Lecture 2: 22.04.2024

Γ-convergence

Main references for this lecture:

- Ch. 1 of [B, 2000] (see also Ch. 1, 7 of [BD, 1998])

In this lecture we see the definition (and some properties) of lower semicontinuity and Γ-convergence. To fix the notation we start by recalling the notion of upper and lower limits.

Let (X, d) be a metric space.
Unless otherwise specified, when we write $x_j \to x$ we mean $d(x_j, x) \to 0$.

<u>Def</u> (upper and lower limits). Let $f: X \to [-\infty, +\infty]$ and $x \in X$

$$\varliminf_{y \to x} f(y) := \inf \left\{ \varliminf_{j \to +\infty} f(x_j) : x_j \to x \right\}$$
$$= \inf \left\{ \lim_{j \to +\infty} f(x_j) : \begin{array}{l} x_j \to x \text{ and} \\ \exists \text{ the limit of } f(x_j) \end{array} \right\},$$

$$\varlimsup_{y \to x} f(y) := \sup \left\{ \varlimsup_{j \to +\infty} f(x_j) : x_j \to x \right\}$$
$$= \sup \left\{ \lim_{j \to +\infty} f(x_j) : \begin{array}{l} x_j \to x \text{ and} \\ \exists \text{ the limit of } f(x_j) \end{array} \right\}.$$

<u>Note</u>: the fact that we can restrict the inf (or sup) to sequences s.t. the limit of $f(x_j)$ exists is consequence of the fact that we can always extract a subsequence converging to the $\varliminf$ (or $\varlimsup$).

Rmk (immediate properties):

(1.1) by choosing $x_j = x$ we get $\underline{\lim}_{y\to x} f(y) \le f(x)$

(1.2) $\underline{\lim}_{y\to x} f(y)+g(y) \ge \underline{\lim}_{y\to x} f(y) + \underline{\lim}_{y\to x} g(y)$

(1.3) $\underline{\lim}_{y\to x} f(y)+g(y) \le \underline{\lim}_{y\to x} f(y) + \overline{\lim}_{y\to x} g(y)$

(1.4) $\overline{\lim}_{y\to x} f(y) = - \underline{\lim}_{y\to x} -f(y)$.

By (1.4) one can get the analogous properties (with opposite sign) for the $\overline{\lim}$.

It may be useful to give an alternative definition of upper and lower limits just in terms of neighbors. This topological definitions coincide with the previous ones in metric spaces but may differ in general topological spaces.

Rmk (topological definitions): it holds that

$$\underline{\lim}_{y\to x} f(y) = \sup_{U\in N(x)} \inf_{y\in U} f(y)$$
$$\overline{\lim}_{y\to x} f(y) = \inf_{U\in N(x)} \sup_{y\in U} f(y).$$

Notice that $U \mapsto \inf_{y\in U} f(y)$ is decreasing (w.r.t. inclusion) so the $\sup_{U\in N(x)}$ resembles a limit process as U shrinks to x. Analogous for the $\overline{\lim}$.

Lower semicontinuity

Def (lsc): $f : X \to [-\infty, +\infty]$ is lower semicontinuous at $x \in X$ iff

$$f(x) \leq \liminf_{j \to +\infty} f(x_j) \quad \forall\, x_j \to x. \qquad \text{(LSC)}$$

If f is lsc $\forall x \in X$ we say that f is lsc.

Note: lsc depends on the metric (more in general topology) we choose. If we want to highlight the role of metric we write d-lsc.

Note: the definition above is that of sequential lsc. In topological spaces f is lsc iff $f(x) \leq \inf_{y \in U} f(y)$ $\forall U \in N(x)$. This coincide with sequential lsc in metric spaces, thus we simply say lsc.

Rmk (characterization): the following are equivalent

(2.1) f is lsc

(2.2) $f(x) = \liminf_{y \to x} f(y)$

(2.3) $\forall t \in \mathbb{R}$, $\{x \in X : f(x) \leq t\} = \{f \leq t\}$ is closed

Indeed: (2.1) $\Rightarrow$ (2.2) by taking the inf over $x_j \to x$ in (LSC) we get $f(x) \leq \liminf_{y \to x} f(y) \overset{(1.1)}{\leq} f(x)$.

(2.2) $\Rightarrow$ (2.1) immediate from definition of lower limit.

(2.1) $\Rightarrow$ (2.3) $\forall\, t \in \mathbb{R}$, take any $x_j \in \{f \le t\}$ s.t.
$x_j \to x \overset{(LSC)}{\Longrightarrow} f(x) \le \liminf_{j\to+\infty} f(x_j) \le t \Rightarrow x \in \{f \le t\}$.
Every converging sequence in $\{f \le t\}$ has limit still in $\{f \le t\} \Rightarrow \{f \le t\}$ is closed.

(2.3) $\Rightarrow$ (2.1) assume by contradiction that $\exists$ $x_j \to x$ s.t. $f(x) > \liminf_{j\to+\infty} f(x_j)$.
Then $\exists\ t_0 \in \mathbb{R}$ s.t. $f(x) > t_0 > \liminf_{j\to+\infty} f(x_j)$.
By properties of $\liminf$ $\exists\ \{x_{j_k}\} \subseteq \{x_j\}$ s.t. $\lim_{k\to+\infty} f(x_{j_k}) = \liminf_{j\to+\infty} f(x_j)$.
For k large enough $x_{j_k} \in \{f \le t_0\}$ but converges to $x \notin \{f \le t_0\} \Rightarrow \{f \le t_0\}$ is not closed, which is a contradiction.

Rmk (immediate properties):

(3.1) f, g are lsc $\Rightarrow f + g$ is lsc

(3.2) let $\mathcal{J}$ be a family of indicies (also uncountable).
Let f_α be lsc $\forall\, \alpha \in \mathcal{J} \Rightarrow \sup_{\alpha\in\mathcal{J}} f_\alpha$ is lsc

Indeed: for ever $x_j \to x$, $\forall\, \alpha \in \mathcal{J}$

$$f_\alpha(x) \overset{(LSC)}{\le} \liminf_{j\to+\infty} f_\alpha(x_j) \le \liminf_{j\to+\infty} \sup_{\alpha\in\mathcal{J}} f_\alpha(x_j),$$

Taking the sup over α we get the result

(3.3) let $E \subseteq X$ and $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$
χ_E is lsc $\Leftrightarrow E$ is open

Indeed: $\{\chi_E \le t\} = \begin{cases} X & t \ge 1 \\ E^c & 0 \le t < 1 \\ \emptyset & t < 0 \end{cases}$ is closed $\forall\, t \in \mathbb{R} \Leftrightarrow$
E is open and the result comes from (2.3).

Def (lsc envelope): let $f: X \to [-\infty, +\infty]$, its lower semicontinuous envelope is the function $sc(f): X \to [-\infty, +\infty]$ defined as

$$sc(f)(x) := \sup \left\{ g: X \to [-\infty, +\infty] : \begin{array}{l} g \le f \\ g \text{ is lsc} \end{array} \right\}.$$

$sc(f)$ is the largest lsc function smaller than f.
$sc(f)$ is also called relaxed functional.

Rmk: by (3.2) $sc(f)$ is lsc

Ex: it is important to consider functions which can attain $-\infty$.
Let $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $sc(f)(x) = \begin{cases} 1/x & x \neq 0 \\ -\infty & x = 0 \end{cases}$.

Γ-convergence

Def (Γ-convergence): let $f_j, f_\infty : X \to [-\infty, +\infty]$, we say that f_j Γ-converges to f_∞ at $x \in X$ iff

(i) liminf inequality: $\forall\, x_j \to x$

$$f_\infty(x) \le \liminf_{j \to +\infty} f_j(x_j)$$

(ii) limsup inequality: $\exists\, \overline{x}_j \to x$ (recovery sequence) s.t.

$$f_\infty(x) \ge \limsup_{j \to +\infty} f_j(\overline{x}_j).$$

If f_j Γ-converges to f_∞ at any $x \in X$ then we say that f_j Γ-converges to f_∞ in X.

We will write $f_j \xrightarrow{\Gamma} f_\infty$ or $\Gamma\text{-}\lim\limits_{j\to+\infty} f_j = f_\infty$. If we want to highlight the metric we write $f_j \xrightarrow{\Gamma(d)} f_\infty$ or $\Gamma(d)\text{-}\lim\limits_{j\to+\infty} f_j = f_\infty$.

If we have a family of functions parametrized by a continuous parameter $f_\varepsilon : X \to [-\infty,+\infty]$, $\varepsilon>0$ $f_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} f_\infty$ iff $\forall\, \varepsilon_j \to 0$ $f_{\varepsilon_j} \xrightarrow{\Gamma} f_\infty$.

<u>Rmk</u>: condition (ii) can be replaced by one of the following conditions.
By this we mean that (i)&(ii) $\Leftrightarrow$ (i) & (ii)' $\Leftrightarrow$ (i) & (ii)''

(ii)' $\exists\, \bar{x}_j \to x$ s.t. $f_\infty(x) = \lim\limits_{j\to+\infty} f_j(\bar{x}_j)$

<u>Indeed</u>: clearly (ii)' $\Rightarrow$ (ii). If (i)&(ii) hold then

$$f_\infty(x) \leq \varliminf_{j\to+\infty} f_j(\bar{x}_j) \leq \varlimsup_{j\to+\infty} f_j(\bar{x}_j) \leq f_\infty(x)$$

so $\exists \lim\limits_{j\to+\infty} f_j(\bar{x}_j) = f_\infty(x)$.

(ii)'' approximate limsup inequality: $\forall\, \varepsilon>0$ $\exists\, x_j^\varepsilon \to x$ s.t.

$$f_\infty(x) \geq \varlimsup_{j\to+\infty} f_j(x_j^\varepsilon) - \varepsilon$$

<u>Indeed</u>: clearly (ii) $\Rightarrow$ (ii)''. If (ii)'' holds then $\forall\, k\in\mathbb{N}$ (take $\varepsilon = 1/k$) $\exists\, x_j^{(k)} \to x$ s.t.

$$f_\infty(x) \geq \varlimsup_{j\to+\infty} f_j(x_j^{(k)}) - 1/k \quad \forall\, k\in\mathbb{N}.$$

We find a sequence "close" to $x_j^{(k)}$ when k is large along which f_j is "close" to the lim. We do it with a sort of diagonal argument.

Let $\sigma_0 := 0$ and define iteratively

$$\sigma_k := \min\left\{ h \geq \sigma_{k-1} : \begin{array}{l} d(x_j^{(k)}, x) \leq 1/k \quad \forall j \geq h \\ f_h(x_h^{(k)}) \leq \overline{\lim}_{j \to +\infty} f_j(x_j^{(k)}) + 1/k \end{array} \right\}.$$

By definition of $\overline{\lim}$ and since $x_j^{(k)} \to x$, σ_k is well defined $\forall k \in \mathbb{N}$.
We then define

$$\bar{x}_j := x_{\sigma_k}^{(k)} \qquad \text{if} \quad \sigma_k \leq j \leq \sigma_{k+1}.$$

Then $\bar{x}_j \to x$ and $f_j(\bar{x}_j) \leq \overline{\lim}_{j \to +\infty} f_j(x_j^{(k)}) + \frac{1}{k}$ $\forall k \in \mathbb{N}$. So

$$f_\infty(x) \geq \overline{\lim}_{j \to +\infty} f_j(x_j^{(k)}) - 1/k$$

$$\geq \overline{\lim}_{j \to +\infty} f_j(\bar{x}_j) - 1/2k.$$

Taking the limit as $k \to +\infty$ we get the result.

We can motivate the definition of Γ-limit as the research of a functions which is a lower bound for f_j along every sequence (condition).
This lower bound is somehow optimal (condition in the sense that f_j converges to f_∞ along at least one sequence.

<u>Note</u>: as for lsc, the definition above is that of sequential Γ-convergence, which coincide with the "topological" definition in metric spaces, but may differ in general topological spaces

Rmk (dependence on the metric): let d and d' be two metrics on X, with d' stronger than d (i.e. if $d'(x_j, x) \to 0 \Rightarrow d(x_j, x) \to 0$) then

$$\Gamma(d) - \lim_{j\to+\infty} f_j \leq \Gamma(d') - \lim_{j\to+\infty} f_j$$

if the Γ-limits exist.

In particular, let $d'(x,y) := \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$ (i.e. $x_j \xrightarrow{d'} x$ $\Leftrightarrow$ $x_j \equiv x$).
It's immediate to prove that $\Gamma(d') - \lim_{j\to+\infty} f_j$ is the pointwise limit $\lim_{j\to+\infty} f_j$.
Since every metric d is weaker than d' we infer that

$$\Gamma(d) - \lim_{j\to+\infty} f_j \leq \lim_{j\to+\infty} f_j .$$

The Γ-limit is always smaller than the pointwise limit (if they exist).

Some easy examples/exercise on the real line (with the metric induced by the modulus).

Exrc 1: let $f_j(x) = \begin{cases} 0 & x \neq 1/j \\ -1 & x = 1/j \end{cases}$, "prove" that

$$\Gamma\text{-}\lim_{j\to+\infty} f_j = \begin{cases} 0 & x \neq 0 \\ -1 & x = 0 . \end{cases}$$

Ex: let $f_j(x) = \begin{cases} 0 & x \notin 1/j\,\mathbb{Z} \\ -1 & x \in 1/j\,\mathbb{Z} \end{cases}$, $\Gamma\text{-}\lim_{j\to+\infty} f_j \equiv -1$.

- the liminf inequality is obvious

- $x \in \mathbb{R}$, $(x - 1/2j, x + 1/2j] \cap 1/j\mathbb{Z} =: \bar{x}_j$ is a recovery sequence.
  Indeed, $|\bar{x}_j - x| < 1/j$ so $\bar{x}_j \to x$ and $f_j(\bar{x}_j) \equiv -1$, so (ii)' holds.

Ex (Γ-limit may not exist): let $f_j(x) = \begin{cases} 0 & x \notin 1/j\mathbb{Z} \\ (-1)^{j+1} & x \in 1/j\mathbb{Z} \end{cases}$.
Let f be a function s.t. (i) holds.
Since, for every $x \in X$, $\tilde{x}_j := (x - 1/2j, x + 1/2j] \cap 1/j\mathbb{Z}$ then $f_j(\tilde{x}_j) = (-1)^{j+1} \Rightarrow f(x) \leq -1$.
But $\forall x_j \to x$, $f_j(x_j) \geq 0$ as j is odd so

$$\overline{\lim_{j \to +\infty}} f_j(x_j) \geq 0 > f(x) \quad \forall x_j \to x, \ \forall f \leq -1$$

so $\nexists f$ which satisfies both (i) and (ii), so f_j does not Γ-converge.

Ex (Γ-converging seq. without pointwise limit): let

$$f_j(x) = \begin{cases} 1 & x \in 1/j(2\mathbb{Z}+1) \\ -1 & x \in 1/j(2\mathbb{Z}) \\ 0 & \text{otherwise.} \end{cases}$$

This sequence does not converge pointwise. Indeed $f_j(1) = \begin{cases} 1 & j \text{ is odd} \\ -1 & j \text{ is even} \end{cases}$ so $\nexists \lim_{j \to +\infty} f_j(1)$. Similar $\forall x \in \mathbb{Q}$.

But $\Gamma\text{-}\lim_{j \to +\infty} f_j \equiv -1$ as in the previous exercise by taking $\bar{x}_j := (x - j, x + j] \cap 1/j(2\mathbb{Z})$ as a recovery sequence.

Alternative definitions

As we have seen from previous examples, Γ- limit not always exist.
It is convenient to introduce quantities that always exists that may help in the computation of the Γ-limit. On the other hand they also provide an alternative (equivalent) definition of Γ- limit.

Def (upper and lower Γ- limits): let $f_j : X \to [-\infty, +\infty]$ we define

$$\Gamma\text{-}\varliminf_{j\to+\infty} f_j(x) := \inf\left\{ \varliminf_{j\to+\infty} f_j(x_j) : x_j \to x \right\}$$

$$\Gamma\text{-}\varlimsup_{j\to+\infty} f_j(x) := \inf\left\{ \varlimsup_{j\to+\infty} f_j(x_j) : x_j \to x \right\}$$

Proposition: $f_j \xrightarrow{\Gamma} f_\infty \iff \Gamma\text{-}\varliminf_{j\to+\infty} f_j = \Gamma\text{-}\varlimsup_{j\to+\infty} f_j = f_\infty$

proof: $(\Rightarrow)$ if $f_j \xrightarrow{\Gamma} f_\infty$ then

$$f_\infty(x) \overset{(i)}{\leq} \Gamma\text{-}\varliminf_{j\to+\infty} f_j(x) \leq \Gamma\text{-}\varlimsup_{j\to+\infty} f_j(x) \leq$$

$$\leq \varlimsup_{j\to+\infty} f_j(\bar{x}_j) \overset{(ii)}{\leq} f_\infty(x).$$

$(\Leftarrow)$ point (i) trivially holds. We prove (ii)", by characterization of inf $\forall \varepsilon > 0 \; \exists \; x_j^\varepsilon \to x$ s.t.

$$f_\infty(x) = \Gamma\text{-}\varlimsup_{j\to+\infty} f_j(x) > \varlimsup_{j\to+\infty} f_j(x_j^\varepsilon) - \varepsilon$$

which is exactly (ii)" so $f_j \xrightarrow{\Gamma} f_\infty$. $\square$

We give also a topological version of upper and lower Γ-limits that might be useful in the sequel.

<u>Rmk</u> (topological definitions): in general topological spaces

$$\Gamma\text{-}\liminf_{j\to+\infty} f_j(x) = \sup_{U\in N(x)} \liminf_{j\to+\infty} \inf_{y\in U} f_j(y)$$

$$\Gamma\text{-}\limsup_{j\to+\infty} f_j(x) = \sup_{U\in N(x)} \limsup_{j\to+\infty} \inf_{y\in U} f_j(y).$$

As for the lsc, these quantities coincide with the previous definitions but may differ in general topological spaces.