Lecture 3: 29.04.2024

We concluded last lecture with an alternative definition of Γ-convergence by means of Γ-upper and lower limit.
It is immediate to deduce some propertie of Γ-conver-gence by exploiting them.

Rmk: (1.1) the Γ-limit (if $\exists$) is unique, e.g. since it coincides with the Γ-liminf, which is a well-defined function

(1.2) if $f_j \leq g_j \ \forall j \Rightarrow \Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j \leq \Gamma\text{-}\underline{\lim}_{j\to+\infty} g_j$.
Same for the Γ-$\overline{\lim}$ (and hence for the Γ-lim if exists)

(1.3) let $\{f_{j_k}\} \subseteq \{f_j\}$ then

$$\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j \leq \Gamma\text{-}\underline{\lim}_{k\to+\infty} f_{j_k}, \quad \Gamma\text{-}\overline{\lim}_{k\to+\infty} f_{j_k} \leq \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j.$$

In particular, if $\exists\ \Gamma\text{-}\lim_{j\to+\infty} f_j = f_\infty$ then $\forall$ subsequence $\{f_{j_k}\} \subseteq \{f_j\}$ $\exists\ \Gamma\text{-}\lim_{k\to+\infty} f_{j_k} = f_\infty$.

Exercise: deduce (1.1) directly from the definition of Γ-limit.

Some properties of Γ-convergence

Now we see some of the main propertes of Γ convergences that will be useful in the sequel.

Proposition 1: $\Gamma\text{-}\varliminf_{j\to+\infty} f_j$, $\Gamma\text{-}\varlimsup_{j\to+\infty} f_j$ are lsc.
In particular if $\exists\ \Gamma\text{-}\lim_{j\to+\infty} f_j$ it is lsc.

Proof: we use the notation $f'(x) := \Gamma\text{-}\varliminf_{j\to+\infty} f_j(x)$ and
and $f''(x) := \Gamma\text{-}\varlimsup_{j\to+\infty} f_j(x)$.

Step 1 (lsc of f'): let $x_k \to x$. $\forall x_k\ \exists x_j^{(k)} \to x_k$ s.t.

$$f'(x_k) \leq \varliminf_{j\to+\infty} f_j(x_j^{(k)}) < f'(x_k) + 1/k.$$

As in the last lecture, we define the seq. of indices

$$\sigma_0' = 0, \quad \sigma_k' := \min\left\{ h \geq \sigma_{k-1}' : \begin{array}{l} d(x_j^{(k)}, x_k) \leq 1/k \quad \forall j \geq h \\ |f'(x_k) - f_h(x_h^{(k)})| < 2/k \end{array} \right\}$$

and define $\bar{x}_j = x^{(k)}_{\sigma_k'}$ when $\sigma_k' \leq j < \sigma_{k+1}'$. By definition
$\bar{x}_j \to x$ and

$$f'(x) \overset{\text{def of } \Gamma\text{-}\varliminf}{\leq} \varliminf_{j\to+\infty} f_j(\bar{x}_j) \leq \varliminf_{k\to+\infty} f_{\sigma_k'}(x^{(k)}_{\sigma_k'}) \leq \varliminf_{k\to+\infty} f'(x_k).$$

Step 2 (lsc of f''): for x_k and $x_j^{(k)}$ as above we have

$$f''(x_k) \leq \varlimsup_{j\to+\infty} f_j(x_j^{(k)}) < f''(x_k) + 1/k.$$

Now we define a slightly different seq. of indeces

$$\sigma_0'' = 0, \quad \sigma_k'' := \min\left\{ h \geq \sigma_{k-1}'' : \begin{array}{l} d(x_j^{(k)}, x_k) \leq 1/k \\ f_j(x_j^{(k)}) - f''(x_k) < 2/k \end{array} \ \forall j \geq h \right\}$$

and $\tilde{x}_j = x^{(k)}_{\sigma_k''}$ when $\sigma_k'' \leq j < \sigma_{k+1}''$. As above

$$f''(x) \overset{\text{def of } \Gamma\text{-}\varlimsup}{\leq} \varlimsup_{j\to+\infty} f_j(\bar{x}_j) \leq f''(x_k) + 2/k \quad \forall k$$

and taking the $\varliminf$ as $k \to +\infty$ we get the result. $\square$

Note: in the previous proof, when dealing with f' it was sufficient that $f'(x_k) \sim f_j(\bar{x}_j)$ along a subsequence (indeed we used monotonicity of $\underline{\lim}$).
Since $\overline{\lim}$ has the inverse monotonicity, we need that the whole sequence $f_j(\bar{x}_j) \lesssim f''(x_k)$.
This is why we used two sligthey different de- -finitions for σ_k' and σ_k''.

Corollary 2: $\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j \leq sc(\underline{\lim}_{j\to+\infty} f_j)$ and $\Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j \leq sc(\overline{\lim}_{j\to+\infty} f_j)$.
In particular, if there exist the Γ-limit and the pointwise limit, then $\Gamma\text{-}\lim_{j\to+\infty} f_j \leq sc(\lim_{j\to+\infty} f_j)$.

proof: by definition of $\Gamma\text{-}\underline{\lim}$, taking $x_j \equiv x$ as a test sequence we get $\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j(x) \leq \underline{\lim}_{j\to+\infty} f_j(x)$. Same for the $\Gamma\text{-}\overline{\lim}$. $\square$

The lsc of the $\Gamma\text{-}\overline{\lim}$ allows to simplify the computation of the Γ-limit, by restring its computation only on a dense set.

Rmk (density argument for limsup inequality): let d' stronger than d (i.e. $d'(x_j,x) \to 0 \Rightarrow d(x_j,x) \to 0$), and let

(2.1) $D \subseteq X$ be a d'-dense set

(2.2) $\Gamma(d')\text{-}\overline{\lim}_{j\to+\infty} f_j(x) \leq f(x) \quad \forall x \in D$, f d'-continuous

then $\Gamma(d)\text{-}\overline{\lim}_{j\to+\infty} f_j(x) \leq f(x) \quad \forall x \in X$.

Indeed: $\forall x \in X \; \exists x_k \in D : x_k \xrightarrow{d'} x \Rightarrow x_k \xrightarrow{d} x$, so

$$f''(x) \overset{\text{lsc}}{\leq} \underline{\lim}_{k\to+\infty} f''(x_k) \overset{(2.2)}{\leq} \lim_{k\to+\infty} f(x_k) \overset{\text{cont}}{=} f(x).$$

We will use this a lot with $D = C^\infty(\bar{\Omega})$, dense in $H^1(\Omega)$ in $\|\cdot\|_{H^1}$, to get $\Gamma(\|\cdot\|_{L^2})$-limit.

In general, the sum of two Γ-limits differs from the Γ-limit of the sum. A trivial example is the following: we saw in the last lecture that

$$f_j(x) = \begin{cases} -1 & x = 1/j \\ 0 & x \neq 1/j \end{cases} \xrightarrow{\Gamma} f_\infty(x) = \begin{cases} -1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

It can be easily checked that $-f_j(x) \xrightarrow{\Gamma} g_\infty \equiv 0$.
But $f_\infty = f_\infty + g_\infty \neq \Gamma\text{-}\lim_{j\to+\infty} f_j + (-f_j) \equiv 0$.

Adding a uniformly converging sequence though, do not affect Γ-convergence.

<u>Proposition</u> 3: let $f_j \xrightarrow{\Gamma} f_\infty$ and $g_j \rightrightarrows g$, and $g : X \to [-\infty, +\infty]$ continuous. Then

$$\Gamma\text{-}\lim_{j\to+\infty} (f_j + g_j) = f_\infty + g.$$

<u>Proof</u>: Since $g_j \rightrightarrows g$, $\forall \varepsilon > 0$ $\sup_{x\in X} |g_j(x) - g(x)| < \varepsilon$ for j large enough.
In particular, $\forall x_j \to x$, $|g_j(x_j) - g(x)| < \varepsilon$.

$$f_\infty(x) + g(x) \overset{(i)}{\leq} \underline{\lim}_{j\to+\infty} f_j(x_j) + \lim_{j\to+\infty} g(x_j)$$

$$\leq \underline{\lim}_{j\to+\infty} \left(f_j(x_j) + g_j(x_j)\right) + \varepsilon,$$

since it holds $\forall \varepsilon > 0$ it yields (i) for $f_j + g_j$.

from (ii)'' for f_j we have $\forall \varepsilon>0\ \exists\, x_j^\varepsilon \to x$ s.t

$$f_\infty(x)+g(x) \overset{\text{(ii)''+ cont}}{\geq} \overline{\lim_{j\to+\infty}} f_j(x_j^\varepsilon)-\varepsilon+\lim_{j\to+\infty} g_j(x_j^\varepsilon)-\varepsilon$$

$$\geq \overline{\lim_{j\to+\infty}} \left(f_j(x_j^\varepsilon)+g_j(x_j^\varepsilon)\right)-2\varepsilon$$

which implies (ii)'' for f_j+g_j. $\square$

Corollary 4 (stability under continuous perturbations): let $g: X\to[-\infty,+\infty]$ be a continuous function and let $f_j \xrightarrow{\Gamma} f_\infty$, then

$$\Gamma\text{-}\lim_{j\to+\infty}(f_j+g)=f_\infty+g.$$

As we have seen, the Γ-limit not always exists. There are some special sequences for which it is easy to prove the existence of the Γ-limit and to compute it.

Proposition 5 (constant sequence): $\Gamma\text{-}\lim\limits_{j\to+\infty} f(x)=sc(f)(x)=\underline{\lim\limits_{y\to x}} f(y)$

proof: we work in two steps

Step 1 ($\exists\, \Gamma\text{-}\lim\limits_{j\to+\infty} f(x)=\underline{\lim\limits_{y\to x}} f(y)$): (i) is obvious by definition of $\underline{\lim\limits_{y\to x}} f(y)$.
From characterization of inf, $\forall\varepsilon>0\ \exists\, x_j^\varepsilon\to x$ s.t

$$\underline{\lim_{y\to x}} f(y) > \lim_{j\to+\infty} f(x_j^\varepsilon)-\varepsilon$$

which proves (ii)''.

Step 2 ($\underline{\lim\limits_{y\to x}} f(y)=sc(f)(x)$): by Step 1 $\underline{\lim\limits_{y\to x}} f(y)$ is lsc

(since it is a Γ-limit) and $\liminf_{y\to x} f(y) \leq f(x)$ so $\liminf_{y\to x} f(y) \leq sc(f)(x)$.
But $\forall\, g \leq f$, g lsc there holds that $g(x) \leq \liminf_{y\to x} g(y) \leq \liminf_{y\to x} f(y)$. Taking the sup over g we get the opposite inequality. $\square$

Another useful result is the following.

<u>Proposition 6</u>: $\Gamma\text{-}\liminf_{j\to+\infty} f_j = \Gamma\text{-}\liminf_{j\to+\infty} sc(f_j)$ (same for $\Gamma\text{-}\limsup$).

<u>proof</u>: by using the topologic definition of $\Gamma\text{-}\liminf$ and $\liminf$ and the fact that $sc(f)(x) = \liminf_{y\to x} f(y)$ we get

$$\Gamma\text{-}\liminf_{j\to+\infty} sc(f_j)(x) = \sup_{U\in N(x)} \liminf_{j\to+\infty} \inf_{y\in U} sc(f_j)(y)$$

$$= \sup_{U\in N(x)} \liminf_{j\to+\infty} \inf_{y\in U} \sup_{V\in N(y)} \inf_{z\in V} f_j(z),$$

since $V \mapsto \inf_V f_j$ is decreasing we can reduce to neighbours that are contained in U so

$$= \sup_{U\in N(x)} \liminf_{j\to+\infty} \inf_{y\in U} \sup_{V\in N(y)} \inf_{z\in U\cap V} f_j(z)$$

$$\geq \sup_{U\in N(x)} \liminf_{j\to+\infty} \inf_{y\in U} f_j(y) = \Gamma\text{-}\liminf_{j\to+\infty} f_j(x).$$

Analogously for the $\Gamma\text{-}\limsup$. $\square$

<u>Note</u>: working similarly as in the proof of the lsc of $\Gamma\text{-}\liminf$ we can prove the result above also reasoning with sequences (same for $\Gamma\text{-}\limsup$).

Proposition 7 (limits of monotone sequences): let $\{f_j\}$ be a a monotone sequence, then the Γ-limit $\exists$ and

(1) if $f_{j+1} \leq f_j$ (decreasing) $\Rightarrow$ $\Gamma\text{-}\lim_{j\to+\infty} f_j = sc(\lim_{j\to+\infty} f_j)$

(2) if $f_{j+1} \geq f_j$ (increasing) $\Rightarrow$ $\Gamma\text{-}\lim_{j\to+\infty} f_j = \lim_{j\to+\infty} sc(f_j)$

proof: my monotonicity, there exists the pointwise limit in both cases.

(1) by Corollary 2, $\Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j \leq sc(\lim_{j\to+\infty} f_j)$.
Moreover by Proposition 5

$$sc(\lim_{j'\to+\infty} f_{j'}) = \Gamma\text{-}\lim_{j\to+\infty} (\lim_{j'\to+\infty} f_{j'}) \leq \Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j$$

since $\{f_j\}$ is decreasing hence $\lim_{j'\to+\infty} f_j \leq f_j \ \forall j$.

(2) since $\{f_j\}$ is increasing, so is $\{sc(f_j)\}$ hence $\forall j$
$sc(f_j) \leq \lim_{j'\to+\infty} sc(f_{j'}) = \sup_{j'} sc(f_{j'})$ which is lsc since it is sup of lsc functions.
By Corollary 2, $\Gamma\text{-}\overline{\lim}_{j\to+\infty} sc(f_j) \leq \lim_{j\to+\infty} sc(f_j)$.
Moreover, $\forall j' < j$ $sc(f_j) \geq sc(f_{j'})$ so

$$\Gamma\text{-}\underline{\lim}_{j\to+\infty} sc(f_j) \geq sc(f_{j'}) \quad \forall j'$$

and the result follows by taking the sup and by Proposition 6. $\square$

Convergence of minimum problems

It is possible to apply the Direct Method to Γ-converging, since the Γ-limit is lsc. It is necessary though to be sure that we can reduce to study infima on a compact set.

<u>Def</u> (coerciveness): $f : X \to [-\infty, +\infty]$ is coercive if $\forall\, t \in \mathbb{R}$, $\{f \leq t\}$ are precompact.
f is mildly-coercive if $\exists\, K \subseteq X$ compact s.t. $\inf_X f = \inf_K f$.
$f_j : X \to [-\infty, +\infty]$ is equi mildly-coercive if $\exists\, K \subseteq X$ cmpt s.t. $\inf_X f_j = \inf_K f_j$.

<u>Rmk</u>: f coercive $\Rightarrow$ f mildly-coercive
indeed, this is obvious if $f \equiv +\infty$, if not take $t \in \mathbb{R}$ s.t. $\{f \leq t\} \neq \emptyset$ and $K := \overline{\{f \leq t\}}$. Then

$$\inf_X f = \inf_{\{f \leq t\}} f \geq \inf_K f \geq \inf_X f.$$

Actually, it is sufficient that $\exists\, t_0$ s.t. $\emptyset \neq \{f \leq t_0\}$ is precompact to imply mild-coerciveness.

<u>Theorem</u> (fundamental Thm of Γ-convergence): let (X, d) be a metric space and $f_j, f_\infty : X \to [-\infty, +\infty]$ be s.t. $\{f_j\}$ is equi mildly coercive and $f_j \xrightarrow{\Gamma} f_\infty$. Then

$$\exists \lim_{j \to +\infty} \inf_X f_j = \min_X f_\infty .$$

Moreover, $\forall\, \{x_j\}$ s.t. $\lim_{j \to +\infty} f_j(x_j) = \min_X f_\infty$, every cluster point of $\{x_j\}$ is a global minimizer of f_∞.

proof: let $K \subset\subset X$ s.t. $\inf_K f_j = \inf_X f_j \; \forall j$. By characterization of inf $\exists \; \tilde{x}_j \in K$ s.t.

$$\inf_X f_j \leq f_j(\tilde{x}_j) < \inf_X f_j + 1/j \quad \forall j.$$

Taking the lim we get

$$(*) \quad \lim_{j\to+\infty} \inf_X f_j = \lim_{j\to+\infty} f_j(\tilde{x}_j).$$

Since K is compact $\exists \{\tilde{x}_{j_k}\} \subseteq \{\tilde{x}_j\}$ and $x_\infty \in K$ s.t. $\tilde{x}_{j_k} \to x_\infty$ and $\lim_{j\to+\infty} f_j(\tilde{x}_j) = \lim_k f_{j_k}(\tilde{x}_{j_k})$.

Now define $x_j := \begin{cases} \tilde{x}_j & j = j_k \text{ for some } k \\ x_\infty & \text{otherwise} \end{cases}$, then

$$(*_1) \quad f_\infty(x_\infty) \overset{(i)}{\leq} \underline{\lim}_{j\to+\infty} f_j(x_j) \leq \underline{\lim}_{k\to+\infty} f_{j_k}(\tilde{x}_{j_k}) = \lim_{j\to+\infty} \inf_X f_j.$$

By def. of Γ-convergence, $\forall x \in X \; \exists \; \bar{x}_j \to x$ a rec. seq., then

$$(*_2) \quad f_\infty(x_\infty) \overset{(ii)}{\geq} \overline{\lim}_{j\to+\infty} f_j(\bar{x}_j) \geq \overline{\lim}_{j\to+\infty} \inf_X f_j, \quad \forall x \in X.$$

By $(*_1)$ and taking the inf in $(*_2)$ we get

$$\inf_X f_\infty \leq f_\infty(x_\infty) \leq \underline{\lim}_{j\to+\infty} \inf_X f_j$$
$$\leq \overline{\lim}_{j\to+\infty} \inf_X f_j \leq \inf_X f_\infty$$

which yields $\exists \lim_{j\to+\infty} \inf_X f_j = \inf_X f_\infty \left(= f_\infty(x_\infty)\right) = \min_X f_\infty$.

We can repeat the argument above $\forall \{x_j\}$ satisfying $(*)$ since then $\forall \, x_0$ cluster point of $\{x_j\}$ we repeat all the passages with x_0 in place of x_∞. This proves the full result. $\square$

Rmk: (3.1) notice that, in the case in which $\inf_x f_j = \min_x f_j$ $\forall j$ and let x_j be a (global) minimizer of f_j, then

$$\lim_{j\to+\infty} f_j(x_j) = \min_x f_\infty .$$

In particular, for j large enough $x_j \in K$ (as in the proof, so $\exists \{x_{j_k}\} \subseteq \{x_j\}$ converging to some x_0 a global minimizers of f_∞.
Loosely speaking, if $f_j \xrightarrow{\Gamma} f_\infty$ then global minimizers converg to global minimizers.

(3.2) Γ-convergence does not give any information about local minimizers.
Local minimizers (in general) do not converge to local minimizers.

Ex: $f_j(x) = \begin{cases} -1 & x \in \frac{1}{j}\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$, we saw that $\Gamma\text{-}\lim_{j\to+\infty} f_j \equiv -1$.
Let $g_j(x) = x^2 + f_j$, by stability under continuous perturbation $g_j \xrightarrow{\Gamma} x^2 - 1$.
In this case every $x \in \frac{1}{j}\mathbb{Z}$ is a local minimizer of g_j but they converge to any point in $\mathbb{R}$, while the unique local minimizer of the Γ-limit is $x = 0$.

Equi (mildly-) coerciveness is essential to prove the fund. thm, since it ensures that min. seq. stay bounded. If minimizers "escape" to infinity, Γ convergence alone is (in general) not sufficient.

Ex (lack of coerciveness): take $f_j(x)=\begin{cases}0 & x\neq j\\ -1 & x=j\end{cases}$. It is immediate to prove that

$$f_\infty := \Gamma\text{-}\lim_{j\to+\infty} f_j \equiv 0.$$

Even though each f_j is mildly-coercive, since $\{f\leq -1/2\}=\{j\}$, the sequence is not equi mildly-coercive, indeed $\forall K \Subset \mathbb{R}$, for j large enough $\inf_K f_j = 0 \neq -1 = \inf_x f_j$.
For such a Γ-converging sequence, the fund. thm does not hold

$$\lim_{j\to+\infty} \inf_x f_j = -1 \neq 0 = \min_x f_\infty.$$