > 0$. Let $y_0 \in N_\delta \cap \Omega \cap B_{1/2}$, and assume by contradiction that $w(y_0) < 0$.

Consider, for $\gamma > 0$ to be chosen later, the following function in $B_{1/4}(y_0)$:

$$v(x) = w(x) - \gamma\left(u_{r_0}(x) - \frac{1}{2n}|x - y_0|^2\right).$$

Then, $-\Delta v = 0$ in $B_{1/4}(y_0) \cap \Omega$, and moreover $v(y_0) < 0$. Thus, v must have a negative minimum on $\partial(B_{1/4}(y_0) \cap \Omega)$.

Let us now prove that this does not happen, if c_1/C_2 and δ are small enough.

To see it, we write

$$\partial(B_{1/4}(y_0)\cap\Omega)\subset\partial\Omega\cup\big(\partial B_{1/4}(y_0)\cap N_\delta\big)\cup\big(\partial B_{1/4}(y_0)\cap(\Omega\setminus N_\delta)\big).$$

On $\partial\Omega$ we have $v\geq 0$.

Moreover, let us write $\|u_{r_0}\|_{C^{1,1}(B_1)}=: C_0$, and choose $\gamma>0$ and δ such that

$$\delta^2\leq\frac{1}{64C_0 n},\qquad 64nc_1\leq\gamma\leq\frac{C_2}{C_0}$$

Then, on $\partial B_{1/4}(y_0)\cap N_\delta$ we have

$$v\geq -c_1-C_0\gamma\delta^2+\frac{\gamma}{2n}\left(\frac{1}{4}\right)^2\geq 0.$$

Moreover, on $\partial B_{1/4}(y_0)\cap\Omega\setminus N_\delta$ we have

$$v\geq C_2-C_0\gamma\geq 0\quad\text{on } \partial B_{1/4}(y_0)\cap\Omega\setminus N_\delta.$$

Hence, $v\geq 0$ on $\partial(B_{1/4}(y_0)\cap\Omega)$, a contradiction. □

Using the previous lemma, we can now show that there is a cone of directions τ in which the solution is monotone near the origin.

Proposition 2.32. *Let u be any solution to (2.15), and assume that $0\in\partial\{u>0\}$ is a regular point. Let $u_r(x)=r^{-2}u(rx)$. Then, there exist $r_0>0$ and $e\in\mathbb{S}^{n-1}$ such that*

$$\partial_\tau u_{r_0}\geq 0\quad\textit{in } B_{1/2}$$

for every $\tau\in\mathbb{S}^{n-1}$ satisfying $\tau\cdot e\geq 1/2$.

Proof. By Proposition 2.30, for any $\varepsilon>0$ there exist $e\in\mathbb{S}^{n-1}$ and $r_0>0$ such that for all $\tau\in\mathbb{S}^{n-1}$,

$$|u_{r_0}(x)-\frac{1}{2}(x\cdot e)_+^2|\leq\varepsilon\quad\text{in } B_1\tag{2.23}$$

$$|\partial_\tau u_{r_0}(x)-(x\cdot e)_+(\tau\cdot e)|\leq\varepsilon\quad\text{in } B_1.\tag{2.24}$$

Next, we claim

$$u_{r_0}>0\text{ in }\{x\cdot e>C_0\sqrt{\varepsilon}\},\quad u_{r_0}=0\text{ in }\{x\cdot e<-C_0\sqrt{\varepsilon}\},\tag{2.25}$$

which means that [the free boundary is contained in a strip]

$$\partial\Omega:=\partial\{u_{r_0}>0\}\subset\{|x\cdot e|\leq C_0\sqrt{\varepsilon}\}\tag{2.26}$$

for some C_0 depending only on n.

To prove the first property in (2.25), note that if $x\cdot e>C_0\sqrt{\varepsilon}$ then, if $C_0\geq\sqrt{2}$,

$$u_{r_0}>\frac{1}{2}(C_0\sqrt{\varepsilon})^2-\varepsilon>0.$$

To prove the second property in (2.25), note that if there was a free boundary point x_0 in $\{x\cdot e<-C_0\sqrt{\varepsilon}\}$ then by nondegeneracy we would get

$$\sup_{B_{C_0\sqrt{\varepsilon}}(x_0)}u_{r_0}\geq c(C_0\sqrt{\varepsilon})^2>2\varepsilon,$$

if $C_0\geq\sqrt{2/c}$, a contradiction with (2.23). Therefore, we have (2.25), and thus also (2.26), as desired.

Next, for each $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \geq 1/2$ we define $w := \partial_\tau u_{r_0}$. Our goal is to apply Lemma 2.31. Note:

(a) w is bounded and harmonic in $\Omega \cap B_1$.
(b) $w = 0$ on $\partial\Omega \cap B_1$.
(c) By (2.24), if $\delta \gg \sqrt{\varepsilon}$ then w satisfies $w \geq -\varepsilon$ in N_δ and $w \geq \delta/4 > 0$ in $(\Omega \setminus N_\delta) \cap B_1$.

[Recall $N_\delta := \{x \in B_1 : \operatorname{dist}(x, \partial\Omega) < \delta\}$.] The first inequality in (c) follows from (2.24), and to check the last inequality in (c), note that by (2.25) and (2.26), we have

$$\{x \cdot e < \delta - C_0\sqrt{\varepsilon}\} \cap \Omega \subset N_\delta.$$

Thus, by (2.24), we get that for all $x \in (\Omega \setminus N_\delta) \cap B_1$, if $\delta \gg \sqrt{\varepsilon}$,

$$w \geq \frac{1}{2}(x \cdot e)_+ - \varepsilon \geq \frac{1}{2}\delta - \frac{1}{2}C_0\sqrt{\varepsilon} - \varepsilon \geq \frac{1}{4}\delta.$$

Using (a)-(b)-(c), we deduce from Lemma 2.31 that $w \geq 0$ in $B_{1/2}$.

Since $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq 1/2$ was arbitrary, the proposition is proved. □

Remark 2.33. The property (2.26) is of fundamental importance in the theory of free boundary problems. It is also known as "flatness" of the free boundary (see also the concept of a "Reifenberg flat domain"). In many free boundary problems, flatness of the free boundary implies that it is smooth. We will also observe it later, when we study the one-phase free boundary problem.

As a consequence of the previous proposition, we find:

Corollary 2.34. *Let u be any solution to* (2.15)*, and assume that $0 \in \partial\{u > 0\}$ is regular. Then, there exists $r_0 > 0$ such that the free boundary $\partial\{u_{r_0} > 0\}$ is Lipschitz in $B_{1/2}$. In particular, the free boundary of u, $\partial\{u > 0\}$, is Lipschitz in $B_{r_0/2}$ (with Lipschitz constant bounded by one).*

Proof. This follows from the fact that for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq 1/2$ (by Proposition 2.32),

$$\partial_\tau u_{r_0} \geq 0 \quad \text{in } B_{1/2}. \tag{2.27}$$

Indeed, let $x_0 \in B_{1/2} \cap \partial\{u_{r_0} > 0\}$ be any free boundary point in $B_{1/2}$, and let

$$\begin{aligned} \Theta &:= \{\tau \in \mathbb{S}^{n-1} : \tau \cdot e > 1/2\}, \\ \Sigma_1 &:= \{x \in B_{1/2} : x = x_0 - t\tau, \text{ with } \tau \in \Theta, t > 0\}, \\ \text{and } \Sigma_2 &:= \{x \in B_{1/2} : x = x_0 + t\tau, \text{ with } \tau \in \Theta, t > 0\}. \end{aligned}$$

We claim that

$$\begin{cases} u_{r_0} &= 0 \text{ in } \Sigma_1, \\ u_{r_0} &> 0 \text{ in } \Sigma_2. \end{cases} \tag{2.28}$$

Indeed, since $u_{r_0}(x_0) = 0$, it follows from (2.27), and since $u_{r_0} \geq 0$ that

$$u_{r_0}(x_0 - t\tau) = 0 \quad \forall t > 0 \quad \text{and } \tau \in \Theta.$$

In particular, there cannot be any free boundary point in Σ_1.

On the other hand, by the same argument, if $u_{r_0}(x_1) = 0$ for some $x_1 \in \Sigma_2$ then we would have

$$u_{r_0} = 0 \quad \text{in } \{x \in B_{1/2} : x = x_1 - t\tau, \text{ with } \tau \in \Theta, t > 0\} \ni x_0.$$

In particular, x_0 would not be a free boundary point. Thus, $u_{r_0}(x_1) > 0$ for all $x_1 \in \Sigma_2$, and (2.28) follows.

Finally, notice that (2.28) yields that the free boundary $\partial\{u_{r_0} > 0\} \cap B_{1/2}$ satisfies both the interior and exterior cone condition, and thus it is Lipschitz (with Lipschitz constant bounded by one). □

Once we know that the free boundary is Lipschitz, we may assume without loss of generality that $e = e_n$ and that

$$\partial\{u_{r_0} > 0\} \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2}$$

for a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. Here, $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Remark 2.35 (C^1 regularity of the free boundary)**.**

(i) It is not difficult to show that the Lipschitz constant can be made as small as desired (in smaller balls) by refining the proof (scaling argument). Basically, this amounts to showing that there is $e \in \mathbb{S}^{n-1}$ such that for any $\delta > 0$ there is $r_\delta > 0$ with

$$\partial_\tau u \geq 0 \quad \text{in } B_{r_\delta}$$

for any $\tau \in \mathbb{S}^{n-1}$ such that $\tau \cdot e \geq \delta$.

(ii) Regularity of a free boundary point is an open property, i.e. if $0 \in \partial\{u > 0\}$ is regular, then there is $\rho > 0$ such that any point in $\partial\{u > 0\} \cap B_\rho$ is also regular. In fact, by Proposition 2.32 and the local C^1 convergence, any blow-up u_0 at $y \in \partial\{u > 0\} \cap B_\rho$ must satisfy $\partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$ whenever $\tau \cdot e \geq \frac{1}{2}$. By the classification of blow-ups from Theorem 2.23, this implies that u_0 is 1D, i.e. y is a regular point.

(iii) From (i) and (ii) one can easily deduce that the free boundary is C^1 near regular points. [We will not need this fact, since we will provide a direct proof of $C^{1,\alpha}$ regularity in the next subsection].
Indeed, by (i), making $\delta > 0$ small, we obtain the existence of a tangent plane to the free boundary at $0 \in \partial\{u > 0\}$. By (ii), all points $z \in B_\rho$ have a tangent plane (and hence a normal vector ν_z), and by (i),

$$|\nu_z - \nu_0| \leq C\delta \qquad \text{for all} \quad z \in \partial\{u > 0\} \cap B_{r_\delta}.$$

This implies that the free boundary is C^1.

As another application of Remark 2.35(i), we get the uniqueness of blow-ups at regular points.

Lemma 2.36. *Let u be a solution to* (2.15) *and assume that $0 \in \partial\{u > 0\}$ is a regular point. Then, the blow-up $\lim_{r\to 0} u_r = u_0$ is unique.*

Proof. By Proposition 2.22 and Corollary 2.29 there exists a subsequence $r_j \to 0$ such that $u_{r_j} \to u_0 = \frac{1}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$. Assume that there is another subsequence $r_j' \to 0$ such that $u_{r_j'} \to u_0' = \frac{1}{2}(x \cdot e')_+^2$ for some $e' \in \mathbb{S}^{n-1}$. Note that u_0' must be 1D by the same argument as in Remark 2.35(ii), namely due to Proposition 2.32 and the local C^1 convergence towards the blow-up limit. Then, as soon as $r_j' < r_\delta$ from Remark 2.35(i), it holds

$$\partial_\tau u_{r_j'} \geq 0$$

for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \delta$. In particular,

$$\partial_\tau u_0' \geq 0.$$

This implies $e' \cdot \tau \geq 0$ for any $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \delta$. Letting $\delta \to 0$, this yields $e = e'$ and therefore $u_0 = u_0'$. Since by Proposition 2.22 any subsequence r_j has a subsubsequence r_{j_k} for which $u_{r_{j_k}}$ converges, this implies convergence of the sequence u_r, as claimed. □

2.8. **Lipschitz implies $C^{1,\alpha}$ regularity of the free boundary.** Now, we want to prove that Lipschitz free boundaries are $C^{1,\alpha}$. A key ingredient in the proof is the following boundary Harnack principle.

Theorem 2.37 (Boundary Harnack principle)**.** *Let Ω be a Lipschitz domain and w_1 and w_2 be non-negative functions such that for $i = 1, 2$,*

$$\begin{cases} -\Delta w_i &= 0 \quad \text{in } B_1 \cap \Omega, \\ w_i &= 0 \quad \text{on } B_1 \cap \partial\Omega, \end{cases}$$

and for some $C_0 > 0$

$$C_0^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_0.$$

Then, it holds

$$\frac{1}{C} w_2 \leq w_1 \leq C w_2 \quad \text{in } \Omega \cap B_{1/2}. \tag{2.29}$$

Moreover,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\Omega \cap B_{1/2})} \leq C \tag{2.30}$$

for some $\alpha > 0$. The constants α and C depend only on n, C_0, and the Lipschitz constant of Ω.

We first explain how Theorem 2.37 implies the $C^{1,\alpha}$ regularity of the free boundary. Later, we will provide a proof of the boundary Harnack principle.

Remark 2.38. It is of central importance that Ω is allowed to be Lipschitz in Theorem 2.37. If $\partial\Omega$ is smooth (i.e. at least $C^{1,\alpha}$) then it follows from a barrier argument that both $w_1 \asymp w_2 \asymp d_\Omega$ (see Remark 1.11). However, in Lipschitz domains the result cannot be proved with a simple barrier argument, and it is much more delicate to establish.

The boundary Harnack is a crucial tool in the study of free boundary problems!

Theorem 2.39. *Let u be any solution to* (2.15)*, and assume that $0 \in \partial\{u > 0\}$ is a regular point. Then, there exists $r_0 > 0$ such that the free boundary $\partial\{u_{r_0} > 0\}$ is $C^{1,\alpha}$ in $B_{1/4}$, for some small $\alpha > 0$. In particular, the free boundary of u, $\partial\{u > 0\}$, is $C^{1,\alpha}$ in $B_{r_0/4}$.*

Proof. Let $\Omega = \{u_{r_0} > 0\}$. By Corollary 2.34, if $r_0 > 0$ is small enough then (possibly after a rotation) we have

$$\Omega \cap B_{1/2} = \{x_n \geq g(x')\} \cap B_{1/2}, \qquad \partial\Omega \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2},$$

where g is Lipschitz.

Let

$$w_2 := \partial_{e_n} u_{r_0} \quad \text{and} \quad w_1 := \partial_{e_i} u_{r_0} + \partial_{e_n} u_{r_0}, \quad i = 1, ..., n-1.$$

Since $\partial_\tau u_{r_0} \geq 0$ in $B_{1/2}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e_n \geq 1/2$ by Proposition 2.32, we have that

$$w_1 \geq 0 \quad \text{in } B_{1/2}, \qquad w_2 \geq 0 \quad \text{in } B_{1/2}$$

The nonnegativity of w_2 is obvious. To see the nonnegativity of w_1, we apply Proposition 2.32 with $\tau = \frac{e_1+e_n}{|e_1+e_n|}$, which satisfies

$$\frac{e_1 + e_n}{|e_1 + e_n|} \cdot e_n = \frac{1}{|e_1 + e_n|} = 1/\sqrt{2} > 1/2.$$

Since w_1 and w_2 are positive harmonic functions in $\Omega \cap B_{1/2}$, and vanish on $\partial\Omega \cap B_{1/2}$, we can use the boundary Harnack (see Theorem 2.37) to get for some $\alpha > 0$

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\Omega \cap B_{1/4})} \leq C.$$

Since $w_1/w_2 = 1 + \partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0}$, this yields

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{0,\alpha}(\Omega \cap B_{1/4})} \leq C. \tag{2.31}$$

We claim that this implies that the free boundary is $C^{1,\alpha}$ in $B_{1/4}$. Indeed, if $u_{r_0}(x) = t$ then the normal vector to the level set $\{u_{r_0} = t\}$ is given by

$$\nu_i(x) = \frac{\partial_{e_i} u_{r_0}}{|\nabla u_{r_0}|} = \frac{\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0}}{\sqrt{1 + \sum_{j=1}^{n-1} (\partial_{e_j} u_{r_0} / \partial_{e_n} u_{r_0})^2}}, \quad i = 1, ..., n.$$

By (2.31), this function is a $C^{0,\alpha}$ function. Taking $t \to 0$, we get that the free boundary is $C^{1,\alpha}$ (since the normal vector to the free boundary is a $C^{0,\alpha}$ function). □

2.9. **Boundary Harnack principle.** The goal of this subsection is to give a proof of Theorem 2.37.

The boundary Harnack principle in Lipschitz domains has a long history. It was first established in [Kem72]. A standard reference for its proof is [CS05], where it was also applied to free boundary problems. In this lecture, however, we will follow a much more recent (and shorter) proof from [DSS20].

Remark. We make a few more comments on the boundary Harnack principle.

- It holds true in much rougher situations than Theorem 2.37. For instance, it holds true in Hölder domains $\partial\Omega \in C^{0,\alpha}$, where $\alpha \in (0, 1)$.
- The following two assumptions on the domain Ω are even sufficient for a BHP to hold:
  - interior corkscrew condition: For any $\xi \in \partial\Omega$ and $r \in (0, 1)$ there are $\kappa > 0$ and $x \in \Omega$ such that $B_{\kappa r}(x) \subset B_r(\xi) \cap \Omega$
  - Harnack chain condition: There is $k \in \mathbb{N}$ such that for any $x, y \in \Omega$ and balls $B_1, \dots, B_k$ such that $x \in B_1$, $y \in B_k$, $B_i \cap B_{i+1} \neq \emptyset$, such that

$$\mathrm{diam}(B_i) \asymp \mathrm{dist}(B_i, \partial\Omega), \qquad k \lesssim \log(1 + |x - y| \min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(y, \partial\Omega)\}^{-1}).$$

- A nontrivial example of a domain satisfying the previous two conditions is the Koch snowflake
- The boundary Harnack principle fails in domains with "exponential cusps", e.g. for

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \quad 0 < y < e^{-1/x}\}.$$

For simplicity, we assume from now on that $0 \in \partial\Omega$ and that $\partial\Omega$ is a Lipschitz graph in the e_n direction where the Lipschitz constant of $\partial\Omega$ is bounded by one, i.e. that

$$\|g\|_{C^{0,1}(\partial\Omega)} \leq 1 \quad \text{where} \quad \Omega \cap B_{1/2} = \{x_n > g(x')\} \cap B_{1/2}. \tag{2.32}$$

Note that in that case, the constants will be independent of Ω. It is not difficult to generalize the proof to domains with arbitrary Lipschitz constants.

Moreover, note that the proof of Proposition 2.43 is a little simpler if the Lipschitz constant is assumed to be small.

We introduce the notation

$$\Omega_\delta := \{x \in \Omega : d(x) := \mathrm{dist}(x, \Omega^c) \geq \delta\}.$$

[Recall the weak Harnack inequality for supersolutions (see Lemma 1.14), the local boundedness estimate with exponent $\varepsilon = 1$ (see Lemma 1.15), and the Harnack inequality (see Theorem 1.16).]

First, we have to improve the local boundedness estimate from Lemma 1.15:

Lemma 2.40 (improved L^∞ bound for weak subsolutions)**.** *Let $u \in C(B_1)$. Then, for any $\varepsilon > 0$*

$$-\Delta u \leq 0 \quad \text{in } B_1 \quad \Longrightarrow \quad \sup_{B_{1/2}} u \leq C\|u\|_{L^\varepsilon(B_{3/4})},$$

for some C depending only on n and ε.

Proof. The result for $\varepsilon = 1$ was already shown in Lemma 1.15. For $\varepsilon > 1$, we deduce the result immediately from Hölder's inequality.

For $\varepsilon \in (0,1)$ and $r \in (0,1/2)$, we can proceed by Young's inequality

$$\sup_{B_{r/2}} u \leq Cr^{-n}\|u\|_{L^1(B_r)} \leq \sup_{B_r} u^{1-\varepsilon} \fint_{B_r} |u|^\varepsilon \,\mathrm{d}x \leq \frac{1}{2}\sup_{B_r} u + C\left(\fint_{B_r} |u|^\varepsilon \,\mathrm{d}x\right)^{\frac{1}{\varepsilon}}.$$

By a standard iteration argument (see [GG82, Lemma 1.1]), this implies

$$\sup_{B_{1/2}} u \leq \frac{1}{2}\sup_{B_{1/2}} u + C\|u\|_{L^\varepsilon(B_1)},$$

which immediately implies the desired result.

Let us give a few more details on the iteration argument. We define

$$S(B_\rho(x)) = (2\rho)^{n/\varepsilon}\|u\|_{L^\infty(B_\rho(x))}, \qquad \gamma := \left(\fint_{B_1} |u|^\varepsilon\right)^{1/\varepsilon}.$$

Moreover, we define

$$Q := \sup_{B_\rho(x_0) \subset B_1} S(B_{\rho/2}(x_0)), \qquad \tilde{Q} := \sup_{B_\rho(x_0) \subset B_1} S(B_{\rho/4}(x_0)).$$

We have already shown that

$$S(B_{\rho/4}(x_0)) \leq \delta Q + C\gamma \ \ \forall B_\rho(x_0) \subset B_1, \qquad \text{i.e.} \qquad \tilde{Q} \leq \delta Q + C\gamma.$$

We claim that also the following holds true:

$$Q \leq c\tilde{Q}.$$

In that case, we could deduce the desired result, since it would yield

$$c^{-1}Q \leq \tilde{Q} \leq \delta Q + C\gamma \qquad \text{i.e.} \qquad Q \leq \tilde{c}\gamma,$$

as desired. To prove the claim, we fix $B_\rho(x_0) \subset B_1$ and cover $B_{\rho/2}(x_0)$ with balls $B_{\rho/8}(z_j)$, for $j \in \{1, \dots, N\}$, and points $z_j \in B_{\rho/2}(x_0)$ such that $B_{\rho/2}(z_j) \subset B_1$. Note that we can choose $N \in \mathbb{N}$ depending only on the dimension. Then, it holds

$$S(B_{\rho/8}(z_j)) \leq \tilde{Q}.$$

By summing over j, we deduce

$$S(B_{\rho/2}(x_0)) \le c \sum_{j=1}^{N} S(B_{\rho/8}(z_j)) \le c\tilde{Q}.$$

This proves the claim and we conclude the proof. $\square$

As a consequence, we show an L^∞ bound for u in terms of its value an interior point in Ω.

Lemma 2.41 (Carleson estimate). *Let $u \in C(\overline{B_1})$ be a nonnegative function such that*

$$\begin{cases} -\Delta u &= 0 \quad \text{in } B_1 \cap \Omega, \\ u &= 0 \quad \text{on } B_1 \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain as in (2.32). Assume, moreover, that $u(\frac{1}{2}e_n) = 1$. Then,

$$\|u\|_{L^\infty(B_{1/2})} \le C,$$

for some constant C depending only on n.

Note that by the assumptions on Ω, we have that $\frac{e_n}{2} \in \Omega$ and $d(\frac{e_n}{2}) \ge (2\sqrt{2})^{-1}$. Moreover, if u does not satisfy the assumption $u(\frac{1}{2}e_n) = 1$, then we get

$$\|u\|_{L^\infty(B_{1/2})} \le Cu\left(\frac{1}{2}e_n\right).$$

Proof. Notice that since $u \ge 0$ is harmonic whenever $u > 0$, and it is continuous, we have $-\Delta u = -\Delta u_+ \le 0$ in B_1 in the weak sense (see the proof of Lemma 2.20, where we have shown that $-u_-$ is superharmonic).

Moreover, since the Lipschitz constant of $\partial\Omega$ is bounded by 1, we have

$$B_\rho\left(\frac{1}{2}e_n\right) \subset \{\Delta u = 0\} \quad \text{with } \rho = \frac{1}{2\sqrt{2}}.$$

In particular, by Harnack's inequality (see Theorem 1.16) we have

$$u \le C_n \quad \text{in } B_{1/4}\left(\frac{1}{2}e_n\right).$$

That is,

$$u(0, x_n) \le C_n \quad \text{for } x_n \in \left[\frac{1}{4}, \frac{1}{2}\right].$$

Repeating iteratively, we get

$$u(0, x_n) \le C_n^k \quad \text{for } x_n \in \left[2^{-k-1}, 2^{-k}\right],$$

so that

$$u(0, t) \le t^{-K} \quad \text{for } t \in \left(0, \frac{1}{2}\right],$$

where K depends only on n.

We can repeat the same procedure at all points in $B_{1/2}$ by iterating successive Harnack inequalities, to deduce that

$$u \le d^{-K} \quad \text{in } B_{1/2}.$$

In particular, for $\varepsilon > 0$ small enough we have

$$\int_{B_{1/2}} |u|^\varepsilon \leq C.$$

By Lemma 2.40, we deduce that $\|u\|_{L^\infty(B_{1/4})} \leq C$, and the result in $B_{1/2}$ follows from a covering argument. $\square$

We need another auxiliary lemma.

Lemma 2.42. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain as in (2.32). Let $u \in C(\overline{B_1})$ satisfy*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} u \geq 1 & \text{in } B_1 \cap \Omega_\delta \\ u \geq -\delta & \text{in } B_1. \end{cases}$$

Then, for all $k \in \mathbb{N}$ such that $k\delta \leq 3/4$, we have

$$u \geq -\delta(1-c_0)^k \quad \text{in } B_{1-k\delta}$$

for some constant c_0 depending only on n.

Proof. Let $u^- = \min\{u, 0\}$. Notice that u^- is superharmonic since $-\Delta u^- = 0$ when $u^- < 0$, and $u^- \leq 0$, so we have $-\Delta u^- \geq 0$ (see the proof of Lemma 2.20). Let $w = u^- + \delta$. By assumption,

$$w \geq 0, \qquad -\Delta w \geq 0.$$

Let $x_0 \in \partial\Omega \cap B_{1-2\delta}$. By the weak Harnack inequality (see Lemma 1.14) applied to $B_{2\delta}(x_0)$, we deduce

$$\inf_{B_\delta(x_0)} w \geq c\delta^{-n} \|w\|_{L^1(B_\delta(x_0))}.$$

Since $\partial\Omega$ is Lipschitz and $w \geq \delta$ in Ω^c, we can bound

$$\|w\|_{L^1(B_\delta(x_0))} \geq \delta |\{w \geq \delta\} \cap B_\delta(x_0)| \geq c\delta^{n+1}$$

for some c depending only on n. Thus,

$$\inf_{B_\delta(x_0)} w \geq c_0 \delta.$$

In particular, since $w \geq \delta$ in $B_1 \cap \Omega_\delta$ we have $w \geq c_0\delta$ in $B_{1-\delta}$ and therefore

$$u \geq -\delta(1-c_0) \quad \text{in } B_{1-\delta}.$$

Applying iteratively this inequality for balls of radius $1-2\delta, 1-3\delta, \ldots$, we obtain the desired result. $\square$

The following result is a key step in the proof of the boundary Harnack inequality.

Proposition 2.43. *There exists $\delta > 0$, depending only on n, such that the following holds. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain as in (2.32). Assume that $u \in C(\overline{B_1})$ satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} u \geq 1 & \text{in } B_1 \cap \Omega_\delta \\ u \geq -\delta & \text{in } B_1. \end{cases}$$

Then, $u \geq 0$ in $B_{1/2}$.

Note that in comparison to Lemma 2.31, where $\partial\Omega = \partial\{u_{r_0} > 0\}$ is the free boundary of a (rescaled) solution to the obstacle problem, here, we assume instead that $\partial\Omega$ is Lipschitz continuous.

Proof. It is enough to show that, for some $a > 0$, we have

$$\begin{cases} u \geq a & \text{in } B_{1/2} \cap \Omega_{\delta/2} \\ u \geq -\delta a & \text{in } B_{1/2}. \end{cases} \tag{2.33}$$

Indeed, iterating (2.33) at all scales, and at all points $z \in \partial\Omega \cap B_{1/2}$, we obtain

$$\begin{cases} u \geq a^k & \text{in } B_{2^{-k}}(z) \cap \Omega_{2^{-k}\delta} \\ u \geq -\delta a^k & \text{in } B_{2^{-k}}(z) \end{cases}$$

for all $k \in \mathbb{N}$. In particular, the first inequality yields that

$$u(z + te_n) \geq 0 \quad \text{for } z \in \partial\Omega \cap B_{1/2}, \quad t > 0,$$

and therefore $u \geq 0$ in $B_{1/2}$.

Let us show (2.33). We start with the first inequality. Let $x_0 \in B_{1/2} \cap \Omega_{\delta/2}$, and let us suppose that $\delta/2 \leq \operatorname{dist}(x_0, \Omega^c) < \delta$ (otherwise, we are done by assumption).

Consider the function $w = u + \delta$, which satisfies $w \geq 0$ in Ω by assumption. Notice that we can connect the points x_0 and $x_0 + \frac{1}{2}\delta e_n$ with a sequence of (three) overlapping balls in Ω, and that

$$w(x_0 + \frac{1}{2}\delta e_n) \geq 1 + \delta,$$

by assumption. Hence, by Harnack's inequality (see Theorem 1.16)

$$w(x_0) \geq \frac{1}{C} w\left(x_0 + \frac{1}{2}\delta e_n\right) \geq \frac{1}{C},$$

for some constant C.

In particular, by taking $\delta > 0$ smaller than $1/(2C) =: a$, we get

$$u(x_0) \geq \frac{1}{C} - \delta \geq \frac{1}{2C} \quad \text{for all } x_0 \in B_{1/2} \cap \Omega_{\delta/2},$$

which yields the first claim in (2.33).

Moreover, by Lemma 2.42, if $k\delta \leq 3/4$, then

$$u \geq -\delta(1 - c_0)^k \quad \text{in } B_{1-k\delta}.$$

Hence, if we take $k = 1/(2\delta)$, we deduce

$$u \geq -\delta(1 - c_0)^{1/(2\delta)} \quad \text{in } B_{1/2},$$

and taking δ small enough such that $(1 - c_0)^{1/(2\delta)} \leq 1/(2C)$ we are done. □

We can now give the proof of the boundary Harnack principle.

Proof of Theorem 2.37. Thanks to Lemma 2.41, up to a constant depending on C_0, we may assume $w_1(\frac{1}{2}e_n) \geq 1$ and $w_2(\frac{1}{2}e_n) \geq 1$. [Since $\|w_1\|_{L^\infty(B_{1/2})} \geq C_0$ by assumption]. Then, let us define

$$v = Mw_1 - \varepsilon w_2$$

for some constants M (large) and ε (small) to be chosen later.

Let $\delta > 0$ be given by Proposition 2.43. Our goal is to apply Proposition 2.43 to v. Clearly,

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \Omega \cap B_1, \\ v &= 0 \quad \text{on } B_1 \cap \partial\Omega. \end{aligned}$$

Moreover, since w_2 is bounded and $w_1 \geq 0$ by assumption,

$$v \geq -\varepsilon w_2 \geq -\delta \quad \text{in } B_{1/2}$$

for $\varepsilon > 0$ small enough.

Moreover, by the Harnack inequality (see Theorem 1.16), and since $w_1(e_n/2) \geq 1$, we can take M large so that

$$Mw_1 \geq 1 + \delta \quad \text{in } B_{1/2} \cap \Omega_\delta.$$

That is,

$$v = Mw_1 - \varepsilon w_2 \geq 1 \quad \text{in } B_{1/2} \cap \Omega_\delta,$$

for M large enough depending only on n. Thus, the hypotheses of Proposition 2.43 are satisfied, and therefore we deduce that $v \geq 0$ in $B_{1/2}$.

This means that,

$$w_2 \leq Cw_1 \quad \text{in } B_{1/4}$$

for some constant C depending only on n. The inequality in $B_{1/2}$ follows by a covering argument. Finally, reversing the roles of w_1 and w_2, we obtain the first claim.

To prove the second claim, let us denote

$$W := \frac{w_1}{w_2},$$

so that we have to prove Hölder regularity for W in $\Omega \cap B_{1/2}$. Notice that, by the first claim, we know that

$$\frac{1}{C} \leq W \leq C \quad \text{in } B_{1/2} \cap \Omega,$$

for some C depending only on n. We start by claiming that, for some $\theta > 0$ and all $k \in \mathbb{N}$, we have

$$\operatorname{osc}_{B_{2^{-k-1}}} W \leq (1-\theta)\operatorname{osc}_{B_{2^{-k}}} W. \tag{2.34}$$

Indeed, let

$$a_k := \sup_{B_{2^{-k}}} W \quad \text{and} \quad b_k := \inf_{B_{2^{-k}}} W.$$

If we denote $p_k = \frac{1}{2^{k+1}} e_n$, then

$$\text{either} \quad W(p_k) \geq \frac{1}{2}(a_k + b_k) \quad \text{or} \quad W(p_k) \leq \frac{1}{2}(a_k + b_k).$$

Suppose first that $W(p_k) \geq \frac{1}{2}(a_k + b_k)$, and let us define

$$v := \frac{w_1 - b_k w_2}{a_k - b_k}.$$

Notice that, by assumption, $\frac{1}{2} w_2(p_k) \leq v(p_k) \leq w_2(p_k)$. In particular, we can apply the first claim to the pair of functions v and w_2 in the ball $B_{2^{-k}}$, to deduce that $v \geq \frac{1}{C} w_2$ in $B_{2^{-k-1}}$, that is,

$$\frac{w_1 - b_k w_2}{a_k - b_k} \geq \frac{1}{C} w_2 \text{ in } B_{2^{-k-1}} \iff \inf_{B_{2^{-k-1}}} W \geq \frac{1}{C}(a_k - b_k) + b_k.$$

Since $\sup_{B_{2^{-k-1}}} W \leq \sup_{B_{2^{-k}}} W \leq a_k$, we deduce that

$$\operatorname{osc}_{B_{2^{-k-1}}} W \leq a_k - \frac{1}{C}(a_k - b_k) - b_k = \left(1 - \frac{1}{C}\right)(a_k - b_k) = (1-\theta)\operatorname{osc}_{B_{2^{-k}}} W,$$

with $\theta = 1/C$, as desired. If we assume instead that $W(p_k) \le \frac{1}{2}(a_k + b_k)$, then the argument is similar taking $v := (a_k w_2 - w_1)/(a_k - b_k)$ instead. Altogether, we have shown (2.34).

In particular, we have shown that, for some small α depending only on n, we have

$$\operatorname{osc}_{B_r(x_0)} W \le C r^{\alpha} \quad \text{for all } r \in (0, 1/4) \text{ and } x_0 \in \partial\Omega \cap B_{1/2}, \tag{2.35}$$

We now need to combine (2.35) with interior estimates for harmonic functions.

Indeed, letting $x, y \in \Omega \cap B_{1/2}$, we want to show that

$$|W(x) - W(y)| \le C|x - y|^{\alpha}, \tag{2.36}$$

for some constant C depending only on n. Let $2r = \operatorname{dist}(x, \partial\Omega) = |x - x^*|$, with $x^* \in \partial\Omega$.

We consider two cases:

- If $|x - y| \ge r/2$, then we apply (2.35) in a ball $B_\rho(x^*)$ with radius $\rho = 2r + |x - y|$ to deduce
$$|W(x) - W(y)| \le \operatorname{osc}_{B_\rho(x^*)} W \le C(2r + |x - y|)^{\alpha} \le C'|x - y|^{\alpha}.$$
- If $|x - y| \le r/2$, then by (2.35) we know that $\operatorname{osc}_{B_r(x)} W \le C r^{\alpha}$. In particular, if we denote $c^* := W(x)$, then
$$\|w_1 - c^* w_2\|_{L^\infty(B_r(x))} = \|w_2(W - c^*)\|_{L^\infty(B_r(x))} \le C r^{\alpha} \|w_2\|_{L^\infty(B_r(x))}.$$
Moreover, since $w_1 - c^* w_2$ is harmonic in $B_r(x)$, by Corollary 1.5 (rescaled and after Hölder interpolation) we know that
$$[w_1 - c^* w_2]_{C^{0,\alpha}(B_{r/2}(x))} \le \frac{C}{r^{\alpha}} \|w_1 - c^* w_2\|_{L^\infty(B_r(x))} \le C\|w_2\|_{L^\infty(B_r(x))}.$$
Hence, using that $w_1(x) - c^* w_2(x) = 0$, we get
$$|W(y) - W(x)| = \left| \frac{w_1(y) - c^* w_2(y)}{w_2(y)} \right| \le C|x - y|^{\alpha} \frac{\|w_2\|_{L^\infty(B_r(x))}}{w_2(y)}.$$
Finally, by Harnack's inequality (see Theorem 1.16) applied to w_2 in $B_{2r}(x)$,
$$\|w_2\|_{L^\infty(B_r(x))} \le C w_2(y)$$
for some C depending only on n.

With these two cases, we have shown (2.36). This proves the result. $\square$

2.10. **Higher regularity of the free boundary.** Summary: So far we have proved

$$\begin{array}{ccccccc} \{u = 0\} \text{ has positive} & & \text{any blow-up is} & & \text{free boundary} & & \text{free boundary} \\ \text{density at the origin} & \Longrightarrow & u_0 = \frac{1}{2}(x \cdot e)_+^2 & \Longrightarrow & \text{is Lipschitz near } 0 & \Longrightarrow & \text{is } C^{1,\alpha} \text{ near } 0 \end{array}$$

As a last step, we prove that $C^{1,\alpha}$ free boundaries are actually C^∞.

Theorem 2.44 (Smoothness of the free boundary near regular points)**.** *Let u be any solution to* (2.15)*, and assume that $0 \in \partial\{u > 0\}$ is a regular free boundary point. Then, the free boundary $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin.*

For this, we need the following result.

Theorem 2.45 (Higher order boundary Harnack)**.** *Let $\Omega \subset \mathbb{R}^n$ be any $C^{k,\alpha}$ domain, with $k \geq 1$ and $\alpha \in (0,1)$. Let w_1, w_2 be two solutions of*

$$\begin{cases} -\Delta w_i &= 0 \quad \text{in } B_1 \cap \Omega, \\ w_i &= 0 \quad \text{on } \partial\Omega \cap B_1, \end{cases}$$

with $w_2 > 0$ in Ω. Assume that

$$C_0^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_0.$$

Then,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{k,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C,$$

where C depends only on n, k, α, C_0, and Ω.

Important comment: Contrary to Theorem 2.37, the proof of Theorem 2.45 is a perturbative argument, in the spirit of (but much more delicate than) the Schauder estimates from Chapter 3. We will not prove the higher order boundary Harnack here; we refer to [DSS15] for the proof of such a result.

Proof of Theorem 2.44. Let $u_{r_0}(x) = r_0^{-2} u(r_0 x)$. By Theorem 2.39 and Proposition 2.32, we know that if $r_0 > 0$ is small enough then the free boundary $\partial\{u_{r_0} > 0\}$ is $C^{1,\alpha}$ in B_1, and (possibly after a rotation)

$$\partial_{e_n} u_{r_0} > 0 \quad \text{in } \{u_{r_0} > 0\} \cap B_1.$$

Thus, using the higher order boundary Harnack (see Theorem 2.45) with $w_1 = \partial_{e_i} u_{r_0}$ and $w_2 = \partial_{e_n} u_{r_0}$, we find

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{1,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C.$$

Actually, by a simple covering argument,

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{1,\alpha}(\overline{\Omega} \cap B_{1-\delta})} \leq C_\delta \tag{2.37}$$

for any $\delta > 0$.

Now, as in the proof of Theorem 2.39, we notice that if $u_{r_0}(x) = t$ then the normal vector to the level set $\{u_{r_0} = t\}$ is given by

$$\nu_i(x) = \frac{\partial_{e_i} u_{r_0}}{|\nabla u_{r_0}|} = \frac{\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0}}{\sqrt{1 + \sum_{j=1}^n (\partial_{e_j} u_{r_0} / \partial_{e_n} u_{r_0})^2}}, \quad i = 1, ..., n.$$

By (2.37), this is a $C^{1,\alpha}$ function in $B_{1-\delta}$ for any $\delta > 0$. Hence, taking $t \to 0$ we see that the normal vector to the free boundary is $C^{1,\alpha}$ inside B_1. Hence, the free boundary is actually $C^{2,\alpha}$.

Repeating now the same argument, and using that the free boundary is $C^{2,\alpha}$ in $B_{1-\delta}$ for any $\delta > 0$, we find

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{2,\alpha}(\overline{\Omega} \cap B_{1-\delta'})} \leq C_{\delta'},$$

which yields that the normal vector is $C^{2,\alpha}$ and thus the free boundary is $C^{3,\alpha}$.

Iterating this argument, we find that the free boundary $\partial\{u_{r_0} > 0\}$ is C^∞ inside B_1, and hence $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin. $\square$

Remark 2.46. Note that near any regular point, u is actually C^∞ up to the free boundary. This follows from the boundary regularity of solutions to the Dirichlet problem in smooth domains (see for instance [Eva10]).

Remark 2.47. There are other ways to prove the C^∞ regularity of the free boundary near regular points. Moreover, it turns out that the free boundary is actually analytic near regular points. This can be proved for instance by applying a so-called partial hodograph-Legendre transformation. This idea goes back to Kinderlehrer-Nirenberg (see [KN77]) and is nicely explained for instance in [PSU12, Chapter 6.4.2].

This completes the study of regular free boundary points. It remains to understand what happens at points where the contact set has density zero. This is the content of the next section.

2.11. **Uniqueness of blow-ups at singular points.** We finally study the behavior of the free boundary at singular points.

Definition 2.48. Let u be any solution to (2.15) satisfying for some $x_0 \in B_{1/2} \cap \partial\{u > 0\}$

$$\lim_{r\to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = 0 \tag{2.38}$$

(i.e., the contact set has zero density at x_0). Then, x_0 is called a *singular free boundary point.* We denote by $\Sigma \subset \partial\{u > 0\}$ the set of all singular points.

The following result is basically a combination of Caffarelli's classification of blow-ups (see Theorem 2.23) and the results of the previous subsections.

Proposition 2.49. *Let u be any solution to* (2.15) *and $0 \in \partial\{u > 0\}$. Then, we have the following dichotomy:*

(a) *Either* (2.21) *holds (i.e. 0 is a regular point) and the blow-up of u at 0 is unique and of the form*

$$u_0(x) = \frac{1}{2}(x\cdot e)_+^2,$$

for some $e \in \mathbb{S}^{n-1}$.

(b) *Or* (2.38) *holds (i.e. 0 is a singular point) and all blow-ups of u at 0 are of the form*

$$u_0(x) = \frac{1}{2}x^T A x,$$

for some matrix $A \geq 0$ with $\mathrm{tr}A = 1$.

To show Proposition 2.49 remains to prove that the blow-up near singular points cannot also be of type (a).

Proof. By the classification of blow-ups (see Theorem 2.23), the possible blow-ups can only have one of the two forms presented. If (2.21) holds, then by Corollary 2.29, there is at least one blow-up of the form $u_0(x) = \frac{1}{2}(x\cdot e)_+^2$. Then, u_0 is unique by Lemma 2.36.

Alternatively, let us assume that (2.38) holds. Let u_0 be a blow-up of u at 0, i.e., $u_{r_k} \to u_0$ in $C^1_{\rm loc}(\mathbb{R}^n)$ along a sequence $r_k \to 0$, where $u_r(x) = r^{-2}u(rx)$. Notice that the functions u_r solve $\Delta u_r = \chi_{\{u_r>0\}}$ in B_1 in the weak sense, i.e.

$$\int_{B_1} \nabla u_r \cdot \nabla \eta \, dx = -\int_{B_1} \chi_{\{u_r>0\}} \eta \, dx \quad \text{for all } \eta \in C^\infty_c(B_1). \tag{2.39}$$

Moreover, by assumption (2.38), we have $|\{u_r = 0\} \cap B_1| \to 0$. Thus taking limits $r_k \to 0$ in (2.39),

$$\int_{B_1} \nabla u_0 \cdot \nabla \eta \, dx = -\int_{B_1} \eta \, \mathrm{d}x \quad \text{for all } \eta \in C^\infty_c(B_1),$$

i.e. $\Delta u_0 = 1$ in B_1. By the classification of blow-ups, this implies that $u_0(x) = \frac{1}{2}x^T Ax$, as desired. □

In the previous section we proved that the free boundary is C^∞ in a neighborhood of any regular point. A natural question then is to understand better the solution u near singular points. The main question is to determine the size of the singular set! The key to proving this is the uniqueness of blow-ups [uniqueness will provide us with expansions].

Theorem 2.50 (Uniqueness of blow-ups at singular points)**.** *Let u be any solution to* (2.15)*. Let $0 \in \partial\{u > 0\}$ be a singular free boundary point. Then, there exists a homogeneous quadratic polynomial $p(x) = \frac{1}{2}x^T Ax$, with $A \geq 0$ and $\Delta p = 1$, such that*

$$u_r \to p \quad \textit{in } C^1_{loc}(\mathbb{R}^n).$$

In particular, the blow-up of u at 0 is unique.

To prove this, we need the following result on Weiss' monotonicity formula, and we will also introduce another monotonicity formula due to Monneau.

Recall $W_u(r)$ as in Theorem 2.18, i.e.

$$W_u(r) = \frac{1}{r^{n+2}} \int_{B_r} \left(\frac{1}{2}|\nabla u|^2 + u \right) - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2.$$

Lemma 2.51. *Let u be any solution to* (2.15) *with $0 \in \partial\{u > 0\}$. Then, any blow-up u_0 of u at 0 satisfies for any $r > 0$*

$$W_{u_0}(r) = W_{u_0}(1) = \begin{cases} \frac{\alpha_n}{2} & \textit{if } u_0 = \frac{1}{2}(x\cdot e)^2_+, \\ \alpha_n & \textit{if } u_0 = \frac{1}{2}x^T Ax, \end{cases}$$

where

$$\alpha_n = \frac{\omega_n}{4n(n+2)}.$$

Proof. First, note that $W_{u_0}(r) = W_{u_0}(1)$ due to (2.19), namely for any $r > 0$,

$$W_{u_0}(r) = \lim_{r_j \to 0} W_{u_{r_j}}(r) = \lim_{r_j \to 0} W_u(rr_j) = W_u(0+).$$

Then, we compute using that $\Delta u_0 = 1$ in $\{u_0 > 0\}$ and that by the 2-homogeneity of u_0 (see Proposition 2.22), $\partial_r u_0 = 2u_0$ (radial derivative),

$$\begin{aligned} W_{u_0}(1) &= \int_{B_1} \left(\frac{1}{2}|\nabla u_0|^2 + u \right) - \int_{\partial B_1} u_0^2 \\ &= \int_{B_1} \left(-\frac{1}{2}\Delta u_0 + 1 \right) u_0 \, \mathrm{d}x + \frac{1}{2}\int_{\partial B_1} \partial_r u_0 u_0 \, \mathrm{d}x - \int_{\partial B_1} u_0^2 \end{aligned}$$

$$= \frac{1}{2} \int_{B_1} u_0 \, \mathrm{d}x.$$

Next, we compute for $u_0 = \frac{1}{2}(x \cdot e)_+^2$,

$$W_{u_0}(1) = \frac{1}{4} \int_{B_1} (x \cdot e)_+^2 \, \mathrm{d}x = \frac{1}{8} \int_{B_1} x_n^2 \, \mathrm{d}x = \frac{\alpha_n}{2},$$

and for $u_0(x) = x^T A x$,

$$W_{u_0}(1) = \frac{1}{2} \int_{B_1} x^T A x \, \mathrm{d}x = \alpha_n \mathrm{Tr}(A) = \alpha_n.$$

□

Theorem 2.52 (Monneau's monotonicity formula)**.** *Let u be any solution to* (2.15)*, and assume that $0 \in \partial\{u > 0\}$ is a singular free boundary point. Let q be any homogeneous quadratic polynomial with $q \geq 0$, $q(0) = 0$, and $\Delta q = 1$. Then, the quantity*

$$M_{u,q}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)^2$$

is monotone in r, that is, $\frac{d}{dr} M_{u,q}(r) \geq 0$.

Proof. A direct computation yields

$$\begin{aligned}
\frac{d}{dr} M_{u,q}(r) &= \frac{d}{dr} \left(\frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)^2 \right) \\
&= \frac{d}{dr} \left(\int_{\partial B_1} \frac{(u - q)^2 (ry)}{r^4} \right) \\
&= \int_{\partial B_1} \frac{2(u - q)(ry)(ry \cdot \nabla(u - q)(ry) - 2(u - q)(ry))}{r^5} \\
&= \frac{2}{r^{n+4}} \int_{\partial B_r} (u - q)\{x \cdot \nabla(u - q) - 2(u - q)\}.
\end{aligned}$$

On the other hand, recall that $W_u(r)$ is monotone increasing in $r > 0$, and that by Lemma 2.51,

$$W_u(0+) = W_q(r) = \alpha_n.$$

Hence,

$$\begin{aligned}
0 &\leq W_u(r) - W_u(0+) \\
&= W_u(r) - W_q(r) \\
&= \frac{1}{r^{n+2}} \int_{B_r} \left(\frac{1}{2} |\nabla(u - q)|^2 + \nabla(u - q) \cdot \nabla q + (u - q) \right) - \frac{1}{r^{n+3}} \int_{\partial B_r} ((u - q)^2 + 2(u - q)q) \\
&= \frac{1}{r^{n+2}} \int_{B_r} \frac{1}{2} |\nabla(u - q)|^2 - \frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)^2 + \frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)(x \cdot \nabla q - 2q) \\
&= \frac{1}{r^{n+2}} \int_{B_r} \frac{1}{2} |\nabla(u - q)|^2 - \frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)^2 \\
&= \frac{1}{2r^{n+2}} \int_{B_r} (-(u - q)\Delta(u - q)) + \frac{1}{2r^{n+3}} \int_{\partial B_r} (u - q)(x \cdot \nabla(u - q) - 2(u - q)).
\end{aligned}$$

Altogether, we have

$$\frac{d}{dr}M_{u,q}(r) \geq \frac{2}{r^{n+3}}\int_{B_r}(u-q)\Delta(u-q).$$

But since $\Delta u = \Delta q = 1$ in $\{u > 0\}$, and $(u-q)\Delta(u-q) = q \geq 0$ in $\{u = 0\}$, we have

$$\frac{d}{dr}M_{u,q}(r) \geq \frac{2}{r^{n+3}}\int_{B_r\cap\{u=0\}} q \geq 0.$$

□

Proof of Theorem 2.50. By Proposition 2.49 and Proposition 2.22, we know that at any singular point we have a subsequence $r_j \to 0$ along which $u_{r_j} \to p$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, where p is a 2-homogeneous quadratic polynomial satisfying $p(0) = 0, p \geq 0$, and $\Delta p = 1$.

By Monneau's monotonicity formula with such polynomial p, we find

$$M_{u,p}(r) := \frac{1}{r^{n+3}}\int_{\partial B_r}(u-p)^2$$

is monotone increasing in $r > 0$. In particular, the limit $\lim_{r\to 0} M_{u,p}(r) := M_{u,p}(0+)$ exists.

Now, recall that we have a sequence $r_j \to 0$ along which $u_{r_j} \to p$. In particular, if $0 \in \Sigma$,

$$r_j^{-2}\{u(r_j x) - p(r_j x)\} \to 0 \quad \text{loc. unif. in } \mathbb{R}^n \qquad \text{i.e.} \quad \frac{1}{r_j^2}\|u-p\|_{L^\infty(B_{r_j})} \to 0$$

as $r_j \to 0$. This yields

$$M_{u,p}(r_j) \leq \frac{1}{r_j^{n+3}}\int_{\partial B_{r_j}}\|u-p\|^2_{L^\infty(B_{r_j})} \to 0$$

along the subsequence $r_j \to 0$, and therefore $M_{u,p}(0+) = 0$.

Let us show that this implies the uniqueness of blow-ups.

Indeed, if there was another subsequence $r_\ell \to 0$ along which $u_{r_\ell} \to q$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, for a 2-homogeneous quadratic polynomial q, then we would repeat the argument above to find that $M_{u,q}(0+) = 0$.

But then, by homogeneity of p and q,

$$\int_{\partial B_1}(p-q)^2 = \frac{1}{r^{n+3}}\int_{\partial B_r}(p-q)^2 \leq 2M_{u,p}(r) + 2M_{u,q}(r) \to 0,$$

This means that $p = q$, and thus the blow-up of u at 0 is unique. □

Summarizing, we have proved the following result:

Theorem 2.53. *Let u be any solution to* (2.15)*. Then, we have the following dichotomy:*

(a) *Either the blow-up of u at 0 is of the form*

$$u_0(x) = \frac{1}{2}(x\cdot e)^2_+ \quad \textit{for some } e \in \mathbb{S}^{n-1},$$

and the free boundary is C^∞ in a neighborhood of the origin.

(b) *Or there is a homogeneous quadratic polynomial p, with $p(0) = 0, p \geq 0$, $\Delta p = 1$, such that*

$$u_0(x) = p(x).$$

In particular, when this happens we have

$$\lim_{r\to 0} \frac{|\{u=0\}\cap B_r|}{|B_r|} = 0,$$

2.12. **The size of the singular set.** The last question that remains to be answered is: How large can the set of singular points be?

To prove it, we establish expansions of u at singular points. This is similar to what we did for regular points (recall Proposition 2.30, where we proved that for any $\varepsilon > 0$, there is $r_0 > 0$ such that

$$|u_{r_0}(x) - \frac{1}{2}(x\cdot e)_+^2| \le \varepsilon \quad \text{in } B_1,$$

or equivalently

$$|u(x) - \frac{1}{2}(x\cdot e)_+^2| \le \varepsilon r_0^2 \quad \text{in } B_{r_0}.$$

We need the following lemma, which mainly follows from the uniqueness of blow-ups and uses again Monneau's formula.

Lemma 2.54. *Let u be any solution to* (2.15)*. Let us denote by $\Sigma \subset \partial\{u>0\}$ the set of singular free boundary points, and denote for $x_0 \in \Sigma$ the blow-up by $p_{x_0} = \frac{1}{2}x^T A_{x_0} x$. Then, for any $\varepsilon > 0$ there is $r > 0$ such that whenever $|x - x_0| \le r$, then*

$$|u(x) - p_{x_0}(x-x_0)| \le \varepsilon |x-x_0|^2, \qquad |\nabla u(x) - \nabla p_{x_0}(x-x_0)| \le \varepsilon |x - x_0|. \tag{2.40}$$

Here, $r > 0$ is independent of x_0 and only depends on $\|u\|_{C^{1,1}}$ and n.
Moreover, the map $\Sigma \ni x_0 \mapsto A_{x_0}$ is continuous.

Remark 2.55.
- The expansion in (2.40) implies that $u \in C^2(x_0)$ (i.e. C^2 at the point x_0)!
- Another way to think about (2.40) would be

$$|u(x) - p_{x_0}(x-x_0)| = o(|x-x_0|^2), \qquad |\nabla u(x) - \nabla p_{x_0}(x-x_0)| = o(|x-x_0|),$$

  where the modulus $o(|x-x_0|^2)$ is independent of x_0! (This is crucial!)

Proof. We will not prove the statement in (2.40) in full detail. Here, we will only show (2.40) with $r > 0$ depending on x_0, because the proof is much simpler. Indeed, let $x_0 = 0 \in \Sigma$ and assume by contradiction that there is a subsequence $r_k \to 0$ along which

$$r_k^{-2}\|u - p\|_{L^\infty(B_{r_k})} \ge c_1 > 0.$$

Then, there would be a subsequence of r_{k_i} along which $u_{r_{k_i}} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, for a certain blow-up u_0 satisfying $\|u_0 - p\|_{L^\infty(B_1)} \ge c_1 > 0$. However, by uniqueness of blow-ups it must be $u_0 = p$, and hence we reach a contradiction.

The proof of the second part of (2.40) is analogous.

Note: It requires a lot more work to prove that the expansions in (2.40) are independent of x_0. In that case, we assume by contradiction that there is a $\varepsilon > 0$ and sequences $r_j \to 0$ and solutions u_j to (2.15) with $\|u_j\|_{C^{1,1}} \le 1$, and $0 \in \Sigma$ for all j such that for any 2-homogeneous quadratic polynomial p with $p \ge 0$, $p(0) = 0$ and $\Delta p = 1$, we have

$$\|u_j - p\|_{L^\infty(B_{r_j})} \ge \varepsilon r_j^2 > 0. \tag{2.41}$$

Consider now the sequence

$$v_j(x) = \frac{u_j(r_j x)}{r_j^2}.$$

By assumption, (v_j) is uniformly bounded in $C^{1,1}$, and hence, by Arzelà-Ascoli's theorem, there exists $v_0 \in C^{1,1}(\mathbb{R}^n)$ such that $v_j \to v_0$ locally in $C^1(\mathbb{R}^n)$ (up to extracting a further subsequence) and v_0 solves $\Delta v_0 = \chi_{\{v_0>0\}}$.

Once we prove that $v_0 = \frac{1}{2}x^T A x$ for some matrix $A \geq 0$ with $\mathrm{tr}A = 1$, i.e. that 0 is a singular point for v_0, then we immediately obtain a contradiction to $v_j \to v_0$ from (2.41) by taking $p = v_0$.

To show it, one can prove that

$$\frac{|\{v_j > 0\} \cap B_r|}{|B_r|} \to 0$$

uniformly in j by establishing Lipschitz estimates of the free boundary near regular points that only depend on the $C^{1,1}$ norm of the solution (see [PSU12, Lemma 7.2]).

Finally, let us prove continuity of $x_0 \mapsto A_{x_0}$.

Let $(x_k) \subset \Sigma$ with $x_k \to 0 \in \Sigma$. Then, let p_k and p_0 be the blow-ups at x_k and 0, respectively. First, by the convergence $u_r \to p_0$ as $r \to 0$, for any $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that

$$\int_{\partial B_1} (u_{r_\varepsilon} - p_0)^2 \leq \varepsilon. \tag{2.42}$$

Next, by the convergence $u_{r,x_k} \to p_k$ and Theorem 2.52, it holds for any $k \in \mathbb{N}$,

$$\begin{aligned}
\int_{\partial B_1} (p_k - p_0)^2 &= \lim_{r\to 0} \int_{\partial B_1} (u_{r,x_k} - p_0)^2 \\
&= \lim_{r\to 0} \frac{1}{r^{n+3}} \int_{\partial B_r} (u(x_k + rx) - p_0(x))^2 \\
&\leq \frac{1}{r_\varepsilon^{n+3}} \int_{\partial B_{r_\varepsilon}} (u(x_k + r_\varepsilon x) - p_0(x))^2 \\
&= \int_{\partial B_1} (r_\varepsilon^{-2} u(x_k + r_\varepsilon x) - p_0(x))^2.
\end{aligned}$$

Hence, taking the limit $k \to \infty$, we deduce from (2.42)

$$\begin{aligned}
\limsup_{k\to\infty} \int_{\partial B_1} (p_k - p_0)^2 &\leq \limsup_{k\to\infty} \int_{\partial B_1} (r_\varepsilon^{-2} u(x_k + r_\varepsilon x) - p_0(x))^2 \\
&= \int_{\partial B_1} (u_{r_\varepsilon} - p_0)^2 \leq \varepsilon.
\end{aligned}$$

This yields the continuity of the map $x_0 \mapsto p_{x_0}$ in $L^2(B_1)$ at 0 (due to the 2-homogeneity of p_{x_0}, and implies the desired result, using again that the p_{x_0} are homogeneous polynomials. In particular, the map $x_0 \mapsto A_{x_0}$ is uniformly continuous on compacts. $\square$

We are now in a position to state a major result on the size of the singular set (due to [Caf98]):

Theorem 2.56. *Let u be any solution to* (2.15). *Let $\Sigma \subset B_1$ be the set of singular points. Then, $\Sigma \cap B_{1/2}$ is locally contained in a C^1 manifold of dimension $n-1$.*

Remark 2.57.
- One can construct examples in which the singular set is $(n-1)$-dimensional. This means that the singular set can be of the same dimension as the regular set!

- Singular points appear in all dimensions $n \geq 2$ (see [Sch75, Sch77]).

We will prove a much finer result in this section.

Note that blow-ups might look very different, depending on the dimension of the set $\{p_{x_0} = 0\}$.

This motivates the following definition:

Definition 2.58. Given any singular point x_0, let p_{x_0} be the blow-up of u at x_0 (a quadratic polynomial). Let $k \in \{0, ..., n-1\}$ be the dimension of the set $\{p_{x_0} = 0\}$ (a proper k-dimensional linear subspace of $\mathbb{R}^n$. We define

$$\Sigma_k := \{x_0 \in \Sigma : \dim(\{p_{x_0} = 0\}) = k\}.$$

Clearly, $\Sigma = \cup_{k=0}^{n-1} \Sigma_k$. This is called *stratification* of the singular set.

The following result gives a more precise description of the singular set.

Theorem 2.59. *Let u be any solution to* (2.15)*. Then, Σ_k is locally contained in a C^1 manifold of dimension k.*

Rough heuristic idea: Assume for simplicity that $n = 2$, so that $\Sigma = \Sigma_1 \cup \Sigma_0$.

- Let $x_0 \in \Sigma_0$. Then, by uniqueness of blow-ups we have the expansion
$$u(x) = p_{x_0}(x - x_0) + o(|x - x_0|^2)$$
- By definition of Σ_0, we have $p_{x_0} > 0$ in $\mathbb{R}^n \setminus \{0\}$, and thus by homogeneity $p_{x_0}(x - x_0) \geq c|x - x_0|^2$, with $c > 0$.
- Hence, by the expansion, u must be positive in a neighborhood of x_0. In particular, all points in Σ_0 are isolated.
- If $x_0 \in \Sigma_1$. Then, by definition of Σ_1 the blow-up must necessarily be of the form $p_{x_0}(x) = \frac{1}{2}(x \cdot e_{x_0})^2$, for some $e_{x_0} \in \mathbb{S}^{n-1}$.
- Hence, by the expansion, u is positive in a region of the form
$$\{x \in B_\rho(x_0) : |(x - x_0) \cdot e_{x_0}| > \omega(|x - x_0|)\},$$
where ω is a certain modulus of continuity, and $\rho > 0$ is small.
Hence, the set Σ_1 has a tangent plane at x_0.
- Now, repeat this at other points $\tilde{x}_0 \in \Sigma_1$ and prove that if $\tilde{x}_0$ is close to x_0 then $e_{\tilde{x}_0}$ must be close to e_{x_0}. This implies that Σ_1 is contained in a C^1 curve.

For the rigorous proof, we require Whitney's extension theorem (see [Whi34] and [PSU12, Lemma 7.10]):

Lemma 2.60 (Whitney's extension theorem)**.** *Let $E \subset \mathbb{R}^n$ be compact, and $f : E \to \mathbb{R}^n$. Assume that for any $x_0 \in E$, there is a polynomial p_{x_0} of degree m such that*

- $q_{x_0}(x_0) = f(x_0)$,
- $|D^k q_{x_0}(x_1) - D^k q_{x_1}(x_1)| = o(|x_0 - x_1|^{m-k})$ *for any $x_0, x_1 \in E$ and $k \in \{0, \dots, m\}$,*

where $o(r) \to 0$ as $r \to 0$ (uniformly in $x_0, x_1 \in E$). Then, f extends to a C^m function on $\mathbb{R}^n$ with

$$f(x) = q_{x_0}(x) + o(|x - x_0|^m) \quad \forall x_0 \in E.$$

Proof of Theorem 2.59. We set $E = \Sigma \cap B_1$. E is compact since Σ is closed. We claim that the polynomials $(q_{x_0})_{x_0 \in \Sigma}$ defined as $q_{x_0}(x) = p_{x_0}(x - x_0)$ satisfy the assumptions of Lemma 2.60 with $f = 0$ and $m = 2$.

Let us first explain how Lemma 2.60 implies the desired result. By Lemma 2.60, there is $f \in C^2(B_1)$ such that $f \equiv 0$ in $\Sigma \cap B_1$, and

$$f(x) = q_{x_0}(x) + o(|x - x_0|^2) \quad \forall x_0 \in \Sigma \cap B_1.$$

This means

$$f(x_0) = \nabla f(x_0) = 0, \qquad D^2 f(x_0) = A_{x_0}.$$

Moreover, for $x_0 \in \Sigma_k$, we can arrange the coordinate vectors so that $e_1, \dots, e_{n-k}$ are the eigenvalues of A_{x_0}, i.e.

$$\det D^2_{(x_1,\dots,x_{n-k})} f(x_0) \neq 0.$$

Since $f \in C^2(B_1)$, by the implicit function theorem,

$$\bigcap_{i=1}^{n-k} \{\partial_i f = 0\}$$

is a k-dimensional C^1 manifold in a neighborhood of x_0. Indeed, we can apply the implicit function theorem to $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^{n-k}$ with $\Phi(x) = (\partial_1 f(x), \dots, \partial_{n-k} f(x))$ which satisfies $\Phi(x_0) = 0$, $D\Phi(x_0)$ is invertible and $\Phi \in C^1$ by construction, to see that $\{\Phi = 0\}$ can be written as a graph expressing the first $n-k$ variables in terms of the remaining k variables locally near x_0, i.e. it is a k-dimensional C^1 manifold.

Since

$$\Sigma \cap B_1 \subset \{\nabla f = 0\} = \bigcap_{i=1}^{n} \{\partial_i f = 0\},$$

this yields the desired result.

Hence, it remains to verify the assumptions of Whitney's extension theorem (see Lemma 2.60). Clearly, $q_{x_0}(x_0) = p_{x_0}(0) = 0 = f(x_0)$ for any $x_0 \in \Sigma$. Hence, it remains to show for any $x_0, x_1 \in \Sigma \cap B_1$

$$|D^k q_{x_0}(x_1) - D^k q_{x_1}(x_1)| = o(|x_0 - x_1|^{2-k}), \quad \forall k \in \{0, 1, 2\}.$$

By Lemma 2.54, and using that $q_{x_1}(x_1) = Dq_{x_1}(x_1) = 0$ we get for $k = 0, 1$

$$|q_{x_0}(x_1) - q_{x_1}(x_1)| = q_{x_0}(x_1) = u(x_1) + o(|x_1 - x_0|^2) = o(|x_1 - x_0|^2),$$
$$|Dq_{x_0}(x_1) - Dq_{x_1}(x_1)| = |Dq_{x_0}(x_1)| = |Du(x_1)| + o(|x_1 - x_0|) = o(|x_1 - x_0|).$$

Moreover, since $D^2 p_{x_0} = D^2 q_{x_0} = A_{x_0}$, the condition for $k = 2$ is equivalent to continuity of the map $x_0 \mapsto A_{x_0}$, which also follows from Lemma 2.54. The proof is complete. □

2.13. **Further results on singular points.** So far, we have proved that the singular set $\Sigma = \cup_{k=0}^{n-1} \Sigma_k$ can be stratified and that the Σ_k are contained in a k-dimensional C^1 manifold.

Question: Is this the best we can do?

The dimension $(n-1)$ of the singular set is optimal.

Natural further questions are the following:

(1) Can we improve the order of the expansion to
$$u(x) = p_2(x) + o(|x|^{2+\alpha})?$$
(2) Is the singular set (or some stratum Σ_k) contained in a $C^{1,\alpha}$ manifold? (This would follow from (1) by Whitney's extension theorem)

(3) How often do singular points occur (generic regularity)?

2.13.1. *More recent results on the size of the set of singular points.*

- Weiss (1999) [Wei99]: In $n = 2$, one has expansion of order $1 + \alpha$, i.e. Σ_1 lies in a $C^{1,\alpha}$ curve
- Colombo, Spolaor, Velichkov (2018) [CSV18]: If $n \geq 3$, one has
$$\|u - p\|_{L^\infty(B_r)} \leq Cr^2 |\log r|^{-\varepsilon},$$
i.e. Σ_m lies in a $C^{1,\log^\varepsilon}$ m-dimensional manifold
- Figalli, Serra (2019) [FS19]: If $n = 2$, one has expansions of order $\alpha = 1$, i.e. Σ_1 lies in a C^2 curve
- Figalli, Serra (2019) [FS19]: If $n \geq 3$, one can write $\Sigma_{n-1} = \Sigma^g_{n-1} \cup \Sigma^a_{n-1}$, where Σ^g_{n-1} is in a $C^{1,1}$ $n-1$-dimensional manifold and Σ^a_{n-1} satisfies $\dim_H(\Sigma^a_{n-1}) \leq n-3$, and Σ_{n-1} lies in a $C^{1,\alpha}$ $n-1$-dimensional manifold. Here, g and a stand for "good" and "anomalous", respectively.
- Figalli, Serra (2019) [FS19]: If $n \geq 3$, one can write $\Sigma_k = \Sigma^g_k \cup \Sigma^a_k$ for any $k = 1, \dots, n-2$ (note that Σ_0 consists of isolated points, i.e. analytic), where Σ^g_k is in a $C^{1,1}$ k-dimensional manifold and Σ^a_k satisfies $\dim_H(\Sigma^a_k) \leq k - 1$, and Σ_m lies in a $C^{1,\log^\varepsilon}$ k-dimensional manifold.
- Franceschini, Zaton (2025) [FZ25b]: There is a closed set $\Sigma_\infty \subset \Sigma$ such that $\dim_H(\Sigma \setminus \Sigma_\infty) \leq n - 2$ and Σ_∞ is contained in a C^∞ $n-1$-dimensional manifold.

2.13.2. *Generic regularity.* It is very natural to understand whether singularities appear often, or if instead most solutions have no singularities. In the context of the obstacle problem, the key question is to understand the generic regularity of free boundaries.

Conjecture (Schaeffer, 1974)**.** Generically, the weak solution of the obstacle problem is also a strong solution, in the sense that the free boundary is a C^∞ manifold.

In other words, the conjecture states that, generically, the free boundary has no singular points.

- Monneau (2003) [Mon03]: The conjecture holds in $\mathbb{R}^2$,
- Figalli, Serra, Ros-Oton (2020) [FROS20]: The conjecture holds in $\mathbb{R}^3$ and $\mathbb{R}^4$ and in $\mathbb{R}^k$, for $k \geq 5$, generically $\dim_H(\Sigma) < n - 4$

It remains an open problem to decide whether or not Schaeffer's conjecture holds in dimensions $n \geq 5$ or not.

3. The Alt-Caffarelli problem

We have seen that the obstacle problem can be written as an unconstrained minimization problem as follows:
$$\text{minimize} \int_\Omega \frac{1}{2}|\nabla u|^2 + u^+ \, dx,$$
This minimization problem contains a non-smooth term u^+ in the functional. The Euler-Lagrange equation for this functional is then
$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } \Omega.$$
One can consider more general minimization problems with non-smooth terms of the following form
$$\text{minimize} \int_\Omega \frac{1}{2}|\nabla u|^2 + (u^+)^\gamma \, dx,$$

for some $\gamma \in (-2, \infty)$. They are also known as the Alt-Phillips problems (see [AP86]). The behavior of minimizers differs widely, depending on the value of γ.

- For $\gamma = 1$, we recover the obstacle problem.
- As $\gamma \to -2$, the functional converges to the perimeter functional and minimzers converge to minimal surfaces (see [DSS23]).
- For $\gamma \geq 2$, minimizers do not exhibit free boundaries.
  Indeed, in that case the Euler-Lagrange equation is of the form $\Delta u = \gamma u^{\gamma-1}$ in all of Ω. By a semilinear version of the strong maximum principle (see [V́84]), if $\Delta u = F'(u)$ in B_1, where F is such that $\int_0^1 F(s)^{-\frac{1}{2}} \,\mathrm{d}s = +\infty$, then if $u \geq 0$ and $u(0) = 0$, it must be $u \equiv 0$.

In this chapter, we will deal with the special case $\gamma = 0$, which is known as the Alt-Caffarelli problem (also known as "one-phase problem"). Its rigorous mathematical study goes back to [AC81]. There are many different ways to motivate the study of this problem. For instance, there are close relations to certain questions in

- fluid equations
- capillarity problems
- shape optimization problems
- optimal eigenvalue
- optimal partition problems
- harmonic measure

A very natural way to motivate the one-phase problem goes as follows:

- Consider a smooth domain $\Omega \subset \mathbb{R}^n$ and a solution u to
  $$\begin{cases} -\Delta u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ |\nabla u| &= 1 \quad \text{on } \partial\Omega. \end{cases}$$
  This problem is known as "Serrin's problem". It is well-known that there only exists a solution to this problem if Ω is a ball (see [Ser71]). (Very recently, it was shown that this result holds true in Lipschitz (and even more general domains) [FZ25a].) The reason for this phenomenon is that the problem is overdetermined. As we have seen, there already exists a unique solution to the Dirichlet problem $-\Delta u = 1$ in Ω with $u = 0$ on $\partial\Omega$. In general, this solution does not satisfy $|\nabla u| = -\partial_\nu u = 1$.

- A more general question in this setting is the following: Consider a domain $\Omega \subset \mathbb{R}^n$ and a solution u to
  $$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \cap B_1, \\ u &= 0 \quad \text{on } \partial\Omega \cap B_1, \\ |\nabla u| &= Q \quad \text{on } \partial\Omega \cap B_1, \end{cases}$$
  where $Q = 1$ (or more generally, $0 \leq Q \in C^\infty$), what can we say about $\partial\Omega \cap B_{1/2}$? Note that also this problem is overdetermined, but now we are asking about local properties of $\partial\Omega$!

- A natural question to ask would be
  $$Q \in C^{k,\alpha} \quad \Rightarrow \partial\Omega \in C^{k+1,\alpha} \ ?$$

This question is equivalent to asking whether the harmonic measure being $C^{k,\alpha}$ implies that $\partial\Omega \in C^{k+1,\alpha}$. [Note that the reverse question, namely whether $\partial\Omega \in C^{k+1,\alpha}$ implies that the harmonic measure is $C^{k,\alpha}$ is a standard consequence of Schauder theory (see [Eva10])].

- It is already a non-trivial question to prove the existence of solutions to the previous problem. It turns out (we will prove it later), that solutions arise from minimizing the following energy functional:
$$u \mapsto \int_{B_1} |\nabla u|^2 + Q^2(x)\mathbb{1}_{\{u>0\}}\,\mathrm{d}x.$$
For $Q \equiv 1$, this problem becomes exactly the Alt-Caffarelli problem and $\Omega = \{u > 0\}$!

3.1. Basic properties of minimizers.

Proposition 3.1. *Let $\Lambda > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $g \in H^1(\Omega)$ be such that $g \geq 0$ in Ω and define*
$$\mathcal{C} := \{w \in H^1(\Omega) : w - g \in H_0^1(\Omega)\}.$$
Then, there exists a minimizer of
$$\mathcal{F}(u) := \mathcal{F}_\Lambda(u) := \mathcal{F}_\Lambda(u,\Omega) := \int_\Omega |\nabla u|^2 + \Lambda|\{u > 0\} \cap \Omega| \quad \text{among all } v \in \mathcal{C}. \tag{3.1}$$
Moreover, any minimizer u satisfies $u \geq 0$ in Ω.

Proof. For any $v \in H^1(\Omega)$ it holds
$$\nabla(\max\{u, 0\}) = \mathbb{1}_{\{u>0\}}\nabla u.$$
Hence,
$$\mathcal{F}_\Lambda(u,\Omega) = \mathcal{F}_\Lambda(\max\{u,0\},\Omega) + \int_{\{u<0\}\cap\Omega} |\nabla u|^2 \geq \mathcal{F}_\Lambda(\max\{u,0\},\Omega), \tag{3.2}$$
which implies that any minimizer must be nonnegative in Ω.
Let $u_k \in H^1(\Omega)$ be a minimizing sequence such that $u_k - g \in H_0^1(\Omega)$ and
$$\mathcal{F}_\Lambda(u_k,\Omega) \leq \mathcal{F}_\Lambda(g,\Omega) \quad \text{for every } k \geq 1.$$
By (3.2), we may assume that, for every $k \in \mathbb{N}$, $u_k \geq 0$ on Ω.
For simplicity, we assume that $n > 2$ (the case $n = 2$ is analogous) and we set $2^* = \frac{2n}{n-2}$.
Then, we have by the Sobolev embedding
$$\begin{aligned}\|u_k - g\|^2_{L^{2^*}(\Omega)} &\leq C\int_\Omega |\nabla(u_k - g)|^2 dx \\ &\leq 2C\left(\int_\Omega |\nabla u_k|^2 dx + \int_\Omega |\nabla g|^2 dx\right) \\ &\leq 2C(\mathcal{F}_\Lambda(u_k,\Omega) + \mathcal{F}_\Lambda(g,\Omega)) \\ &\leq 4C\mathcal{F}_\Lambda(g,\Omega).\end{aligned}$$
Now, we estimate, [using that if $u_k \neq g$, then $u_k > 0$ or $g > 0$ for the second estimate]
$$\begin{aligned}\|u_k - g\|^2_{L^2(\Omega)} &\leq |\{u_k - g \neq 0\} \cap \Omega|^{\frac{2}{n}} \|u_k - g\|^2_{L^{2^*}(\Omega)} \\ &\leq (|\{u_k > 0\} \cap \Omega| + |\{g > 0\} \cap \Omega|)^{\frac{2}{n}}\, 4C\mathcal{F}_\Lambda(g,\Omega)\end{aligned}$$

$$\leq 8C\Lambda^{-2/n}\mathcal{F}_\Lambda(g,\Omega)^{\frac{2+n}{n}},$$

which implies that the sequence u_k is uniformly bounded in $H^1(\Omega)$.

Then, up to a subsequence, u_k converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function $u \in H^1(\Omega)$.

Now, the semi-continuity of the H^1 norm (with respect to the weak H^1 convergence) gives

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 dx.$$

On the other hand, passing again to a subsequence, we get that $u_k \to u$ pointwise a.e. This implies

$$\mathbb{1}_{\{u>0\}} \leq \liminf_{k\to\infty} \mathbb{1}_{\{u_k>0\}},$$

and so,

$$|\{u>0\}\cap\Omega| \leq \liminf_{k\to\infty} |\{u_k>0\}\cap\Omega|,$$

which finally gives that

$$\mathcal{F}_\Lambda(u,\Omega) \leq \liminf_{k\to\infty} \mathcal{F}_\Lambda(u_k,\Omega),$$

and so, u is a solution to (3.1). □

Remark 3.2. Note that the functional $\mathcal{F}_\Lambda$ is not convex, i.e. for $u_t(x) = (1-t)u_1(x)+tu_2(x)$ it holds

$$\mathbb{1}_{\{u_t>0\}} \not\leq (1-t)\mathbb{1}_{\{u_1>0\}} + t\mathbb{1}_{\{u_2>0\}}.$$

Therefore, minimizers are in general not unique! For instance, consider $\Omega = (-2,2)$, $\Lambda = 1$ and minimize $\mathcal{F}_\Lambda(u,\Omega)$ among functions $u \in H^1(\Omega)$ with $u(-2) = u(2) = 1$. Define

$$u_1(x) = 1, \qquad u_2(x) = \max(0, 1-|x+2|) + \max(0, 1-|x-2|).$$

Then, it holds

$$\mathcal{F}_\Lambda(u_1,\Omega) = \int_{-2}^{2} \mathbb{1}_{\{u_1>0\}} = 4, \qquad \mathcal{F}_\Lambda(u_2,\Omega) = \int_{-2}^{-1} 1+1 + \int_1^2 1+1 = 4.$$

One can show (by using the Euler Lagrange equation (see Proposition 3.4)) that there is no $u \in H^1((-2,2))$ with $u(-2) = u(2) = 1$ with $\mathcal{F}_\Lambda(u;\Omega) < 4$. Hence, u_1, u_2 are both minimizers.

We introduce the concept of local minimizers. This allows us to consider the problem without explicitly prescribing boundary data.

Definition 3.3 (Local minimizers)**.** Let $\Omega \subset \mathbb{R}^n$. We say that $u : \Omega \to \mathbb{R}$ is a local minimizer of $\mathcal{F}_\Lambda$ in Ω, if $u \in H^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$, and for any bounded open set $B \Subset \Omega$, we have

$$\mathcal{F}_\Lambda(u,B) \leq \mathcal{F}_\Lambda(v,B) \quad \text{for every } v \in H^1_{\mathrm{loc}}(\Omega) \text{ such that } u-v \in H^1_0(B).$$

If Ω is bounded and smooth, we can equivalently take $B = \Omega$ in the above definition.

The goal of this subsection is to prove the following basic properties of (local) minimizers

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then,*

(i) *$u \geq 0$ a.e. in Ω.*
(ii) *u is weakly subharmonic, i.e. $-\Delta u \leq 0$, in Ω.*
(iii) *If u is continuous, then $u \in L^\infty_{loc}(\Omega)$.*
(iv) *If u is continuous, then $-\Delta u = 0$ in $\Omega \cap \{u>0\}$.*

Remark 3.5.

- We will prove that any minimizer is continuous in the next subsection.
- Proposition 3.4(ii) yields that the distributional Laplacian
$$\Delta u(\phi) = -\int_{\Omega} \nabla u \cdot \nabla \phi \quad \forall \phi \in C_c^1(\Omega) \tag{3.3}$$
is representable as a positive Borel measure, namely given an open set $A \subset \Omega$, one defines
$$\Delta u(A) := \int_A \Delta u \,\mathrm{d}x := \sup\left\{\Delta u(\phi) : \phi \in C_c^1(\Omega), \quad 0 \le \phi \le 1, \quad \operatorname{supp}(\phi) \subset A\right\}.$$
In fact, since (3.3) is defined for any $\phi \in C_c^1(\Omega)$, by density of $C_c^1(\Omega) \subset C_c(\Omega)$ with respect to the supremum norm, we can first extend $\Delta u : C_c(\Omega) \to [0, \infty)$ and then apply Riesz' representation theorem. Note that the idea behind the definition of $\Delta u(A)$ is to approximate $\mathbb{1}_A$ by functions $\phi \in C_c^1(\Omega)$. Unlike for the obstacle problem, we will see that the Laplacian measure is not absolutely continuous with respect to Lebesgue measure, and instead is concentrated on the free boundary $\partial\{u > 0\}$.

First, we see that local minimizers are subharmonic, and in particular they are locally bounded in Ω.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then u is weakly subharmonic, i.e. $-\Delta u \le 0$, in Ω. In particular, if u is continuous, then $u \in L^\infty_{loc}(\Omega)$.*

Note that we assume ontinuity of u in order to deduce $u \in L^\infty_{loc}(\Omega)$ from Lemma 1.15. One can prove that any weak subharmonic function is locally bounded, without assuming it to be continuous.

Proof. Let $B \subset \Omega$ and $\phi \in H_0^1(B)$ be a given non-negative function. Suppose that $t \ge 0$ and $v = u - t\phi$. Then we have that $v_+ \le u = u_+$, and therefore
$$\{v > 0\} \cap B = \{v_+ > 0\} \cap B \subset \{u_+ > 0\} \cap B = \{u > 0\} \cap B.$$
In particular, since u is a minimizer and v is a competitor, we have
$$\int_B |\nabla u|^2 - \int_B |\nabla v|^2 \le -\Lambda\big(|\{u > 0\} \cap B| - |\{v > 0\} \cap B|\big) \le 0$$
This implies that
$$\begin{aligned}\int_B |\nabla u|^2 dx &\le \int_B |\nabla(u - t\phi)|^2 dx \\ &= \int_B |\nabla u|^2 dx - 2t \int_B \nabla u \cdot \nabla \phi \, dx + t^2 \int_B |\nabla \phi|^2 dx,\end{aligned}$$
This yields
$$\int_B \nabla u \cdot \nabla \phi \, dx \le \frac{t}{2} \int_B |\nabla \phi|^2 dx,$$
and the first claim follows by taking the limit $t \to 0$.

Since $u \ge 0$ and $-\Delta u \le 0$, we can apply the local boundedness estimate from Lemma 1.15 to deduce that $u \in L^\infty_{loc}(\Omega)$. $\square$

Remark 3.7 (Pointwise definition of minimizers)**.** By Lemma 1.13, we know that for every $x_0 \in \Omega$, we have
$$r \mapsto \fint_{B_r(x_0)} u \, dx \text{ is non-decreasing.}$$

As a consequence, we can define the following pointwise representative $\tilde{u}$ of u

$$\tilde{u}(x_0) := \lim_{r\to 0^+} \fint_{B_r(x_0)} u(x)\,dx \quad \text{for every } x_0 \in \Omega.$$

Note that $\tilde{u} = u$ a.e. in Ω and $\tilde{u} \geq 0$ in Ω.
From now on, we will identify any solution u of (3.1) with its representative $\tilde{u}$ (and for simplicity, we will always write u instead of $\tilde{u}$).

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then u is weakly harmonic in the interior of $\{u > 0\}$.*

Proof. Let $t \in \mathbb{R}$ and $\phi \in H^1_0(B)$ for some open set $B \subset \{u > 0\} \cap \Omega$. Such a set exists since we assume the interior of $\{u > 0\}$ to be non-empty (otherwise, there is nothing to prove).

Then, it holds

$$\{u + t\phi > 0\} \cap B \subset \{u > 0\} \cap B$$

and hence, since u is a local minimizer,

$$0 \geq \int_B |\nabla u|^2 - \int_B |\nabla(u + t\phi)|^2 = t^2 \int_B |\nabla \phi|^2 + 2t \int_B \nabla u \cdot \nabla \phi.$$

Dividing by $|t|$, we deduce

$$\operatorname{sgn}(t) \int_B \nabla u \cdot \nabla \phi \leq \frac{t}{2} \int_B \nabla u \cdot \nabla \phi.$$

Hence, by taking the limits $t \searrow 0$ and $t \nearrow 0$, we obtain the desired result. $\square$

Proof of Proposition 3.4. Property (i) follows as in Proposition 3.1. Properties (ii) and (iii) follow from Lemma 3.6 and (iv) follows from Lemma 3.8. $\square$

3.2. **Optimal regularity of solutions.** The goal of this section is to prove the following theorem

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then, $\{u > 0\}$ is open, and $u \in C^{0,1}_{loc}(\Omega)$ and for any $B_r(x_0)$ with $B_{2r}(x_0) \subset \Omega$,*

$$\|u\|_{C^{0,1}(B_r(x_0))} \leq C\left(\sqrt{\Lambda} + r^{-n}\|u\|_{L^1(B_{2r}(x_0))}\right),$$

where C only depends on n.

Since Theorem 3.9 yields openness of $\{u > 0\}$, we obtain in particular that u is weakly harmonic in $\{u > 0\}$ from Lemma 3.8.

There are several ways to prove this result. We refer to [Vel23] for a discussion of three different proofs. Here, we will follow an approach that is due to Alt-Caffarelli-Friedman and also works for free boundary problems with two phases (this means that solution are also allowed to be negative).

Lemma 3.10 (The Laplacian estimate)**.** *Let u be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then, for every ball $B_r(x_0)$ such that $B_{2r}(x_0) \subset \Omega$ we have*

$$\Delta u(B_r(x_0)) \leq C\sqrt{\Lambda} r^{n-1}.$$

Proof. Without loss of generality we can assume that $x_0 = 0$. Moreover, by scaling [replace u by $\sqrt{\Lambda}^{-1}u$], we can assume $\Lambda = 1$. We now notice that by Lemma 3.6 the distributional Laplacian (see (3.3))

$$\Delta u(\phi) := -\int_{\Omega} \nabla u \cdot \nabla \phi \, dx \quad \text{for every } \phi \in C_c^1(\Omega),$$

is a positive Radon measure. We first prove that

$$\Delta u(\phi) \leq C r^{n/2} \|\nabla \phi\|_{L^2(B_r)} \quad \text{for every } \phi \in C_c^1(B_r) \text{ and every } B_r \subset \Omega. \tag{3.4}$$

Indeed, for every $\psi \in C_c^1(B_r)$, the optimality of u gives

$$\int_{B_r} |\nabla u|^2 dx \leq \int_{B_r} |\nabla u|^2 dx + |\{u > 0\} \cap B_r| \leq \int_{B_r} |\nabla (u + \psi)|^2 dx + |B_r|.$$

This implies

$$-\int_{B_r} \nabla u \cdot \nabla \psi \, dx \leq \frac{1}{2} \int_{B_r} |\nabla \psi|^2 dx + C r^n.$$

Setting

$$\psi = \frac{r^{n/2}}{\|\nabla \phi\|_{L^2(B_r)}} \phi$$

we get

$$-\int_{B_r} \nabla u \cdot \nabla \phi \, dx \leq C r^{n/2} \|\nabla \phi\|_{L^2(B_r)},$$

which proves (3.4), as desired.

Let now $\phi \in C_c^1(B_{2r})$ be such that $\phi \geq 0$ in B_{2r}, $\phi = 1$ on B_r, and $\|\nabla \phi\|_{L^\infty(B_{2r})} \leq 2/r$. Thus, by the positivity of Δu we have

$$\Delta u(B_r) \leq \Delta u(\phi) \leq C r^{n/2} \|\nabla \phi\|_{L^2(B_{2r})} \leq C r^{n-1}.$$

The first inequality follows because for any $\phi_r \in C_c^1(\Omega)$ with $0 \leq \phi_r \leq 1$ and $\operatorname{supp}(\phi_r) \subset B_r$, it holds $\phi - \phi_r \geq 0$, since $\phi \geq \mathbb{1}_{B_r}$ by construction. ☐

The following lemma yields a useful consequence of the Laplacian estimate.

Lemma 3.11. *Suppose that $u \in H^1(B_R)$ is a nonnegative subharmonic function in the ball $B_R \subset \mathbb{R}^n$ such that $u(0) = 0$. Suppose that there is a constant $C_0 > 0$ such that*

$$\Delta u(B_r) \leq C_0 r^{n-1} \quad \textit{for every } 0 < r < R. \tag{3.5}$$

Then we have

$$\fint_{\partial B_r} u \, dx \leq \frac{C_0}{n\omega_n} r \quad \textit{for every } 0 < r < R.$$

Proof. We first notice that for every smooth u_ε, we have

$$\frac{d}{dr}\left(\fint_{\partial B_r} u_\varepsilon \, dx \right) = \fint_{\partial B_r} \frac{\partial u_\varepsilon}{\partial \nu} \, dx = \frac{1}{n\omega_n r^{n-1}} \int_{B_r} \Delta u_\varepsilon(x) \, dx.$$

Integrating in r and passing to the limit as $\varepsilon \to 0$ and using that $\int_{B_r} \Delta u_\varepsilon \to \Delta u(B_r)$, and (3.5) we get

$$\fint_{\partial B_r} u \, dx \leq u(0) + \int_0^r \frac{\Delta u(B_s)}{n\omega_n s^{n-1}} ds \leq \frac{C_0}{n\omega_n} r.$$

□

Applied to a minimizer of the one-phase problem, the estimate (3.6) yields an upper estimate on the growth of solutions near a free boundary point (compare Lemma 2.8 for the obstacle problem). It is the main ingredient in the proof of the optimal Lipschitz regularity:

Proof of Theorem 3.9. The proof is divided into several steps.

Step 1: Suppose that $x_0 \in \Omega \cap \partial\{u > 0\}$ such that $B_{2R}(x_0) \subset \Omega$ for some $R > 0$. We claim that

$$\fint_{\partial B_r(x_0)} u \, \mathrm{d}x \le C\sqrt{\Lambda} r \quad \forall 0 < r < R. \tag{3.6}$$

Step 1a: To prove it, we first observe that

$$\partial\{u > 0\} \cap \Omega = \big\{x \in \Omega : 0 < |\{u > 0\} \cap B_r(x)| < |B_r| \quad \forall r < \mathrm{dist}(x_0, \partial\Omega)\big\}$$

Since $\partial\{u > 0\} \cap \Omega = \overline{\{u > 0\}} \cap \overline{\Omega \setminus \{u > 0\}} \cap \Omega$, the inclusion $\supset$ is trivially satisfied. To prove the inclusion $\subset$ we observe two things.

First, if $|B_r(x) \cap \{u = 0\}| = 0$, then u is harmonic in $B_r(x)$ and therefore, $B_r(x) \cap \{u = 0\} = \emptyset$. Indeed, we have seen in the proof of Lemma 3.8 that by the minimality of u, it holds

$$\int_{B_r(x)} |\nabla u|^2 \le \int_{B_r(x)} |\nabla v|^2 \quad \forall v \in H^1_0(B_r(x)) \quad \text{s.t. } v = 0 \text{ a.e. in } \{u = 0\} \cap B_r(x).$$

Therefore, by choosing v to be the unique weak solution to $-\Delta v = 0$ in $B_r(x)$ with $v = u$ on $\partial B_r(x)$, we get a contradiction with the previous display [note that v is admissible since by $|B_r(x) \cap \{u = 0\}| = 0$, it holds $v = 0$ a.e. when $u = 0$ in $B_r(x)$], since v has less energy than u, unless $u = v$.
This means that u is harmonic in $B_r(x)$, and therefore, by the strict maximum principle, it must be $u > 0$ in $B_r(x)$.

Second, if $|B_r(x) \cap \{u > 0\}| = 0$, then by Remark 3.7, it must be $u \equiv 0$ in $B_r(x)$, which means that $B_r(x) \cap \{u > 0\} = \emptyset$.

This proves the remaining inclusion $\subset$ and yields the claim.

Step 1b: Note that if we knew that $u(x_0) = 0$, then the claim (3.6) would immediately follow from Lemma 3.10 and Lemma 3.11. [At this point, we don't know that $u(x_0) = 0$ since we don't know continuity of u, yet.]

In particular, by the claim in Step 1b, we can find a sequence (x_n) with $u(x_n) = 0$ such that $x_n \to x_0$. Since (3.6) holds true at x_n (as a consequence of Lemma 3.10 and Lemma 3.11), we can deduce (3.6) at x_0 by using the continuity of the function

$$x \mapsto \fint_{\partial B_r(x)} u,$$

for any fixed $r > 0$, which follows from the fact that $u \in H^1_{loc}(\Omega)$. This proves (3.6).

Step 2: Passing the estimate (3.6) on both sides to the limit as $r \to 0$, in particular, we obtain that $u(x_0) = 0$, recalling that we identify u with its pointwise representative (see Remark 3.7).

Thus $\{u > 0\} \cap \partial\{u > 0\} = \emptyset$ and so $\{u > 0\}$ is open.

Step 3: Let $x_0 \in \Omega$ be such that $B_{2R}(x_0) \subset \Omega$. To prove the Lipschitz estimate, we distinguish between two cases.

- Case 1: If $\operatorname{dist}(x_0, \partial\{u > 0\}) \geq R/4$, then u is harmonic in the ball $B_{R/4}(x_0)$ and so, by gradient estimates (see for instance Corollary 1.5) and using also Lemma 1.15 (or rather Remark 3.7), we have
$$|\nabla u(x_0)| \leq \frac{C}{R^n} \|u\|_{L^\infty(B_R(x_0))} \leq \frac{C}{R^{n+1}} \int_{B_R(x_0)} u \, dx.$$
- Case 2: If $\operatorname{dist}(x_0, \partial\{u > 0\}) < R/4$, then we suppose that the distance to the free boundary is realized by some $y_0 \in \partial\{u > 0\}$ and we set
$$r = \operatorname{dist}(x_0, \partial\{u > 0\}) = |x_0 - y_0|.$$
Since u is harmonic in $B_r(x_0)$, we can again apply the gradient estimate and (3.6), obtaining
$$\begin{aligned} |\nabla u(x_0)| &\leq \frac{C}{r^{n+1}} \int_{B_r(x_0)} u \, dx \\ &\leq \frac{C}{r^{n+1}} \int_{B_{2r}(y_0)} u \, dx \\ &\leq \frac{C}{r^{n+1}} \int_0^{2r} \left(\int_{\partial B_s(y_0)} u \, dx \right) \mathrm{d}s \leq \frac{C\sqrt{\Lambda}}{r^{n+1}} \int_0^{2r} s^n \, \mathrm{d}s \leq C\sqrt{\Lambda}. \end{aligned}$$
where we used that $u \geq 0$ and the inclusion $B_r(x_0) \subset B_{2r}(y_0)$.

By combining the results from both cases we deduce the desired result. $\square$

3.3. **Nondegeneracy.** In this section we prove the non-degeneracy of the solutions to the one-phase problem (2.1). Our main result is the following:

Proposition 3.12 (Non-degeneracy of the solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Let $x_0 \in \overline{\{u > 0\}} \cap \Omega$. Then for every ball $B_{2r}(x_0) \subset \Omega$, we have*
$$\|u\|_{L^\infty(B_r(x_0))} \geq \Lambda^{1/2} cr,$$
where $c > 0$ depends only on n.

The result will follow from the following lemma. In its proof, we will use the property (3.6). Note that there are also direct proofs of nondegeneracy which do not use the Lipschitz continuity of minimizers.

Lemma 3.13. *Let $\Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$ be a local minimizer of $\mathcal{F}_\Lambda$ in Ω. Then, there is a constant $\kappa_0 > 0$, depending on n and Λ, such that:*
$$\text{If } x_0 \in \Omega \text{ and } r \in (0, \operatorname{dist}(x_0, \partial\Omega)) \text{ are s.t. } \fint_{\partial B_r(x_0)} u \, dx \leq \kappa_0 r, \text{ then } u = 0 \text{ in } B_{r/8}(x_0).$$

We first explain how Lemma 3.13 implies Proposition 3.12.

Proof of Proposition 3.12. By scaling [replace u by $\sqrt{\Lambda}^{-1} u$], it suffices to assume $\Lambda = 1$. Then, by the previous lemma, there is $\kappa > 0$ such that for any $x_0 \in \Omega$ and $r > 0$ with $B_{2r}(x_0) \subset \Omega$ it holds
$$\text{either} \qquad u \equiv 0 \quad \text{in } B_{r/8}(x_0) \qquad \text{or} \qquad \fint_{\partial B_r(x_0)} u \, \mathrm{d}x \geq \kappa r.$$
In particular, if $x_0 \in \overline{\{u > 0\}}$, then $u \not\equiv 0$ in $B_{r/8}(x_0)$ for any $r > 0$ such that $B_{2r}(x_0) \subset \Omega$.

Hence, for any such $r > 0$,

$$\max_{B_r(x_0)} u \geq \fint_{B_r(x_0)} u \,\mathrm{d}x = cr^{-n} \int_0^r \int_{\partial B_\rho(x_0)} u \,\mathrm{d}x \,\mathrm{d}\rho \geq c\kappa r^{-n} \int_0^r \rho^n \,\mathrm{d}\rho \geq c\kappa r,$$

as desired. $\square$

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