

Introduction (informal) :

$$-\Delta u = -\sum_{i=1}^n \partial_{ii}^2 u \text{ Laplacian } (u: \mathbb{R}^n \rightarrow \mathbb{R} \text{ } (u \in C^2))$$

→ harmonic fcts $(-\Delta u = 0 \text{ in } \overset{\mathbb{R}^n}{\Omega})$ have nice properties

→ maximum principle: u attains max & min on $\partial\Omega$
→ mean value property: $u(x) = \int u \, dy$
→ Harnack inequality: $\sup u \leq \inf u \cdot B_1(x)$
→ regularity: $u \in C_c^\infty, -\Delta u \in C^\alpha \Rightarrow u \in C^{2+\alpha}$

• $-\Delta$ arises in many settings:

$$\rightarrow \text{Calculus of Variations: } \min_{\substack{u \in H^1(\Omega) \\ u=g \text{ on } \partial\Omega}} \int |\nabla u|^2 \rightarrow \begin{cases} -\Delta u = 0 \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

→ more general models

$$\rightarrow \text{heat equation } \partial_t u - \Delta u = 0$$

→ connection to Brownian motion

$$u(t, x) = \mathbb{E}[u_0(x + X_t)] \text{ solves h.e. with } u(0, \cdot) = u_0.$$

$$\rightarrow \text{Fourier analysis: } -\Delta u = c \mathcal{F}^{-1}(|\xi|^2 \mathcal{F}(u))$$

↖ multiplier

The Laplacian is a local differential operator in the following sense:

Given $u, v \in C^2$ with $u=v$ in some nbh. of $x_0 \in \mathbb{R}^n$. Then, $-\Delta u(x_0) = -\Delta v(x_0)$.

This course: Study differential operators similar to $-\Delta$ that are not local = nonlocal.

Main example: The fractional Laplacian $(-\Delta)^s u$, $s \in (0, 1)$

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+h) - u(x-h)}{|h|^{n+2s}} dh, \quad C_{n,s} \text{ constant} \\ &= \int_{\mathbb{R}^n} \underbrace{\left(\frac{2u(x) - u(x+h) - u(x-h)}{|h|^{2s}} \right)}_{\leq |h|^{2-2s} \text{ if } u \in C_c^2(\mathbb{R}^n)} |h|^{-n} dh < \infty \quad 2^{2s} s \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)} \pi^{-n/2} \end{aligned}$$

→ in this way it is a differential operator (it only takes suff. smooth fcts)
(and $-\Delta u$ is less smooth $\rightarrow$ local)

• not local: $u \equiv 0$, $v = \frac{1}{|x|^{n-2s}}$ (1.2)
 $u \equiv v$ in $(-1,1)$, $(-\Delta)^s u(0) = 0$, $(-\Delta)^s v(0) = - \int_{\mathbb{R}^n} \frac{v(h) + v(-h)}{|h|^{n+2s}} dh < 0$
 $\rightarrow$ a distant bump is felt at the origin

Main question of this lecture:

How does nonlocality influence regularity properties in PDEs?

$\rightarrow$ Consider a solution u to $(-\Delta)^s u = 0$ in Ω .
 • Is $u \in C^\infty$? $\rightarrow$ need to know values $\rightarrow u \in C^\infty$ in Ω (✓)
 $u \uparrow \partial\Omega$? (X)
 • Do we have max/min principle / max value prop / Harnack / ...?

$\rightarrow$ Sometimes, new phenomena occur due to nonlocality
 $\rightarrow$ Sometimes, proofs are much easier or much more complicated.

Example: Strong maximum principle.

Theorem 0.1: (SMP)

Let $\Omega \subseteq \mathbb{R}^n$ connected, bd., open.
 Suppose $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$
 $(-\Delta)^s u = 0$ in Ω , $u \geq 0$ in $\mathbb{R}^n$.
 If $u(x_0) = 0$ for some $x_0 \in \Omega$
 $\Rightarrow u = 0$ in $\mathbb{R}^n$

$\hookrightarrow$ recall local: $\Omega \subseteq \mathbb{R}^n$ connected,
 $u \in C^2(\Omega)$, $u \geq 0$ in Ω ,
 $-\Delta u = 0$ in Ω .
 If $u(x_0) = 0$ for some $x_0 \in \Omega$
 $\Rightarrow u \equiv 0$ in Ω .

local: at x_0 .
 $\Delta^2 u(x_0) = 0$
 but $\nabla u \neq 0$
 e.g. $u(x) = |x|^4$.

Proof: $0 = (-\Delta)^s u(x_0) = - \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x_0+h) + u(x_0-h)}{|h|^{n+2s}} dh$
 $= - c_{n,s} \int_{\mathbb{R}^n} \frac{u(z)}{|x_0-z|^{n+2s}} dz$
 $\rightarrow \geq 0 \quad \forall z$

$\Rightarrow u \equiv 0$ a.e.

Remark 0.2: If we only assume $u \geq 0$ in Ω , then the result fails.

→ Why should we care?

$(-\Delta)^S$ also appears in

- Calculus of Variations

$$\begin{aligned} & \min_{u \in H^s(\mathbb{R}^n)} [u]_{H^s(\mathbb{R}^n)}, \\ & u = g \text{ in } \Omega^c \end{aligned}$$

- fractional heat equation

$$\partial_t u + (-\Delta)^s u = 0$$

→ connections to 2S-state ^{rad. syn.} Levy process
(jumps $\hat{=}$ nonlocal)

• • •

- Boltzmann equation $f: (0, \infty) \times \overset{x}{\mathbb{R}^n} \times \overset{v}{\mathbb{R}^n} \rightarrow (0, \infty)$

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \quad \text{in } (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$$

→ one of the most fundamental eq. in nat. physics
(kinetic theory, evolution of a gas).

- Plan of the course :

- Basic facts about the fractional Laplacian (2s related operators)

- Properties of solutions to weakly $TP\bar{E}_5$

- Interior & boundary regularity of solutions

- Purely unphysical features of solution

- Applications to the Boltzmann eq. / other topics ...