

ADVANCED TOPICS LECTURE: FREE BOUNDARY PROBLEMS

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The goal of this lecture is to give an introduction to free boundary problems. These are partial differential equations which exhibit an a priori unknown interface. A prototype example is given by the melting of ice in water, but free boundary problems also exist in various other contexts such as, physics, material sciences, biology, finance, etc.

Typical questions:

- optimal regularity of solutions (across the free boundary)
 - regularity of the free boundary
 - singular free boundary points
- (1) Basic properties of harmonic functions
 - mean value property, maximum principle
 - basic regularity results
 - (2) The obstacle problem [FRRO22, PSU12]
 - optimal regularity
 - Caffarelli's dichotomy: regular and singular points
 - $C^{1,\alpha}$ regularity of the free boundary near regular points
 - higher regularity of the free boundary
 - properties of singular points
 - outlook
 - (3) The Alt-Caffarelli problem [Vel23, CS05]
 - optimal regularity
 - improvement of flatness
 - higher regularity of the free boundary
 - singular points
 - outlook
 - (4) Further topics
 - thin obstacle problem and nonlocal operators
 - time-dependent free boundary problems
 - free boundary problems with multiple phases
 - ...

1. BASIC PROPERTIES OF HARMONIC FUNCTIONS

The Dirichlet problem for the Laplace equation is given as follows

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &= g & \text{in } \partial\Omega, \end{cases} \quad (1.1)$$

where the boundary condition g and the source term f are given and $\Omega \subset \mathbb{R}^n$ is a bounded (Lipschitz) domain. There are different ways to make sense of solutions to this problem. Under suitable assumptions on f, g , there exists a unique solution.

From now on, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We recall several important facts and definitions.

- We have the following function space

$$H^1(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega) \text{ for } i \in \{1, \dots, n\}\},$$

where $\partial_i u$ are the weak partial derivatives of u and $\nabla u = (\partial_1 u, \dots, \partial_n u)$.

- When equipped with the following scalar product, $H^1(\Omega)$ is a Hilbert space

$$(u, v)_{H^1(\Omega)} = \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \nabla v \, dx, \quad (u, u)_{H^1(\Omega)} = \|u\|_{H^1(\Omega)}^2.$$

- Recall the following integration by parts formula: if $u, v \in H^1(\Omega)$, then

$$\int_{\Omega} \partial_i uv \, dx = - \int_{\Omega} u \partial_i v \, dx + \int_{\partial\Omega} uv \nu_i \, dx, \quad i = 1, \dots, n,$$

where $\nu \in \mathbb{S}^{n-1}$ is the unit outward normal vector to $\partial\Omega$.

- There is a compact trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$, such that $\text{Tr} u = u|_{\partial\Omega}$ whenever $u \in H^1(\Omega) \cap C(\overline{\Omega})$. We define

$$H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}_{H^1(\Omega)}$$

as the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{H^1(\Omega)}$. It holds

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \text{Tr}(u) = 0\}.$$

- Sobolev embedding

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega), \quad \text{if } 2 < n,$$

Moreover, the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact, whenever $q < \frac{2n}{n-2}$. In particular, $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

- Poincaré inequality: for any $u \in H^1(\Omega)$ it holds

$$\begin{aligned} \int_{\Omega} |u - (u)_{\Omega}|^2 \, dx &\leq C_1 \int_{\Omega} |\nabla u|^2 \, dx, \\ \int_{\Omega} |u|^2 \, dx &\leq C_2 \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} |\text{Tr} u|^2 \, dx. \end{aligned}$$

The constants C_1, C_2 only depend on n, Ω .

- Hölder spaces: Let $\alpha \in (0, 1]$. We define for $u \in C(\overline{\Omega})$

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \quad \|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\alpha}(\overline{\Omega})}.$$

Moreover, for $k \in \mathbb{N} \cup \{0\}$, we set

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \|u\|_{C^k(\Omega)} + [D^k u]_{C^{0,\alpha}(\overline{\Omega})}, \quad \|u\|_{C^k(\Omega)} = \sum_{j=1}^k \|D^j u\|_{L^\infty(\Omega)}.$$

Note that by Hölder interpolation, it holds

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \asymp \|u\|_{L^\infty(\overline{\Omega})} + [D^k u]_{C^{0,\alpha}(\overline{\Omega})}, \quad \|u\|_{C^{k,1}(\overline{\Omega})} \asymp \|u\|_{L^\infty(\overline{\Omega})} + \|D^{k+1} u\|_{L^\infty(\Omega)}.$$

We define the spaces

$$C^{k,\alpha}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \|u\|_{C^{k,\alpha}(\overline{\Omega})} < \infty\}.$$

Sometimes, when $0 < k + \alpha = \beta \notin \mathbb{N}$, we define $C^\beta(\overline{\Omega}) := C^{k,\alpha}(\overline{\Omega})$. Note

$$C^\infty(\overline{\Omega}) \subset \dots \subset C^{k,\alpha}(\overline{\Omega}) \subset C^{1,\alpha}(\overline{\Omega}) \subset C^1(\overline{\Omega}) \subset C^{0,1}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega}) \subset C(\overline{\Omega}).$$

- Arzelà-Ascoli's theorem: Given a sequence $(f_i)_i \subset C^{k,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$ and $k \in \mathbb{N} \cup \{0\}$ satisfying $\|f_i\|_{C^{k,\alpha}(\overline{\Omega})} \leq C$ for some $C > 0$. Then, there exists a subsequence $(f_{i_j})_j \subset (f_i)_i$ which converges uniformly (if $k = 0$) and in $C^k(\overline{\Omega})$ (if $k \in \mathbb{N}$) to some $f \in C^{k,\alpha}(\overline{\Omega})$ and $\|f\|_{C^{k,\alpha}(\overline{\Omega})} \leq C$.

Literature recommendation: [Eva10]. Also recall functional analysis and PDE lecture.

Definition 1.1. Let $f \in L^2(\Omega)$. We say that u satisfies $-\Delta u = f$ in Ω in the weak sense whenever $u \in H^1(\Omega)$ and

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \quad \text{for all } v \in H_0^1(\Omega). \quad (1.2)$$

Let $g \in L^2(\partial\Omega)$. We say that u is a weak solution of the Dirichlet problem (1.1) if $u \in H^1(\Omega)$ satisfies $\text{Tr } u = g$, and (1.2).

We say that u is weakly superharmonic (resp. weakly subharmonic) in Ω , or satisfies $-\Delta u \geq 0$ in Ω in the weak sense (resp. $-\Delta u \leq 0$ in the weak sense) if

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx \geq 0 \quad \text{resp.} \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx \leq 0 \quad \text{for all } v \in H_0^1(\Omega), v \geq 0.$$

We say that $u \geq g$ on $\partial\Omega$ if $\text{Tr } u \geq g$ on $\partial\Omega$.

Remark 1.2. If $u \in C^2(\overline{\Omega})$, then it holds $-\Delta u = f$ in Ω in the classical sense, if and only if it holds in the weak sense. Proof: integration by parts.

1.1. Regularity of solutions and the maximum principle. Throughout this section, whenever we say that $\Omega \subset \mathbb{R}^n$ is a domain, we mean that Ω is a connected, bounded, open set with $\partial\Omega \in C^{0,1}$. The latter assumption can usually be relaxed, but we assume it here for simplicity in order to have a well-defined trace operator.

Theorem 1.3 (Existence and uniqueness of weak solutions). *Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in L^2(\Omega)$ and*

$$\{w \in H^1(\Omega) : \text{Tr } w = g\} \neq \emptyset. \quad (1.3)$$

Then, there exists a unique weak solution to the Dirichlet problem (1.1).

Proof. Lax Milgram. (We expect this to be well-known.) □

Remark 1.4. • A sufficient condition for (1.3) to hold true is if $g \in C^{0,1}(\partial\Omega)$.

- (1.3) holds true if and only if there exists $G \in H^1(\Omega)$ such that $\text{Tr } G = g$. One can show that this is the case if and only if $g \in H^{1/2}(\partial\Omega)$.

The unique weak solution to the Dirichlet problem in a ball is explicit:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases} \implies u(x) = \omega_{n-1} \int_{\partial B_1} \frac{(1 - |x|^2)g(y)}{|x - y|^n} dy,$$

where $\omega_{n-1} = |\mathbb{S}^{n-1}|$.

By a rescaling argument, a similar formula holds in any ball $B_r(x_0) \subset \mathbb{R}^n$. Thus, we deduce that for any harmonic function $\Delta u = 0$ in Ω , with $B_r(x_0) \subset \Omega$, we have (Poisson kernel representation)

$$u(x) = \omega_{n-1} r^{-1} \int_{\partial B_r(x_0)} \frac{(r^2 - |x - x_0|^2)u(y)}{|x - y|^n} dy. \quad (1.4)$$

An immediate consequence of (1.4) is the following result.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^n$ be any open set, and $u \in H^1(\Omega)$ be any function satisfying $\Delta u = 0$ in Ω in the weak sense. Then, u is C^∞ inside Ω and u is a classical solution.*

Moreover, if u is bounded and $\Delta u = 0$ in B_1 in the weak sense, then we have the estimates

$$\|u\|_{C^k(B_{1/2})} \leq C_k \|u\|_{L^\infty(B_1)}, \quad (1.5)$$

for all $k \in \mathbb{N}$, and for some constant C_k depending only on k and n .

Proof. For any ball $B_r(x_0) \subset \Omega$ it holds (1.4). By differentiating this formula it is immediate to see that $u \in C^\infty(B_{r/2}(x_0))$ and that (1.5) holds. Since this can be done for any ball $B_r(x_0) \subset \Omega$, we deduce that u is C^∞ inside Ω . $\square$

Next, we prove the maximum principle for weak solutions.

Proposition 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in H^1(\Omega)$ satisfies, in the weak sense,*

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $u \geq 0$ in Ω .

Proof. Notice that since $-\Delta u \geq 0$ in Ω we have

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx \geq 0 \quad \text{for all } v \geq 0, \quad v \in H_0^1(\Omega). \quad (1.6)$$

Let us consider $u^- := \max\{-u, 0\}$ and $u^+ := \max\{u, 0\}$, so that $u = u^+ - u^-$. It is easy to check that $u^\pm \in H^1(\Omega)$ whenever $u \in H^1(\Omega)$, and that $u^- \in H_0^1(\Omega)$ since $\text{Tr } u \geq 0$ on $\partial\Omega$. Hence we can choose $v = u^- \geq 0$ in (1.6). Then, using that $\nabla u = \nabla u^+ - \nabla u^-$ and $\nabla u^+ \cdot \nabla u^- = 0$, we get

$$0 \leq \int_{\Omega} \nabla u \cdot \nabla u^- \, dx = \int_{\Omega} \nabla u^+ \cdot \nabla u^- \, dx - \int_{\Omega} |\nabla u^-|^2 \, dx = - \int_{\Omega} |\nabla u^-|^2 \, dx.$$

Hence, $\nabla u^- \equiv 0$ in Ω . Since $\text{Tr } u^- \equiv 0$ this implies $u^- \equiv 0$ in Ω , that is, $u \geq 0$ in Ω . $\square$

Remark 1.7. • comparison principle: If $-\Delta u \geq -\Delta v$ in Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω .

• in particular, superharmonic functions have their minimum on the boundary.

• Analogously, if $-\Delta u \leq 0$ in Ω and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω .

A useful consequence of the maximum principle is the following.

Lemma 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let u be any weak solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)},$$

for a constant C depending only on the diameter of Ω .

Proof. Let us consider the function

$$\tilde{u}(x) := u(x) / (\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}).$$

We want to prove that $|\tilde{u}| \leq C$ in Ω . Notice that $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } \Omega \\ \tilde{u} = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

with $|\tilde{g}| \leq 1$ and $|\tilde{f}| \leq 1$.

Let us choose R large enough so that $B_R \supset \Omega$; after a translation, we can take $R = \frac{1}{2}\text{diam}(\Omega)$. In B_R , let us consider the function

$$w(x) = \frac{R^2 - |x|^2}{2} + 1.$$

The function w satisfies

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega \\ w \geq 1 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by the comparison principle, we deduce that

$$\tilde{u} \leq w \quad \text{in } \Omega.$$

Since $w \leq C$ (with C depending only on R), we deduce that $\tilde{u} \leq C$ in Ω . Finally, repeating the same argument with $-\tilde{u}$ instead of $\tilde{u}$, we find that $|\tilde{u}| \leq C$ in Ω , and thus we are done. $\square$

The following result follows from the maximum principle and states how solutions to the Dirichlet problem behave near the boundary.

We say that Ω satisfies the *interior ball condition* whenever there exists $\rho_0 > 0$ such that every point on $\partial\Omega$ can be touched from inside with a ball of radius ρ_0 contained in Ω . That is, for any $x_0 \in \partial\Omega$ there exists $B_{\rho_0}(y_0) \subset \Omega$ with $x_0 \in \partial B_{\rho_0}(y_0)$. It is not difficult to see that any C^2 domain satisfies such condition, and also any domain which is the complement of a convex set.

Lemma 1.9 (Hopf lemma). *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the interior ball condition. Let $u \in C(\overline{\Omega})$ be a positive weakly superharmonic function in $\Omega \cap B_2$, with $u \geq 0$ on $\partial\Omega \cap B_2$. Then, $u \geq c_0 d$ in $\Omega \cap B_1$ for some $c_0 > 0$, where $d(x) := \text{dist}(x, \Omega^c)$.*

Note that c_0 in general depends on u !

Proof. Since u is positive and continuous in $\Omega \cap B_2$, we have that

$$u \geq c_1 > 0 \quad \text{in } \{d \geq \rho_0/2\} \cap B_{3/2}$$

for some $c_1 > 0$. Let us consider the solution of

$$\begin{cases} -\Delta w &= 0 & \text{in } B_{\rho_0} \setminus B_{\rho_0/2}, \\ w &= 0 & \text{on } \partial B_{\rho_0}, \\ w &= 1 & \text{on } \partial B_{\rho_0/2}. \end{cases}$$

One can check

$$\begin{aligned} w(x) &= \frac{|x|^{2-n} - \rho_0^{2-n}}{(\rho_0/2)^{2-n} - \rho_0^{2-n}} \quad \text{if } n \geq 3, \\ w(x) &= \frac{\ln(\rho_0/|x|)}{\ln 2} \quad \text{if } n = 2, \\ w(x) &= \max \left\{ 1, \frac{2}{\rho_0}(\rho_0 - |x|) \right\} \quad \text{if } n = 1. \end{aligned}$$

In particular, it is immediate to check that $w \geq c_2(\rho_0 - |x|)$ in B_{ρ_0} for some $c_2 > 0$.

Let us take $x_0 \in \partial\Omega$, and apply the comparison principle to the functions u and $c_1 w(y_0 + x)$ in $(B_{\rho_0}(y_0) \setminus B_{\rho_0/2}(y_0)) \subset \Omega \cap B_{3/2}$, where y_0 is from the definition of the interior ball condition. (We are using that $u \in C(\overline{\Omega})$ to guarantee $u \geq 0$ on $\partial B_{\rho_0}(y_0)$). Hence, we deduce that

$$u(x) \geq c_1 w(y_0 + x) \geq c_1 c_2 (\rho_0 - |x - y_0|) \geq c_1 c_2 d(x) \quad \text{in } B_{\rho_0}(y_0).$$

Setting $c_0 = c_1 c_2$ and using the previous inequality for $x_0 \in \partial\Omega$ and the corresponding ball $B_{\rho_0}(y_0) \subset \Omega \cap B_{3/2}$, the result follows. $\square$

If Ω satisfies the *exterior ball condition*, i.e. there exists $\rho_0 > 0$ such that every point on $\partial\Omega$ can be touched from outside with a ball of radius ρ_0 contained in Ω , we also have the following result:

Lemma 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the exterior ball condition. Let $u \in C(\overline{\Omega})$ be a harmonic function in $\Omega \cap B_2$, with $u = 0$ on $\partial\Omega \cap B_2$. Then, $u \leq c_0 d$ in $\Omega \cap B_1$ for some $c_0 > 0$, where $d(x) := \text{dist}(x, \Omega^c)$.*

Proof. We employ a similar barrier argument as before. $\square$

Remark 1.11. In particular, in nice domains (i.e. those satisfying the interior and exterior ball condition, e.g. if $\partial\Omega \in C^{1,1}$), harmonic functions with $u = 0$ on $\partial\Omega$ behave like linear functions near the boundary, i.e.

$$c_1 d \leq u \leq c_2 d \quad \text{close to } \partial\Omega$$

This property remains true in domains with $\partial\Omega \in C^{1,\alpha}$. However, it is dramatically different in bad domains. For instance,

$$u_1(x) = x_1 x_2 \quad \text{solves } -\Delta u_1 = 0 \quad \text{in } \Omega_1 = \{x_1 x_2 > 0\} \quad \text{with } u_1 = 0 \quad \text{on } \partial\Omega_1,$$

$$u_2(x) = r^{2/3} \sin(2\phi/3) \quad \text{solves } -\Delta u_2 = 0 \quad \text{in } \Omega_2 = \{x_1 < 0 \text{ or } x_2 < 0\} \quad \text{with } u_2 = 0 \quad \text{on } \partial\Omega_2.$$

More generally, for any $\alpha > 0$, the function $u_\alpha(x) = r^\alpha \sin(\alpha\phi)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ and satisfies $u_\alpha = 0$ on $\partial\{(r \cos \phi, r \sin \phi) : \phi \in [0, \pi/\alpha]\}$.

Hence, in free boundary problems (where the boundary of the solution domain is unknown), it is a delicate question to analyze the behavior of the solution close to the boundary.

Remark 1.12. One can prove that solutions to the Dirichlet problem in Ω (1.1) always satisfy $u \in C(\overline{\Omega})$ if Ω satisfies the interior or exterior ball condition.

1.2. The mean value property.

Lemma 1.13. *Let $\Omega \subset \mathbb{R}^n$ be any open set. If $-\Delta u = 0$ in Ω , then*

$$u(x) = \int_{\partial B_r(x)} u(y) \, dy = \int_{B_r(x)} u(y) \, dy \quad \text{for any ball } B_r(x) \subset \Omega. \quad (1.7)$$

Moreover, it holds for any weakly superharmonic (subharmonic) function $u \in H^1(\Omega)$,

$$r \mapsto \int_{B_r(x)} u(y) \, dy \quad \text{is monotone non-increasing (non-decreasing) for } r \in (0, \text{dist}(x, \partial\Omega)). \quad (1.8)$$

The property in (1.7) is called the *mean value property*.

Proof. If u is harmonic, the first equality in the mean value property follows by setting $x_0 = x$ in (1.4). The second equality follows by integrating the first one, namely

$$\int_{B_r(x)} u(y) \, dy = nr^{-n} \int_0^r \rho^{n-1} \int_{B_\rho(x)} u(y) \, dy \, d\rho.$$

The claim for weakly subharmonic functions goes as follows. Fix $0 < \rho < r$ such that $B_r(x) \subset \Omega$. Let v be the solution to $-\Delta v = 0$ in $B_r(x)$ with $v = u$ on $\partial B_r(x)$. Then, by the maximum principle $u \leq v$ in $B_r(x)$. Hence, by the mean value property

$$S(\rho) := \int_{\partial B_\rho(x)} u(y) \, dy \leq \int_{\partial B_\rho(x)} v(y) \, dy = v(x) = \int_{\partial B_r(x)} v(y) \, dy = \int_{\partial B_r(x)} u(y) \, dy = S(r).$$

Then, by integrating over $(0, r)$,

$$A(r) := \int_{B_r(x)} u(y) \, dy = nr^{-n} \int_0^r \rho^{n-1} S(\rho) \, d\rho \leq S(r) nr^{-n} \int_0^r \rho^{n-1} \, d\rho = S(r).$$

However, this yields

$$A'(r) = -n^2 r^{n-1} \int_0^r \rho^{n-1} S(\rho) \, d\rho + nr^{-n} S(r) r^{n-1} = \frac{n}{r} (S(r) - A(r)) \geq 0,$$

as desired. □

The following two lemmas yield the Harnack inequality for harmonic functions.

Lemma 1.14 (Weak Harnack inequality for weak supersolutions). *Let $u \in C(B_1)$. Then,*

$$\begin{cases} -\Delta u \geq 0 & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \end{cases} \implies \inf_{B_{1/2}} u \geq c \|u\|_{L^1(B_{1/2})},$$

for some $c > 0$ depending only on n .

Proof. By the Lebesgue differentiation theorem and (1.8), we have for any $x_0 \in B_{1/3}$

$$u(x_0) \geq \frac{1}{|B_{2/3}|} \int_{B_{2/3}(x_0)} u = c \|u\|_{L^1(B_{2/3}(x_0))} \geq c \|u\|_{L^1(B_{1/3})}$$

for some $c = c(n) > 0$, so that we have proved the property in a ball of radius $1/3$.

To prove it in $B_{1/2}$, consider $\bar{x}_0 \in \partial B_{1/3}$ and the ball $B_{1/6}(\bar{x}_0)$. We can repeat the previous steps to derive

$$\inf_{B_{1/6}(\bar{x}_0)} u \geq c \|u\|_{L^1(B_{1/6}(\bar{x}_0))}.$$

Moreover, if we denote $B := B_{1/3} \cap B_{1/6}(\bar{x}_0)$, then

$$\inf_{B_{1/6}(\bar{x}_0)} u \geq c \|u\|_{L^1(B_{1/6}(\bar{x}_0))} \geq c \int_B u \geq |B| \inf_B u \geq c \inf_{B_{1/3}} u.$$

This implies

$$\inf_{B_{1/2}} u \geq \inf_{B_{1/3}} u \wedge \inf_{x_0 \in \partial B_{1/3}} \inf_{B_{1/6}(\bar{x}_0)} u \geq c \inf_{B_{1/3}} u.$$

Similarly,

$$\|u\|_{L^1(B_{1/2})} \leq \|u\|_{L^1(B_{1/3})} + c \max_{x_0 \in \partial B_{1/3}} \|u\|_{L^1(B_{1/6}(\bar{x}_0))} \leq c \|u\|_{L^1(B_{1/3})}.$$

Altogether, from the first result in this proof, we can conclude

$$\inf_{B_{1/2}} u \geq c_1 \inf_{B_{1/3}} u \geq c_2 \|u\|_{L^1(B_{1/3})} \geq c_3 \|u\|_{L^1(B_{1/2})}$$

for some $c_3 = c_3(n) > 0$. In the last step we have used again (1.8). $\square$

Lemma 1.15 (L^∞ bound for weak subsolutions). *Let $u \in C(B_1)$. Then,*

$$-\Delta u \leq 0 \quad \text{in } B_1 \quad \implies \quad \sup_{B_{1/2}} u \leq C \|u\|_{L^1(B_{3/4})},$$

for some C depending only on n .

We will see later that the L^1 norm in this estimate can be replaced by the L^ε norm for any $\varepsilon > 0$. This follows from Young's inequality and a covering argument.

Proof. The result follows from the the mean value property (1.8) in the same way as Lemma 1.14. $\square$

Theorem 1.16 (Harnack inequality). *Let $u \in C(B_1)$.*

$$\begin{cases} -\Delta u = 0 & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \end{cases} \implies \sup_{B_{1/2}} u \leq c \inf_{B_{1/2}} u,$$

for some $c > 0$ depending only on n .

Proof. Combine Lemma 1.15 and Lemma 1.14. $\square$

Remark 1.17. In particular, we have the following strict maximum principle: If $-\Delta u \geq 0$ in Ω with $u \geq 0$ in Ω and $u \not\equiv 0$, then $u > 0$ in Ω .

We end this subsection with three auxiliary lemmas that all follow from the mean value property and that will be used later in the lecture.

The first lemma says that the pointwise limit of a sequence of superharmonic uniformly bounded functions is superharmonic (in the sense that (1.8) holds).

Lemma 1.18. *Let $\Omega \subset \mathbb{R}^n$, and let $(w_k)_k$ be a sequence of uniformly bounded functions $w_k : \Omega \rightarrow \mathbb{R}$ satisfying (1.8), converging pointwise to some $w : \Omega \rightarrow \mathbb{R}$. Then w satisfies (1.8).*

Proof. The proof is immediate. In fact, let $w_\infty := w$ and let us define for $k \in \mathbb{N} \cup \{\infty\}$, $\phi_{x,k}(r) := \int_{B_r(x)} w_k$. Notice that $\phi_{x,k}(r)$ is non-increasing in r for all $k \in \mathbb{N}$. In particular, given $0 < r_1 < r_2 < R_x$, we have that $\phi_{x,k}(r_1) \geq \phi_{x,k}(r_2)$ for $k \in \mathbb{N}$. Now we let $k \rightarrow \infty$ and use that $w_k \rightarrow w$ pointwise to deduce, by the dominated convergence theorem (notice that w_k are uniformly bounded), that $\phi_{x,\infty}(r_1) \geq \phi_{x,\infty}(r_2)$. That is, $w_\infty = w$ satisfies (1.8). $\square$

The second lemma shows that superharmonic functions are lower semicontinuous.

Lemma 1.19. *Let us assume that $w \in L^1_{loc}(\Omega)$ and satisfies (1.8) in $\Omega \subset \mathbb{R}^n$. Then, up to changing w in a set of measure 0, w is lower semicontinuous in Ω .*

Proof. We define $w_0(x) := \lim_{r \downarrow 0} \fint_{B_r(x)} w$ (which is well defined, since the average is monotone non-increasing). Then $w_0(x) = w(x)$ if x is a Lebesgue point, and thus $w_0 = w$ almost everywhere in Ω . Let us now consider $x_0 \in \Omega$, and let $x_k \rightarrow x_0$ as $k \rightarrow \infty$. Then, by the dominated convergence theorem,

$$\fint_{B_r(x_0)} w = \lim_{k \rightarrow \infty} \fint_{B_r(x_k)} w \leq \liminf_{k \rightarrow \infty} w_0(x_k) \quad (1.9)$$

for $0 < r < \frac{1}{2} \text{dist}(x_0, \partial\Omega)$. Now, by letting $r \downarrow 0$ on the left-hand side, we reach that

$$w_0(x_0) \leq \liminf_{k \rightarrow \infty} w_0(x_k), \quad (1.10)$$

that is, w_0 is lower semi-continuous at x_0 . $\square$

The next result yields a classification of global harmonic functions.

Theorem 1.20 (Liouville's theorem). *Any bounded solution of $\Delta u = 0$ in $\mathbb{R}^n$ is constant.*

Proof. Let u be any global bounded solution of $\Delta u = 0$ in $\mathbb{R}^n$. Since u is smooth (by Corollary 1.5), each derivative $\partial_i u$ is well-defined and is harmonic. Thus, thanks to the mean-value property and the divergence theorem, for any $x \in \mathbb{R}^n$ and $R \geq 1$ we have

$$|\partial_i u(x)| = \left| \frac{c_n}{R^n} \int_{B_R(x)} \partial_i u \right| = \left| \frac{c_n}{R^n} \int_{\partial B_R(x)} u(y) \frac{y_i}{|y|} dy \right| \leq \frac{C}{R^n} \int_{\partial B_R(x)} |u|. \quad (1.11)$$

Thus, using that $|u| \leq M$ in $\mathbb{R}^n$, we find

$$|\partial_i u(x)| \leq \frac{c_n}{R^n} |\partial B_R(x)| M = \frac{c_n}{R^n} |\partial B_1| R^{n-1} M = \frac{c'_n M}{R} \rightarrow 0, \quad \text{as } R \rightarrow \infty. \quad (1.12)$$

Therefore, $\partial_i u(x) = 0$ for all $x \in \mathbb{R}^n$, and u is constant. $\square$

2. THE OBSTACLE PROBLEM

In this chapter, we deal with our first free boundary problem: the obstacle problem.

There is a wide variety of problems in physics, industry, biology, finance, and other areas which can be described by PDEs that exhibit free boundaries. Many of such problems can be written as variational inequalities, for which the solution is obtained by minimizing a constrained energy functional. The obstacle problem is one of the most important and canonical examples.

Given smooth functions $\phi : \Omega \rightarrow \mathbb{R}$ and $g : \partial\Omega \rightarrow \mathbb{R}$, the obstacle problem is the following:

$$\text{minimize } \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \quad \text{among all functions } v \geq \phi \quad \text{in } \Omega \quad \text{with } v = g \quad \text{on } \partial\Omega.$$

- Interpretation: we look for the least energy function v , but the set of admissible functions consists only of functions that are above a certain “obstacle” ϕ .
- in 2D: Think of v as an elastic membrane that is constrained to be above ϕ

- We will see that the Euler-Lagrange equation is given as follows:

$$\begin{cases} v \geq \phi & \text{in } \Omega \\ -\Delta v \geq 0 & \text{in } \Omega \\ -\Delta v = 0 & \text{in the set } \{v > \phi\}, \end{cases}$$

Intuition: Maybe you already know that the unconstrained problem leads to harmonic functions! Hence, if we denote $E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx$, then we will have $E(v + \varepsilon \eta) \geq E(v)$ for every $\varepsilon \geq 0$ and $\eta \geq 0$, $\eta \in C_c^\infty(\Omega)$, which yields $-\Delta v \geq 0$ in Ω . That is, we can perturb v with nonnegative functions ($\varepsilon \eta$) and we always get admissible functions ($v + \varepsilon \eta$). However, due to the constraint $v \geq \phi$, we cannot perturb v with negative functions in all of Ω , but only in the set $\{v > \phi\}$. This is why we get $-\Delta v \geq 0$ everywhere in Ω , but $-\Delta v = 0$ only in $\{v > \phi\}$. (We will show later that any minimizer v is continuous, so that $\{v > \phi\}$ is open.)

- Short form of the Euler-Lagrange equation:

$$\min\{-\Delta v, v - \phi\} = 0 \quad \text{in } \Omega.$$

- Consider $u := v - \phi$. Then, the obstacle problem is equivalent to

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ \Delta u \leq f & \text{in } \Omega \\ \Delta u = f & \text{in the set } \{u > 0\}, \end{cases}$$

where $f := -\Delta \phi$. This way, we can assume without loss of generality that the obstacle is zero.

- The previous problem is the Euler-Lagrange equation associated to the following minimization problem:

$$\text{minimize } \int_{\Omega} \frac{1}{2} |\nabla u|^2 + f u \, dx \quad \text{among all functions } u \geq 0 \quad \text{with } u = g - \phi \quad \text{on } \partial\Omega.$$

- A key feature of the obstacle problem is that it has two unknowns:

the solution u , and the contact set $\{u = 0\}$.

In other words, there are two regions in Ω , characterized by the minimization problem:

one in which $u = 0$, and one in which $-\Delta u = f$.

Moreover, we denote the **free boundary** by

$$\Gamma := \partial\{u > 0\} \cap \Omega,$$

- We will see that since u is a nonnegative supersolution, it will hold $\nabla u = 0$ on Γ , that is, we will have that $u \geq 0$ solves

$$\begin{cases} \Delta u = f & \text{in } \{u > 0\} \\ u = 0 & \text{on } \Gamma \\ \nabla u = 0 & \text{on } \Gamma. \end{cases}$$

This is yet another way to write the Euler Lagrange equation (this time explicitly including the interface Γ).

- We see that we have both Dirichlet and Neumann conditions on Γ . This would usually be an over-determined problem (too many boundary conditions on Γ , recall Lax-Milgram), but since Γ is also free, it turns out that the problem has a unique solution (where Γ is part of the solution).

Some applications of the obstacle problem

- Dam problem,
- Stefan problem,
- Hele-Shaw flow,
- optimal stopping, finance,
- interacting particle systems,
- elasticity

2.1. Well-posedness and the Euler Lagrange equation. Existence and uniqueness of solutions follows easily from the fact that the functional $\int_{\Omega} |\nabla v|^2 dx$ is convex, and that we want to minimize it in the closed convex set $\{v \in H^1(\Omega) : v \geq \phi\}$. The following proof is standard in the calculus of variations

Proposition 2.1 (Existence and uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let $g : \partial\Omega \rightarrow \mathbb{R}$ and $\phi \in H^1(\Omega)$ be such that*

$$\mathcal{C} = \{w \in H^1(\Omega) : w \geq \phi \text{ in } \Omega, \text{Tr } w = g\} \neq \emptyset.$$

Then, there exists a unique minimizer of

$$E(v) := \int_{\Omega} |\nabla v|^2 dx \quad \text{among all } v \in \mathcal{C}. \quad (2.1)$$

Proof. Let us define

$$\theta_0 := \inf \left\{ E(w) := \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx : w \in K \right\},$$

that is, the infimum value of $E(w)$ among all admissible functions $w \in \mathcal{C}$. Let us take a sequence of functions $\{v_k\}$ such that

- (i) $v_k \in H^1(\Omega)$,
- (ii) $\text{Tr } v_k = g$ and $v_k \geq \phi$ in Ω ,
- (iii) $E(v_k) \rightarrow \theta_0$ as $k \rightarrow \infty$.

By (i), $\|v_k\|_{L^2(\Omega)}$ is uniformly bounded, and by the Poincaré inequality,

$$\|v_k\|_{L^2(\Omega)} \leq C \|\nabla v_k\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)},$$

i.e., the sequence $\{v_k\}$ is uniformly bounded in $H^1(\Omega)$. Therefore, a subsequence $\{v_{k_j}\}$ will converge to a certain function v strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$.

Moreover, by compactness of the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$, we will have $\text{Tr } v_{k_j} \rightarrow \text{Tr } v$ in $L^2(\partial\Omega)$, so that $\text{Tr } v = g$.

Furthermore, v satisfies (weak lower semi-continuity of $\|\cdot\|_{H^1(\Omega)}$ and compactness of $H^1(\Omega) \subset L^2(\Omega)$)

$$\|v\|_{H^1(\Omega)} \leq \liminf_{j \rightarrow \infty} \|v_j\|_{H^1(\Omega)}, \quad \|v\|_{L^2(\Omega)} = \lim_{j \rightarrow \infty} \|v_j\|_{L^2(\Omega)},$$

and therefore,

$$E(v) = \frac{1}{2}[v]_{H^1(\Omega)} \leq \frac{1}{2} \liminf_{j \rightarrow \infty} [v_j]_{H^1(\Omega)} = \liminf_{j \rightarrow \infty} E(v_{k_j}).$$

Hence, v is a minimizer of the energy functional. Since $v_{k_j} \geq \phi$ in Ω and $v_{k_j} \rightarrow v$ in $L^2(\Omega)$, we have $v \geq \phi$ in Ω . Thus, we have proved the existence of a minimizer v .

The uniqueness of the minimizer follows from the strict convexity of the functional $E(v)$, as follows: First, observe that the set $\mathcal{C}$ is convex, i.e. if $u, v \in \mathcal{C}$ are both minimizers, then for $t \in (0, 1)$, we have

$$w_t := tu + (1 - t)v \in \mathcal{C}.$$

By minimality of u and v ,

$$E(u) = E(v) \leq E(w_t). \quad (2.2)$$

On the other hand, for the gradients we have the identity

$$\begin{aligned} |\nabla w_t|^2 &= t^2 |\nabla u|^2 + (1 - t)^2 |\nabla v|^2 + 2t(1 - t) \nabla u \cdot \nabla v \\ &= t^2 |\nabla u|^2 + (1 - t)^2 |\nabla v|^2 - t(1 - t) (|\nabla u - \nabla v|^2 - |\nabla u|^2 - |\nabla v|^2) \\ &= t |\nabla u|^2 + (1 - t) |\nabla v|^2 - t(1 - t) |\nabla u - \nabla v|^2. \end{aligned}$$

Integrating over Ω yields

$$E(w_t) = tE(u) + (1 - t)E(v) - \frac{1}{2}t(1 - t) \int_{\Omega} |\nabla u - \nabla v|^2 dx.$$

Since $E(u) = E(v)$, this simplifies to

$$E(w_t) = E(u) - \frac{1}{2}t(1 - t) \int_{\Omega} |\nabla u - \nabla v|^2 dx \leq E(u). \quad (2.3)$$

Combining (2.2) and (2.3) gives equality, and therefore it must be,

$$\int_{\Omega} |\nabla u - \nabla v|^2 dx = 0. \quad (2.4)$$

Therefore $\nabla u = \nabla v$ a.e. in Ω , so $u - v$ is constant a.e. Since $u - v = 0$ on $\partial\Omega$, the constant must be zero. Hence $u = v$. $\square$

From now on, we will always assume that $\phi \in C^\infty(\overline{\Omega})$ for simplicity. One gets analogous results under much weaker regularity assumptions on ϕ , but the proofs might be more technical.

Our goal is to derive the Euler-Lagrange equation for minimizers v of (2.1).

We start with the following lemma.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in C^\infty(\overline{\Omega})$, and $v \in H^1(\Omega)$ be any minimizer of (2.1). Then, $-\Delta v \geq 0$ in Ω .*

Proof. Since v minimizes E among all functions above the obstacle ϕ (and with fixed boundary conditions on $\partial\Omega$), we have that

$$E(v + \varepsilon\eta) \geq E(v) \quad \text{for every } \varepsilon \geq 0 \text{ and } \eta \geq 0, \eta \in C_c^\infty(\Omega).$$

This yields

$$\varepsilon \int_{\Omega} \nabla v \cdot \nabla \eta + \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla \eta|^2 dx \geq 0 \quad \text{for every } \varepsilon \geq 0 \text{ and } \eta \geq 0, \eta \in C_c^\infty(\Omega),$$

and thus

$$\int_{\Omega} \nabla v \cdot \nabla \eta \geq 0 \quad \text{for every } \eta \geq 0, \eta \in C_c^\infty(\Omega).$$

This means that $-\Delta v \geq 0$ in Ω in the weak sense, as desired. $\square$

From here, by showing first that $\{v > \phi\}$ is open, we obtain the Euler-Lagrange equations for the functional:

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in C^\infty(\bar{\Omega})$, and $v \in H^1(\Omega)$ be any minimizer of (2.1). Then, $v \in C_{loc}(\Omega)$ and it holds*

$$\begin{cases} v & \geq \phi & \text{in } \Omega \\ -\Delta v & \geq 0 & \text{in } \Omega \\ \Delta v & = 0 & \text{in } \{v > \phi\} \cap \Omega. \end{cases} \quad (2.5)$$

Proof. By construction, we already know that $v \geq \phi$ in Ω and, thanks to Lemma 2.2, $-\Delta v \geq 0$ in Ω , i.e., v is (weakly) superharmonic. Up to replacing v in a set of measure zero, we may also assume that v is lower semi-continuous (by Lemma 1.19). Thus, we only need to prove that $\Delta v = 0$ in $\{v > \phi\} \cap \Omega$ and that v is continuous.

First, we show that $\{v > \phi\} \cap \Omega$ is open. Let $x_0 \in \{v > \phi\} \cap \Omega$ be such that $v(x_0) - \phi(x_0) > \varepsilon_0 > 0$. Since v is lower semi-continuous and ϕ is continuous, there exists some $\delta > 0$ such that

$$v(x) - \phi(x) > \varepsilon_0/2 \quad \forall x \in B_\delta(x_0).$$

Hence $B_\delta(x_0) \subset \{v > \phi\}$. Since x_0 was arbitrary, this means that $\{v > \phi\}$ is open.

This implies, also, that $\Delta v = 0$ weakly in $\{v > \phi\} \cap \Omega$. Indeed, for any $x_0 \in \{v > \phi\}$ and $\eta \in C_c^\infty(B_\delta(x_0))$ with $|\eta| \leq 1$, we have $v \pm \varepsilon \eta \geq \phi$ in Ω for all $|\varepsilon| < \varepsilon_0/2$, and therefore it is an admissible competitor. Thus, we have

$$E(v + \varepsilon \eta) \geq E(v) \quad \forall |\varepsilon| < \varepsilon_0.$$

In particular, the map $\varepsilon \rightarrow E(v + \varepsilon \eta)$ has a critical point at $\varepsilon = 0$, i.e.

$$\frac{d}{d\varepsilon} E(v + \varepsilon \eta)|_{\varepsilon=0} = 0.$$

Equivalently,

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_{\Omega} |\nabla(v + \varepsilon \eta)|^2 dx \\ &= \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_{\Omega} |\nabla v|^2 + \varepsilon^2 |\nabla \eta|^2 + 2\varepsilon \nabla v \nabla \eta dx \\ &= 2 \int_{\Omega} \nabla v \nabla \eta dx, \end{aligned}$$

i.e. v is weakly harmonic in $B_\delta(x_0)$. Hence, we deduce that v is harmonic in $\{v > \phi\} \cap \Omega$.

Finally, let us show that v is continuous. We already know, by the regularity of harmonic functions (see Corollary 1.5), that v is continuous in $\{v > \phi\} \cap \Omega$. Let us now show that v is continuous in $\{v = \phi\} \cap \Omega$, as well.

Let $y_0 \in \{v = \phi\} \cap \Omega$, and let us argue by contradiction. Since v is lower semi-continuous, it suffices to assume that there is a sequence $y_k \rightarrow y_0$ such that

$$v(y_k) \rightarrow v(y_0) + \varepsilon_0 = \phi(y_0) + \varepsilon_0$$

for some $\varepsilon_0 > 0$.

Since ϕ is continuous, we may assume also that $y_k \in \{v > \phi\}$. Let us denote by z_k the projection of y_k towards $\{v = \phi\}$, so that $\delta_k := |z_k - y_k| \leq |y_0 - y_k| \downarrow 0$ and

$$v(z_k) \rightarrow v(y_0) = \phi(y_0). \quad (2.6)$$

Now, since v is superharmonic, by (1.8),

$$v(z_k) \geq \int_{B_{2\delta_k}(z_k)} v = (1 - 2^{-n}) \int_{B_{2\delta_k}(z_k) \setminus B_{\delta_k}(y_k)} v + 2^{-n} \int_{B_{\delta_k}(y_k)} v = I_1 + I_2.$$

For the first equality, we used that $B_{\delta_k}(y_k) \subset B_{2\delta_k}(z_k)$. Observe that, for I_1 , since v is lower semi-continuous and $\delta_k \downarrow 0$, we can assume that, for k large enough, $v \geq \phi(y_0) - 2^{-n}\varepsilon_0$ in $B_{2\delta_k}(z_k)$, so that

$$I_1 \geq (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0].$$

On the other hand, since v is harmonic in $B_{\delta_k}(y_k)$, we have by the mean-value property that

$$I_2 = 2^{-n}v(y_k).$$

Combining everything, we get

$$v(z_k) \geq (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0] + 2^{-n}v(y_k) \rightarrow \phi(y_0) + 2^{-2n}\varepsilon_0,$$

which contradicts (2.6). Hence, v is continuous in Ω . $\square$

Remark 2.4. As in the case of harmonic functions, it is easy to show that if a function v satisfies

$$\begin{cases} v \geq \phi & \text{in } \Omega, \\ \Delta v \leq 0 & \text{in } \Omega, \\ \Delta v = 0 & \text{in the set } \{v > \phi\}, \end{cases}$$

then it must actually be a minimizer of (2.1).

We next prove the following result, which says that v can be characterized as the least supersolution above the obstacle.

Proposition 2.5 (Least supersolution). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\phi \in H^1(\Omega)$, and $v \in H^1(\Omega)$ be any minimizer of (2.1). Then, for any function w satisfying $-\Delta w \geq 0$ in Ω , $w \geq \phi$ in Ω , and $\text{Tr } w \geq \text{Tr } v$, we have $w \geq v$ in Ω . In other words, if w is any supersolution above the obstacle ϕ , then $w \geq v$.*

Proof. If w is any function satisfying $-\Delta w \geq 0$ in Ω , $w \geq \phi$ in Ω , and $\text{Tr } w \geq \text{Tr } v$, it simply follows from the maximum principle that $w \geq v$. Indeed, we have $-\Delta w \geq -\Delta v$ in $\Omega \cap \{v > \phi\}$, and on the boundary of Ω we have $\text{Tr } w \geq \text{Tr } v$ and $w \geq \phi = v$ on $\{v = \phi\}$. $\square$

2.2. Optimal regularity of solutions. Thanks to Proposition 2.3, we know that any minimizer of (2.1) is continuous and solves (2.5).

From now on, we will restrict our study to solutions of the Euler Lagrange equation without any boundary conditions on $\partial\Omega$. This means, we localize the problem and study it in a ball:

For $\phi \in C^\infty(B_1)$, we consider

$$\begin{cases} v & \geq \phi & \text{in } B_1, \\ -\Delta v & \geq 0 & \text{in } B_1, \\ -\Delta v & = 0 & \text{in } \{v > \phi\} \cap B_1. \end{cases} \quad (2.7)$$

Our next goal is to answer the following question:

Question: What is the optimal regularity of solutions?

Remark 2.6. Notice that in the set $\{v > \phi\}$ we have $\Delta v = 0$, while in the interior of the set $\{v = \phi\}$ we have $\Delta v = \Delta \phi$ (since $v = \phi$ there). Thus, since $\Delta \phi$ is in general not zero, Δv is discontinuous across the free boundary $\partial\{v > \phi\}$ in general. In particular, $v \notin C^2$.

Example: in 1D, consider $v(x) = -x_+^2$, which solves (2.7) in $(-1, 1)$ with $\phi = -x^2$.

We will now prove that any minimizer of (2.1) is actually $C^{1,1}$, which by the previous remark is the optimal regularity.

Theorem 2.7 (Optimal regularity). *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.7). Then, v is $C^{1,1}(B_{1/2})$, with the estimate*

$$\|v\|_{C^{1,1}(B_{1/2})} \leq C\|v\|_{L^\infty(B_{3/4})} + \|\phi\|_{C^{1,1}(B_{3/4})}.$$

The constant C depends only on n .

To prove this, the main step is the following lemma, which establishes that solutions detach at most quadratically from the free boundary.

Lemma 2.8. *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.7). Let $x_0 \in B_{1/2}$ be any point on $\{v = \phi\}$. Then, for any $r \in (0, 1/4)$ we have*

$$0 \leq \sup_{B_r(x_0)} (v - \phi) \leq C\|\phi\|_{C^{1,1}(B_{3/4})}r^2,$$

with C depending only on n .

In particular, Lemma 2.8 implies that $v \in L^\infty(B_{3/4})$.

Proof. After dividing v by a constant if necessary, we may assume that $\|\phi\|_{C^{1,1}(B_1)} \leq 1$. Let

$$\ell(x) := \phi(x_0) + \nabla \phi(x_0) \cdot (x - x_0)$$

be the linear part of ϕ at x_0 . Let $r \in (0, 1/4)$. Then, by the $C^{1,1}$ regularity of ϕ , in $B_r(x_0)$ we have

$$\ell(x) - r^2 \leq \phi(x) \leq v(x). \quad (2.8)$$

Next, we consider

$$w(x) := v(x) - \ell(x) + r^2.$$

Our goal is to show that in the ball $B_r(x_0)$, we have

$$w \leq Cr^2.$$

This function w satisfies $w \geq 0$ in $B_r(x_0)$ by (2.8), and $-\Delta w = -\Delta v \geq 0$ in $B_r(x_0)$. Let us split w into $w = w_1 + w_2$, with

$$\begin{cases} -\Delta w_1 = 0 & \text{in } B_r(x_0) \\ w_1 = w & \text{on } \partial B_r(x_0) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_2 \geq 0 & \text{in } B_r(x_0) \\ w_2 = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Notice that by the maximum principle, $0 \leq w_1 \leq w$ and $0 \leq w_2$, and hence $0 \leq w_2 \leq w$.

Moreover, note that

$$w_1(x_0) \leq w(x_0) = v(x_0) - \ell(x_0) + r^2 = r^2,$$

and thus by the Harnack inequality (see Theorem 1.16),

$$\|w_1\|_{L^\infty(B_{r/2}(x_0))} \leq Cr^2.$$

For w_2 , notice that $-\Delta w_2 = -\Delta v$, and in particular $-\Delta w_2 = 0$ in $\{v > \phi\}$. This means that w_2 attains its maximum on $\{v = \phi\}$. But in the set $\{v = \phi\}$ we have

$$w_2 \leq w = \phi - \ell + r^2 \leq Cr^2,$$

and therefore we deduce that

$$\|w_2\|_{L^\infty(B_r(x_0))} \leq Cr^2.$$

Combining the bounds for w_1 and w_2 , we get

$$\|w\|_{L^\infty(B_r(x_0))} \leq Cr^2,$$

as desired. Recalling the definition of w , and using that $\|\phi\|_{C^{1,1}(B_1)} \leq 1$, we find by (2.8),

$$v - \phi = w + \ell - \phi + r^2 \leq Cr^2 \quad \text{in } B_{r/2}(x_0),$$

as desired. $\square$

As shown next, the previous lemma easily implies the $C^{1,1}$ regularity.

Proof of Theorem 2.7. Dividing v by a constant if necessary, we may assume that

$$\|v\|_{L^\infty(B_{3/4})} + \|\phi\|_{C^{1,1}(B_{3/4})} \leq 1.$$

We already know that $v \in C_{\text{loc}}^\infty(\{v > \phi\})$, since v is harmonic there. Moreover, v is $C^\infty(\{v = \phi\})$, since $\phi \in C^\infty$. Hence, it remains to show smoothness of v across the interface $\Gamma = \partial\{v > \phi\}$. For this, we will use Lemma 2.8.

Let $x_1 \in \{v > \phi\} \cap B_{1/2}$, and let $x_0 \in \Gamma$ be the closest free boundary point. Denote $\rho = |x_1 - x_0|$. Then, we have $-\Delta v = 0$ in $B_\rho(x_1)$, and thus we have also $-\Delta(v - \ell) = 0$ in $B_\rho(x_1)$, where ℓ is the linear part of ϕ at x_0 . By estimates for harmonic functions (see Corollary 1.5), the quadratic growth from Lemma 2.8, and since $\phi \in C^{1,1}$ (arguing as in (2.8)), we find

$$\begin{aligned} \|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} &= \|D^2(v - \ell)\|_{L^\infty(B_{\rho/2}(x_1))} \leq \frac{C}{\rho^2} \|v - \ell\|_{L^\infty(B_\rho(x_1))} \\ &\leq \frac{C}{\rho^2} \|v - \phi\|_{L^\infty(B_\rho(x_1))} + \frac{C\rho^2}{\rho^2} \leq \frac{C\rho^2}{\rho^2} = C. \end{aligned}$$

[The factor ρ^{-2} in the second step comes from rescaling Corollary 1.5, i.e. applying it to $v_\rho(x) := v(\rho x)$ and using that $\|D^2 v\|_{L^\infty(B_{\rho/2})} = \rho^{-2} \|D^2 v_\rho\|_{L^\infty(B_{1/2})}$.]

In particular, $|D^2 v(x_1)| \leq C$. We can do this for all $x_1 \in \{v > \phi\} \cap B_{1/2}$. Moreover, for $x_1 \in \partial\{v > \phi\}$, we deduce $|D^2 v(x_1)| \leq C$ from Lemma 2.8. Altogether, it follows $\|v\|_{C^{1,1}(B_{1/2})} \leq C$, as desired. $\square$

2.3. Nondegeneracy. Next, we want to prove that, at all free boundary points, v separates from ϕ at least quadratically (we already know at most quadratically). That is, we want

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \tag{2.9}$$

for all free boundary points $x_0 \in \partial\{v > \phi\}$. This property is essential in order to study the free boundary later.

We will prove it under an additional assumption:

Assumption: The obstacle ϕ satisfies

$$-\Delta\phi \geq c_0 > 0 \quad \text{in } B_1. \quad (2.10)$$

Remark 2.9. The assumption (2.10) is quite mild.

- Since $-\Delta v \geq 0$ everywhere, it is clear that if $x_0 \in \partial\{v > \phi\}$, then $-\Delta\phi(x_0) \geq 0$.
In fact, if $-\Delta\phi(x_0) < 0$, then, since v touches ϕ from above at x_0 , the function $v - \phi$ has a global minimum there, i.e. $(-\Delta)(v - \phi) \leq 0$, i.e. $-\Delta v(x_0) < 0$, a contradiction).
- It can be proved that, in fact, if $\Delta\phi$ and $\nabla\Delta\phi$ do not vanish simultaneously, then $-\Delta\phi > 0$ near all free boundary points [Caf98].
- The assumption (2.10) is somewhat necessary. Without it, the lower bound in (2.9) actually fails and one can construct counterexamples in which the free boundary is a fractal set with infinite perimeter (see [Caf98]).
Idea: Just choose $u = 0$ and note that given any fractal set, we can find ϕ such that $\{\phi = 0\}$ is this set. Then, $u = 0$ solves the obstacle problem with obstacle ϕ .

Proposition 2.10 (Nondegeneracy). *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.7). Assume that ϕ satisfies $-\Delta\phi \geq c_0 > 0$ in B_1 . Then, for every free boundary point $x_0 \in \partial\{v > \phi\} \cap B_{1/2}$, we have*

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \quad \text{for all } r \in (0, 1/4),$$

with a constant $c > 0$ depending only on n and c_0 .

Proof. Let $x_1 \in \{v > \phi\}$ be any point close to x_0 (we will let $x_1 \rightarrow x_0$ at the end of the proof). Consider the function [we will see that the r^2 essentially comes from the fact that $\Delta(|x - x_1|^2) = 2n$.]

$$w(x) := v(x) - \phi(x) - \frac{c_0}{2n}|x - x_1|^2.$$

Then, in $\{v > \phi\} \cap B_r(x_1)$, we have

$$-\Delta w = -\Delta v + \Delta\phi + c_0 = \Delta\phi + c_0 \leq 0,$$

Moreover, $w(x_1) > 0$. Hence, by the maximum principle, w attains a positive maximum on $\partial(\{v > \phi\} \cap B_r(x_1))$. But on the free boundary $\partial\{v > \phi\}$ we clearly have $w < 0$. Therefore, there is a point on $\partial B_r(x_1)$ at which $w > 0$. In other words,

$$0 < \sup_{\partial B_r(x_1)} w = \sup_{\partial B_r(x_1)} (v - \phi) - \frac{c_0}{2n}r^2.$$

Letting now $x_1 \rightarrow x_0$, we find $\sup_{\partial B_r(x_0)} (v - \phi) \geq cr^2 > 0$, as desired. $\square$

Remark 2.11. Note that we have used the fact that $-\Delta v \geq 0$ in B_1 only for continuity of v in the proof of the nondegeneracy!

This ends the study of basic properties of the obstacle problem. Before we continue, let us quickly summarize:

Summary of basic properties. Let $\phi \in C^\infty(B_1)$ and v be any solution to the obstacle problem

$$\begin{cases} v \geq \phi & \text{in } B_1 \\ -\Delta v \geq 0 & \text{in } B_1 \\ \Delta v = 0 & \text{in } \{v > \phi\} \cap B_1. \end{cases}$$

Then, we have:

- **Optimal regularity:** $\|v\|_{C^{1,1}(B_{1/2})} \leq C(\|v\|_{L^\infty(B_1)} + \|\phi\|_{C^{1,1}(B_1)}).$

- **Quadratic growth:** If $-\Delta\phi \geq c_0 > 0$, then

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \quad \text{for all } r \in (0, 1/2)$$

at all free boundary points $x_0 \in \partial\{v > \phi\} \cap B_{1/2}$.

2.4. An alternative way to formulate the obstacle problem. Recall the obstacle problem (2.7) problem

$$\begin{cases} v \geq \phi & \text{in } B_1, \\ \Delta v \leq 0 & \text{in } B_1, \\ \Delta v = 0 & \text{in } \{v > \phi\} \cap B_1 \end{cases}$$

for some $\phi \in C^\infty(B_1)$ with $-\Delta\phi \geq c_0 > 0$. Clearly, this problem is equivalent to

$$\begin{cases} u \geq 0 & \text{in } B_1, \\ \Delta u \leq f & \text{in } B_1, \\ \Delta u = f & \text{in } \{u > 0\} \cap B_1, \end{cases} \quad (2.11)$$

where $f = -\Delta\phi \geq c_0 > 0$.

Let us quickly explain that this problem arises as the Euler-Lagrange equation of an alternative energy functional, without going into too much detail.

Proposition 2.12 (An alternative energy functional). *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \rightarrow \mathbb{R}$ be such that*

$$\mathcal{C} = \{u \in H^1(\Omega) : u \geq 0 \text{ in } \Omega, u|_{\partial\Omega} = g\} \neq \emptyset.$$

Then, for any $f \in L^2(\Omega)$ with $f \geq 0$ there exists a unique minimizer of

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} f u \quad (2.12)$$

among all functions $u \in \mathcal{C}$.

Moreover, the following are equivalent.

- (i) *u minimizes $\frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} f u$ among all functions satisfying $u \geq 0$ in Ω and $\text{Tr } u = g$.*
- (ii) *u minimizes $\frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} f u^+$ among all functions satisfying $\text{Tr } u = g$.*

Proof. We skip the proof of the existence and uniqueness. The equivalence of (i) and (ii) follows once we show that minimizers to (ii) are nonnegative. (Note that $\mathcal{C} \neq \emptyset$ implies that $g \geq 0$ on $\partial\Omega$.)

To show this, recall that $|\nabla u|^2 = |\nabla u^+|^2 + |\nabla u^-|^2$, and therefore, since $f \geq 0$ in Ω ,

$$\frac{1}{2} \int_{\Omega} |\nabla u^+|^2 + \int_{\Omega} f u^+ \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} f u^+,$$

with strict inequality unless $u = u^+$. Hence, any minimizer u of the functional in (ii) must be nonnegative. $\square$

The equivalence of (i) and (ii) will help us understand the connection between the obstacle problem and the Alt-Caffarelli free boundary problem later.

The Euler-Lagrange equation associated to (2.12) is given as follows:

Proposition 2.13. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $f \in C^\infty(\Omega)$, and $u \in H^1(\Omega)$ be any minimizer of (2.12) subject to the boundary conditions $\text{Tr } u = g$. Then, u solves*

$$\begin{cases} \Delta u &= f\chi_{\{u>0\}} & \text{in } \Omega, \\ u &\geq 0 & \text{in } \Omega \end{cases}$$

in the weak sense.

Proof. Notice that, by Proposition 2.12, u is actually a minimizer of

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} f u^+$$

subject to the boundary conditions $\text{Tr } u = g$. Hence, for any $\eta \in H_0^1(\Omega)$ and $\varepsilon > 0$ we have

$$E(u + \varepsilon \eta) \geq E(u).$$

In particular, we obtain

$$0 \leq \lim_{\varepsilon \downarrow 0} \frac{E(u + \varepsilon \eta) - E(u)}{\varepsilon} = \int_{\Omega} \nabla u \cdot \nabla \eta + \lim_{\varepsilon \downarrow 0} \int_{\Omega} f \frac{(u + \varepsilon \eta)^+ - u^+}{\varepsilon}.$$

Notice that

$$\lim_{\varepsilon \downarrow 0} \frac{(u + \varepsilon \eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in } \{u > 0\}, \\ \eta^+ & \text{in } \{u = 0\}, \end{cases}$$

so that we have

$$\int_{\Omega} \nabla u \cdot \nabla \eta + \int_{\Omega} f \eta \chi_{\{u>0\}} + \int_{\Omega} f \eta^+ \chi_{\{u=0\}} \geq 0 \quad \text{for all } \eta \in H_0^1(\Omega).$$

Assume first that $\eta \geq 0$, so that

$$\int_{\Omega} \nabla u \cdot \nabla \eta + \int_{\Omega} f \eta \geq 0 \quad \text{for all } \eta \in H_0^1(\Omega), \eta \geq 0,$$

which implies that $\Delta u \leq f$ in the weak sense. On the other hand, if $\eta \leq 0$, then

$$\int_{\Omega} \nabla u \cdot \nabla \eta + \int_{\Omega} f \eta \chi_{\{u>0\}} \geq 0 \quad \text{for all } \eta \in H_0^1(\Omega), \eta \leq 0,$$

which implies that $\Delta u \geq f\chi_{\{u>0\}}$ in the weak sense. Hence, (recall that $f \geq 0$),

$$f\chi_{\{u>0\}} \leq \Delta u \leq f \quad \text{in } \Omega.$$

In particular, notice that $\Delta u = f$ in $\{u > 0\}$.

Now, since f is smooth, this implies that $\Delta u \in L_{\text{loc}}^\infty(\Omega)$. One can show (elliptic regularity theory and Calderón-Zygmund estimates) that this implies $u \in C_{\text{loc}}^{1,1-\varepsilon}(\Omega) \cap W_{\text{loc}}^{2,2}(\Omega)$. Thus, $\Delta u = 0$ almost everywhere in the level set $\{u = 0\}$ and we have

$$\Delta u = f\chi_{\{u>0\}} \quad \text{a.e. in } \Omega.$$

From here, one can easily deduce that $\Delta u = f\chi_{\{u>0\}}$ in Ω in the weak sense. $\square$

As we mentioned before, the formulation of the obstacle problem (2.12) is equivalent to the one from (2.1). One can also deduce the $C^{1,1}$ regularity and nondegeneracy from the Euler-Lagrange equation in Proposition 2.13. This is a little shorter, however, more complicated tools like Schauder theory and the Harnack inequality for equations of the form $-\Delta u = f$ have to be used. For more details see [FRRO22].

Summary of basic properties. Let $f \in C^\infty(B_1)$ and u be any solution to the obstacle problem

$$\begin{cases} u \geq 0 & \text{in } B_1, \\ \Delta u = f \chi_{\{u>0\}} & \text{in } B_1. \end{cases}$$

Then, we have:

- **Optimal regularity:** $\|u\|_{C^{1,1}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,1}(B_1)})$.
- **Quadratic growth:** If $f \geq c_0 > 0$, then

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \quad \text{for all } r \in (0, 1/2)$$

at all free boundary points $x_0 \in \partial\{u > 0\} \cap B_{1/2}$.

2.5. Regularity of free boundaries: an overview. The next goal of this chapter is to understand properties of the free boundary in the obstacle problem.

We will from now on consider solutions to

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \text{ in } B_1, \\ \Delta u = f \text{ in } \{u > 0\} \cap B_1, \end{cases} \quad (2.13)$$

with

$$f \geq c_0 > 0 \quad \text{and} \quad f \in C^\infty(B_1).$$

Note that all of these properties are in particular satisfied by solutions to the obstacle problem, as we have seen before.

Remark 2.14. Several remarks are in order:

- Note that on the interface

$$\Gamma = \partial\{u > 0\} \cap B_1,$$

since $u \in C^{1,1}$ and $u \geq 0$, we have that

$$u = 0 \text{ on } \Gamma, \quad \nabla u = 0 \text{ on } \Gamma.$$

(if $\nabla u \neq 0$ on Γ , there would be a sign change).

- Due to Remark 2.11, the nondegeneracy from Proposition 2.10 still holds true. Hence, under (2.13), we still have for some $0 < c < C$ (now with C depending on $\|u\|_{C^{1,1}(B_1)}$),

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \quad \forall x_0 \in \partial\{u > 0\}. \quad (2.14)$$

- Since $u \in C^{1,1}$, we have that $\Delta u \in L^\infty$, i.e. it holds $\Delta u = f$ a.e. in $\{u > 0\} \cap B_1$. Moreover, since $u \in C^{1,1}$, we have that $\nabla u \in H^1$, it holds that $\Delta u = 0$ a.e. on $\{\nabla u = 0\} \supset \{u = 0\}$ (It is a general fact that derivatives of an H^1 function v vanish a.e. on $\{v = 0\}$, and it follows from the fact that $\nabla v = \nabla v_+ - \nabla v_-$ a.e.). From here, we can deduce that for any $\eta \in C_c^\infty(B)$ and $B \Subset B_1$,

$$\int_B \nabla u \nabla \eta = - \int_B \Delta u \eta + \int_{\partial B} \partial_\nu u \eta = - \int_B f \chi_{\{u>0\}} \eta \, dx,$$

i.e. u solves in the weak sense

$$\Delta u = f \chi_{\{u>0\}} \quad \text{in } B_1.$$

For simplicity, we will assume from now on that

$$f \equiv 1,$$

i.e. we will consider solutions u to

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \text{ in } B_1, \\ \Delta u = 1 \text{ in } \{u > 0\} \cap B_1, \end{cases} \quad (2.15)$$

It is also possible to study the problem with a general $f \in C^\infty$, but it is more technically involved.

The central mathematical challenge in the obstacle problem is to understand the geometry/regularity of the free boundary Γ . Clearly, despite knowing that $u \in C^{1,1}$, Γ could still be a very irregular object, even a fractal set with infinite perimeter.

Our goal will be to prove Caffarelli's dichotomy, which splits the free boundary Γ into a set of **regular points** and a set of **singular points**. We will show that

- (i) Γ is C^∞ near regular points
- (ii) Characterize the set of singular points and prove that they are contained in an $(n-1)$ -dimensional C^1 manifold.

These are the main and most important result in the obstacle problem. (i) was proved by Caffarelli in 1977 (see [Caf77]), and it is one of the major results for which he received the Wolf Prize in 2012, the Shaw Prize in 2018, and the Abel Prize in 2023.

Definition 2.15 (blow-up). We say that u_0 is a blow-up of u (satisfying (2.15)) at $x_0 \in \partial\{u > 0\} \cap B_1$, if there is a sequence $r_k \searrow 0$ such that

$$u_{r_k, x_0}(x) := \frac{u(x_0 + r_k x)}{r_k^2}$$

satisfies

$$u_{r_k} \rightarrow u_0 \quad \text{in } C_{loc}^1(\mathbb{R}^n).$$

If $x_0 = 0$, we denote $u_{r_k, x_0} = u_{r_k}$.

Clearly, blow-ups always exist by Arzelà-Ascoli's theorem and the $C^{1,1}$ regularity of u . Moreover, it is not difficult to see that they are global solutions to the obstacle problem (2.15).

Overview of the strategy.

- Given any free boundary point x_0 , one considers the rescalings u_{r_k, x_0} ("zooming in" at a free boundary point).
- By $C^{1,1}$ estimates, a subsequence of $u_{r_k} \rightarrow u_0$ (blow-up) in $C_{loc}^1(\mathbb{R}^n)$ as $r_k \rightarrow 0$.
- Main issue: **classify blow-ups**:
 - either $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ (regular points)
 - or $u_0(x) = \frac{1}{2}x^T A x$ (singular points).

Here, $e \in \mathbb{S}^{n-1}$ and $A \geq 0$ is a positive semi-definite matrix satisfying $\text{tr} A = 1$.

- transfer information from u_0 to u :

- free boundary is $C^{1,\alpha}$ near regular points (for some small $\alpha > 0$).
- $C^{1,\alpha}$ implies C^∞ (reminiscent of Hilbert's XIX problem).

2.6. Classification of blow-ups. The aim of this section is to classify all possible blow-ups u_0 . For this, we proceed in three steps:

- prove that blow-ups are 2-homogeneous, i.e. $u_0(\lambda x) = \lambda^2 u_0(x)$ for all $\lambda \geq 0$.
- prove that blow-ups are convex, i.e. $D^2 u_0 \geq 0$.
- complete classification of blow-ups

Proposition 2.16 (Homogeneity of blow-ups). *Let u be any solution to (2.15) with $0 \in \partial\{u > 0\}$. Then, any blow-up of u at 0 is 2-homogeneous.*

Remark 2.17. Note that not all global solutions to the obstacle problem in $\mathbb{R}^n$ are homogeneous. There exist global solutions u_0 that are convex, $C^{1,1}$, and whose contact set $\{u_0 = 0\}$ is an ellipsoid. In fact, it was shown recently in [EFW25] (it was a conjecture for more than 90 years) that the coincidence set of a global solution with non-empty interior has to be either a half-space, an ellipsoid, a paraboloid, or a cylinder with an ellipsoid or paraboloid as base.

The result Proposition 2.16 says that such non-homogeneous solutions cannot appear as blow-ups.

Our proof uses a very important tool in the theory of free boundaries, namely a monotonicity formula.

Theorem 2.18 (Weiss' monotonicity formula). *Let u be any solution to (2.15) with $0 \in \partial\{u > 0\}$. Then, the quantity*

$$W_u(r) := \frac{1}{r^{n+2}} \int_{B_r} \left(\frac{1}{2} |\nabla u|^2 + u \right) - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 \quad (2.16)$$

is monotone in r , i.e.

$$\frac{d}{dr} W_u(r) = \frac{1}{r^{n+4}} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2 dx \geq 0 \quad \forall r \in (0, 1).$$

Proof. Let $u_r(x) = r^{-2} u(rx)$, and observe that by scaling

$$W_u(r) = \int_{B_1} \left(\frac{1}{2} |\nabla u_r|^2 + u_r \right) - \int_{\partial B_1} u_r^2. \quad (2.17)$$

Using this, together with $\frac{d}{dr}(\nabla u_r) = \nabla \frac{d}{dr} u_r$, we find

$$\frac{d}{dr} W_u(r) = \int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r.$$

Now, integrating by parts we get

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = - \int_{B_1} \Delta u_r \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu(u_r) \frac{d}{dr} u_r.$$

Now, note that

$$\frac{d}{dr} u_r = -2r^{-3} u(rx) + r^{-2} x \cdot \nabla u(rx) = \frac{1}{r} \{x \cdot \nabla u_r - 2u_r\}. \quad (2.18)$$

Thus, $\frac{d}{dr} u_r = 0$ in $\{u_r = 0\}$ (recall that $\nabla u_r = u_r = 0$ on $\{u_r = 0\}$ by Remark 2.14). Moreover, since $\Delta u_r = 1$ in $\{u_r > 0\}$, we have

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = - \int_{B_1} \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu(u_r) \frac{d}{dr} u_r.$$

Thus, we deduce, using also that $\partial_\nu = x \cdot \nabla$ on ∂B_1 together with (2.18)

$$\begin{aligned} \frac{d}{dr} W_u(r) &= \int_{\partial B_1} \partial_\nu(u_r) \frac{d}{dr} u_r - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r \\ &= \int_{\partial B_1} x \cdot \nabla u_r r^{-1} \{x \cdot \nabla u_r - 2u_r\} - 2 \int_{\partial B_1} u_r r^{-1} \{x \cdot \nabla u_r - 2u_r\} \\ &= \frac{1}{r} \int_{\partial B_1} (x \cdot \nabla u_r - 2u_r)^2, \end{aligned}$$

which gives the desired result after scaling back from u_r to u . $\square$

Proof of Proposition 2.16. Let $u_r(x) = r^{-2}u(rx)$, and notice that we have the scaling property

$$W_{u_r}(\rho) = W_u(\rho r),$$

for any $r, \rho > 0$. Indeed,

$$\begin{aligned} W_{u_r}(\rho) &= \rho^{-n-2} \int_{B_\rho} \left(\frac{1}{2} |\nabla u_r|^2 + u_r \right) - \rho^{-n-3} \int_{\partial B_\rho} u_r^2 \\ &= \rho^{-n-2} r^{-2} \int_{B_\rho} \left(\frac{1}{2} |\nabla u|^2 + u \right) - \rho^{-n-3} r^{-4} \int_{\partial B_\rho} u^2 \\ &= (r\rho)^{-n-2} \int_{B_{r\rho}} \left(\frac{1}{2} |\nabla u|^2 + u \right) - (r\rho)^{-n-3} \int_{\partial B_{r\rho}} u^2 = W_u(r\rho). \end{aligned}$$

If u_0 is any blow-up of u at 0 then there is a sequence $r_j \rightarrow 0$ satisfying $u_{r_j} \rightarrow u_0$ in $C_{\text{loc}}^1(\mathbb{R}^n)$. Thus, for any $\rho > 0$ we have

$$W_{u_0}(\rho) = \lim_{r_j \rightarrow 0} W_{u_{r_j}}(\rho) = \lim_{r_j \rightarrow 0} W_u(\rho r_j) = W_u(0+). \quad (2.19)$$

Notice that the limit $W_u(0+) := \lim_{r \rightarrow 0} W_u(r)$ exists by monotonicity of W and since $u \in C^{1,1}$ implies $W_u(r) \geq -C$ for all $r \geq 0$. Moreover, the second equality follows by scaling (see (2.17)).

Hence, the function $W_{u_0}(\rho)$ is constant in ρ . However, by Theorem 2.18 this yields that

$$x \cdot \nabla u_0 - 2u_0 = 0 \quad \text{in } \mathbb{R}^n,$$

and therefore u_0 is 2-homogeneous. (Note that u_0 is a global solution to (2.15), and therefore we can take any $r > 0$ in Theorem 2.18.) Indeed, this property implies that

$$\psi(\lambda) = \lambda^{-2} u_0(\lambda x)$$

satisfies

$$\psi'(\lambda) = \lambda^{-3} (-2u_0(\lambda x) + (\lambda x) \cdot \nabla u_0(\lambda x)) = 0 \quad \forall \lambda \geq 0,$$

which implies that

$$\lambda^{-2} u_0(\lambda x) = \psi(\lambda) = \psi(1) = u_0(x).$$

$\square$

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