Lecture 1: 15.04.2024

Γ- convergence of integral functionals

Main references for the course:

- Γ- convergence for Beginners, Braides (2000)
- Homogenization of Multiple Integrals, Braides and Defranceschi (1998)

The framework of this course is the Calculus of Variations. Let's give a quick overview of the standard problem in CalcVar and what is the usual strategy to attack it.

Calculus of Variations: study of minimum problems.

Let X be a topological space (for simplicity a sequential space, e.g. a metric space), consider a function

$$F : X \longrightarrow (-\infty, +\infty], \text{ so called energy (functional).}$$

We may be interested in minimizing the energy F in X:

$$\begin{cases} \inf \{ F(u) : u \in X \} =: M > -\infty, \\ \text{(if possible) find } \bar{u} \in X : F(\bar{u}) = M. \end{cases}$$

An immediate comment is that we are not solving minimum problems "per se" but solving problems that can be rewritten as minimization of a certain functional.

Rmk (solving PDEs): since minimizers satisfies "$DF(\bar{u})=0$" we can relate minimum problems to PDEs.

solving (variational) PDEs $\Leftrightarrow$ solving a min. pb.

A classical example is Poisson's equation

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \Leftrightarrow \min_{u|_{\partial\Omega}=0} \int_\Omega \tfrac{1}{2}|\nabla u|^2 - fu \, dx.$$

In general, solving (distributionally) elliptic PDEs in divergence form corresponds to minimize a suitable "Lagrangean".

Direct Method (of CalcVar): this is the main tool to solve minimum problems. It consists in an application of (a variant of) Weierstrass' Theorem: if

$\{u \in X : F(u) \leq t\}, t \in \mathbb{R}$ — the closure is compact

- F is coercive (i.e. sublevels are precompact)
- F is lower-semicontinuous (i.e. sublevels are closed)

then $X_T := \{u \in X : F(u) \leq T\}$ is compact, for $T \in \mathbb{R}$ (e.g. T chosen so that $X_T \neq \emptyset$).

By Weierstrass Thm $\exists\, \bar{u} \in X_T : F(\bar{u}) = \min_{X_T} F = \min_X F$.

Note: compactness and lower-semicontinuity are crucial notions throughout the whole course. They depend on the choice of the topology.

Γ-convergence ...

Many mathematical problems are characterized by the presence of a parameter that may be artificial (coming from an approximation or discretization) or natural (e.g. a geometric quantity).

Let $\varepsilon > 0$ and consider $F_\varepsilon : X \longrightarrow (-\infty, +\infty]$ and study

$$\begin{cases} \inf\{F_\varepsilon(u) : u \in X\} =: M_\varepsilon > -\infty \\ u_\varepsilon \in X : F_\varepsilon(u_\varepsilon) = M_\varepsilon \end{cases}$$

Due to the presence of an additional parameter, these problems are usually difficult.
We look for an "effective" energy $F_0 : X \longrightarrow (-\infty, +\infty]$ whose min. pb. is easier to solve but is "close" to that of F_ε:

$$\begin{cases} \inf\{F_0(u) : u \in X\} =: M_0 > -\infty \\ u_0 \in X : F_0(u_0) = M_0 \end{cases} \qquad \text{[easier]}$$

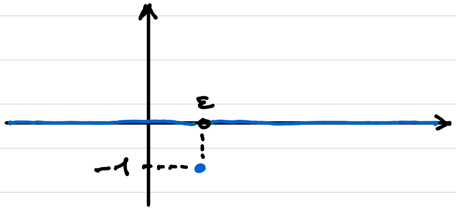
and such that $\lim_{\varepsilon \to 0} M_\varepsilon = M_0$ and $\lim_{\varepsilon \to 0} u_\varepsilon = u_0$.

The research of such an effective energy can be formalized with a notion of convergence which implies the convergence of minimum problems and of (global) minimizers.

For these reasons De Giorgi and Franzoni (1975) introduce the notion of Γ-convergence.

Do we need a "new" notion of convergence for this porpouse? Is any elementary convergence (e.g. pointwise convergence) doing this job?

Ex (What about pointwise convergence?): consider



$$F_\varepsilon(u) = \begin{cases} 0 & u \neq \varepsilon \\ -1 & u = \varepsilon, \end{cases}$$

clearly $\inf_{\mathbb{R}} F_\varepsilon = -1$. Let's compute the pointwise limit of F_ε:

- $\forall u \neq 0$, $F_\varepsilon(u) = 0$ when $\varepsilon < |u| \Rightarrow \lim_{\varepsilon \to 0} F_\varepsilon(u) = 0$
- $u = 0$, $F_\varepsilon(u) \equiv 0$ $\forall \varepsilon > 0 \Rightarrow \lim_{\varepsilon \to 0} F_\varepsilon(u) = 0$

so $F_\varepsilon \to G$, $G \equiv 0$, pointwise. But $\inf_{\mathbb{R}} G = 0$, so the pointwise limit of F_ε is <u>not</u> a good approximation for the minimum problem of F_ε.

Note: pointwise convergence is not good enough. We have to check how F_ε behaves along converging sequences, since the characterize the topology.

Def (Γ-convergence, but Naïve): let $F_\varepsilon, F_0 : X \to (-\infty, +\infty]$ we say that F_ε Γ-converges to F_0 ($F_\varepsilon \xrightarrow{\Gamma} F_0$) as $\varepsilon \to 0$ if the following to properties hold:

(i) F_0 is a lower bound, i.e. $\forall u \in X$ and $\forall u_\varepsilon \to u$

$$F_0(u) \leq F_\varepsilon(u_\varepsilon) + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0 \; ;$$

(ii) F_0 is the "optimal" lower bound, i.e. $\forall u \in X$ $\exists\, \bar{u}_\varepsilon \to u$ (so called recovery sequence) such that

$$F_\varepsilon(\bar{u}_\varepsilon) \leq F_0(u) + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0.$$

Naively (for this we would need additional assumptions)

$$\text{(i)} \Rightarrow \inf F_0 \leq \inf F_\varepsilon + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0$$

$$\text{(ii)} \Rightarrow \inf F_\varepsilon \leq \inf F_0 + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0$$

Rmk: from the previous comment, a consequence (up to requesting extra assumptions on F_ε) of the fact that $F_\varepsilon \xrightarrow{\Gamma} F_0$ is that

$$\inf F_0 \overset{(i)}{\leq} \underline{\lim}_{\varepsilon \to 0} \inf F_\varepsilon \leq \overline{\lim}_{\varepsilon \to 0} \inf F_\varepsilon \overset{(ii)}{\leq} \inf F_0$$

which yields

the Γ-limit is always lsc

$$\lim_{\varepsilon \to 0} \inf F_\varepsilon = \inf F_0 \left(= \min F_0\right).$$

This is the property we wanted to be ensured by the notion of convergence we were looking for.

Note: again, all of this depend heavily on the choice of the topology.

... of integral functionals

In this course we will mainly focus on the case

$$X \subseteq L^1(\Omega;\mathbb{R}^m), \quad \text{e.g. } \begin{array}{l} X = L^p(\Omega;\mathbb{R}^m) \\ X = W^{1,p}(\Omega;\mathbb{R}^m) \\ X = W^{1,p}_g(\Omega;\mathbb{R}^m) \end{array}$$

endowed with different topology (strong, weak etc.). The type of functionals we will deal with is integral functionals, i.e.

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, u, \nabla u)\, dx.$$

Note: techniques to find the Γ-limit changes from problem to problem. Intuition is important but sometimes might be misleading

This is an idea of the program of the course:

1. definiton and properties of Γ-convergence
2. Relaxation in Lebesgue/Sobolev spaces
3. (variational) Homogenization
4. discrete-to-continuum limit
5. gradient theory of phase transitions

We see one of the "applications" of Γ-convergence and how we cannot trust too much our intuition.

Ex (Homogenization): asymptotic description of variational problems with highly oscillating solutions.

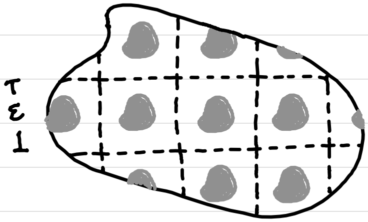
In its main formulation, we have an integral functional whose density depends explicity on the space variable x;

$$F_\varepsilon(u) = \int_\Omega g_\varepsilon(x, u, \nabla u)\, dx.$$

We say that it is nonhomogeneous, since in different points of Ω, the way in which g_ε acts on u and ∇u is different.
If the dependence in x is e.g. periodic, with period ε, when taking $\varepsilon \to 0$ we expect an "homogeneous" behavior (i.e. not depending on x);

$$F_\varepsilon \xrightarrow[\varepsilon \to 0]{\Gamma} F_0(u) = \int_\Omega g_{hom}(u, \nabla u)\, dx.$$

(i) nonhomogeneous Poisson's equation: Suppose we want to study the conductivity of a nonhogeneous material $\Omega \subseteq \mathbb{R}^n$.



$f : \Omega \to \mathbb{R}$ source term

$a : \mathbb{R}^n \to \{\alpha, \beta\}$ $\alpha, \beta > 0$ conductivity $(0,1)^n$- periodic.

The conductivity of the material is α in the gray regions and β elsewhere.
As $\varepsilon \to 0$ the periodicity of the conductivity increases.

The way electricity is conducted through Ω can be described by

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \\ u_\varepsilon|_{\partial\Omega} = 0 \end{cases} \iff \min_{u|_{\partial\Omega}=0} \overset{=: F_\varepsilon(u)}{\int_\Omega \frac{1}{2} a\left(\frac{x}{\varepsilon}\right)|\nabla u|^2 - fu\,dx}$$

The dependence on x inside the divergence, and the fact that $a(\cdot/\varepsilon)$ is oscillating faster and faster, makes the problem hard.

Via homog. techniques, we can prove that

$$F_\varepsilon(u) \xrightarrow{\Gamma} F_0(u) := \int_\Omega (A_{hom}\nabla u)\cdot\nabla u - fu\,dx$$

for some explicit, elliptic $A_{hom} \in \mathbb{R}^{n\times n}$. So that, as $\varepsilon \ll 1$ the electricity is conducted (up to small errors) as the solution to the following effective problem

$$\begin{cases} -\operatorname{div}(A_{hom}\nabla u_0) = f \\ u_0|_{\partial\Omega} = 0 \end{cases}.$$

(ii) a 1-dimensional case: let $n=1$ and $\Omega = (0,1)$,

$$F_\varepsilon(u) = \int_0^1 \frac{1}{2} a\left(\frac{x}{\varepsilon}\right)|u'|^2 - fu\,dx, \quad u \in H^1_0(0,1),\ f \in L^2(0,1).$$

(using weak H^1 topology)
Assume that $F_\varepsilon \xrightarrow{\Gamma} F_0$ and try to find how F_0 looks like.

- (wrong) intuition: by Riemann-Lebesgue lemma, we know that $a(\cdot/\varepsilon) \overset{*}{\underset{L^\infty(0,1)}{\rightharpoonup}} \fint a =: \bar{a}$.

Since $u \in H^1_0(0,1)$, $|u'|^2 \in L^1(0,1)$ so, for every fixed u we have

$$\int_0^1 \frac{1}{2} a\left(\frac{x}{\varepsilon}\right)|u'|^2 - fu \, dx \xrightarrow[\varepsilon \to 0]{} \int_0^1 \frac{1}{2} \bar{a} |u'|^2 - fu \, dx =: G(u).$$

So we may bet on the fact that G is the Γ limit of F_ε.
That's not true, G is only the pointwise limit.

- (correct) intuition: since Γ-convergence implies conv. of (global) minimizers, we study the convergence of minimizers of F_ε ($\exists!$ because of convexity).

$$\begin{cases} -\left(a\left(\frac{x}{\varepsilon}\right) u_\varepsilon'(x)\right)' = f(x) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases}$$ is the unique min. of F_ε in H^1_0.

By integration

$$u_\varepsilon'(x) = \frac{1}{a\left(\frac{x}{\varepsilon}\right)} F(x), \quad \text{where} \quad F(x) = -\int_0^x f(t)\,dt + c.$$

By Riemann-Lebesgue (and continuity of F)

$$u_\varepsilon'(x) \underset{L^2}{\rightharpoonup} \fint \frac{1}{a} F(x).$$

Since $\|u_\varepsilon\|_{L^2} \leq C$ (easy to check) $\Rightarrow$ $u_\varepsilon \underset{H^1}{\rightharpoonup} u_0$ for some u_0 such that $u_0' = \fint \frac{1}{a} F$ and $u_0(0) = u_0(1) = 0$.
Thus u_0 solves

$$\begin{cases} -\left(\left(\fint \frac{1}{a}\right)^{-1} u_0'(x)\right)' = f(x) \\ u_0(0) = u_0(1) = 0 \end{cases} \Leftrightarrow u_0 \text{ is the (unique] min. of } F_0.$$

But F_0 s.t. this equation is its Euler-Lagrange eq. is

$$F_0(u) = \int_0^1 \frac{1}{2} a^* |u'|^2 - fu \, dx, \quad a^* := \left(\fint \frac{1}{a}\right)^{-1}$$ harmonic mean of a.

So, by studying the minimizers of F_ε we found its Γ-limit, which is a bit counterintuitive since it is different from the pointwise limit, $F_0 \neq G$ e.g. when $\{\alpha, \beta\} = \{1,2\}$.