Lecture 8: 03.06.2024

We continue our study of (seq.) weak lsc for functionals acting on the gradient (defined in Sobolev spaces).
We saw that this is somehow related to quasiconvexity of the density.

We prove that (under growth conditions) quasiconvexity $\Rightarrow$ rank-1 convexity. This we will be used to deduce local Lipschitzia--nity of f (since rank-1 convex functions are actually locally Lipschitz.
Since this is a bit beyond the goal of this course, we only see some ideas of the proof. You are encouraged to fill the gaps.

Lemma 1: f quasiconvex and $f(z) \leq c(|z|^p+1) \Rightarrow f$ rank-1 convex.

Proof (ideas): we can reduce to the case $A = (1-\lambda) e_1 \otimes e_1$ and $B = -\lambda e_1 \otimes e_1$, $\lambda \in (0,1)$. Consider $E = (0,1)^n$ and define

$$h(t) = \begin{cases} (1-\lambda)t & 0 \leq t < \lambda \\ \lambda - \lambda t & \lambda \leq t \leq 1 \end{cases} \quad \text{extended it 1-periodically,}$$

and $u_j(x) := e_1 \frac{1}{j} h(jx_1)$, thus $\nabla u_j \in \{A, B\}$ a.e..
This u_j does not attain zero boundary conditions so we use a cut-off close to the boundary of the cube; define

$$v_j(x) = \min\{u_j(x), \operatorname{dist}(x, E^c)\} \in W_0^{1,\infty}(E; \mathbb{R}^n).$$

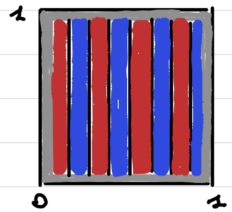
Let $E_j := \{x \in E : \operatorname{dist}(x, E^c) > 1/j\}$, then $v_j = u_j$ on E_j since $\|h\|_{L^\infty} < 1$. By quasiconvexity and growth conditions we get

$$0 = f\left(⨍_E \nabla v_j\right) \leq ⨍_E f(\nabla v_j(x))\,dx$$

$$= \int_{E \setminus E_j} f(\nabla v_j(x))\,dx + \int_{E_j} f(\nabla u_j(x))\,dx$$

taking the limit, since $f(\nabla u_j) \rightharpoonup \lambda f(A) + (1-\lambda) f(B)$ we get the result. $\square$

Rmk (heuristic explanation for fine lamination): in the proof above from two rank-1 connected matrices A, B, we defined an oscillating function u_j whose gradient is either A or B. This is called a laminate.



Laminates cannot attain affine boundary conditions, for this reason (and since quasiconvexity works with test functions with boundary conditions) we used a cut-off argument to define v_j.
To recover rank-1 convexity we want that

$$\int_E f(\nabla v_j(x))\,dx \sim \int_E f(\nabla u_j(x))\,dx,$$

which means that the "energy" of the cut-off should be negligible. Since this goes as $\|u_j\|_{L^\infty} \cdot r^{-1} \cdot |E_j|$, where r is the "thickness" of the cut-off region E_j, this term goes to zero only if $\|u_j\|_{L^\infty} \to 0$.
This is possible only if the number of oscillations goes to ∞.

Rmk: if f is rank-1 convex and $0 \leq f(A) \leq C(|A|^p + 1)$ $\forall A \in \mathbb{R}^{m\times n}$ for some $C > 0$, the f is locally Lipschitz and

$$|f(A) - f(B)| \leq \tilde{C}(1 + |A|^{p-1} + |B|^{p-1})|A - B|, \quad \forall A, B \in \mathbb{R}^{m\times n}$$

for som $\tilde{C} > 0$.

We need the following version of Vitali covering. This is some-times called Vitali-Lebesgue covering. Since it is a standard re-sult, we omit the proof.

Lemma 2 (Thm 4.4 [BD,1998]): let $\Omega \subseteq \mathbb{R}^n$ bounded, open. Let $\mathcal{U}$ be a family of closed balls in Ω.
If for a.e. $x \in \Omega$, $\inf\{\operatorname{diam}(B) : B \ni x, B \in \mathcal{U}\} = 0$ then $\exists \{B_j\} \subseteq \mathcal{U}$ disjoint s.t. $|\Omega \setminus \bigcup_j B_j| = 0$.

Theorem 3: let $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ be Borel s.t. $0\le f(A)\le c(|A|^p+1)$ $\forall A\in\mathbb{R}^{m\times n}$ and let $F(u)=\int_\Omega f(\nabla u(x))\,dx$.
Then F is (seq.) weak-lsc $\iff$ f is quasiconvex.

Proof: ($\Rightarrow$) by Theorem 1 of last lecture, for every $Q\subseteq\mathbb{R}^n$ cube $f(⨍_Q\nabla v)\le ⨍_Q f(\nabla v)$ $\forall v\in W^{1,p}_{loc}$ with Q-periodic gradient. Let $E\subseteq\Omega$ open, bounded and $\varphi\in C^\infty_c(E;\mathbb{R}^m)$. $\exists z\in\mathbb{R}^m$, $\lambda>0$ s.t. $z+\lambda E\subseteq Q$ (for some cube Q), define

$$\psi(x)=\begin{cases}\lambda\varphi\left(\frac{x-z}{\lambda}\right) & x\in(z+\lambda E)\\ 0 & \text{otherwise}\end{cases}$$

and extend ψ Q-periodically.
Taking $v(x):=Ax+\psi(x)\in W^{1,p}_{loc}$, $\nabla v=A+\nabla\psi$ is Q-periodic, by divergence Theorem $\int_Q\nabla\psi(x)\,dx=0\Rightarrow ⨍_Q\nabla v=A$, then

$$|Q|f(A)\le\int_Q f(A+\nabla\psi(x))\,dx$$

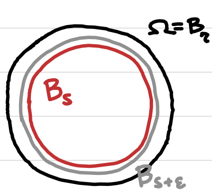
$$=|Q\setminus(z+\lambda E)|f(A)+\int_{z+\lambda E}f\left(A+\nabla\varphi\left(\tfrac{x-z}{\lambda}\right)\right)dx.$$

By the change of variable $y=\frac{x-z}{\lambda}$ we obtain

$$\lambda^n|E|f(A)=|\lambda E|f(A)\le\lambda^n\int_E f(A+\nabla\varphi(y))\,dy.$$

($\Leftarrow$) we work in two steps.

Step 1 (u linear, Ω ball): let $\Omega=B_r$, $u(x)=Ax$, $A\in\mathbb{R}^{m\times n}$.
Let $u_j\underset{W^{1,p}}{\rightharpoonup}u$ in Ω.



Again, since quasiconvexity works with affine boundary conditions, we want to glue u_j with u close to $\partial\Omega$.
For $0<s<r$ and $\varepsilon>0$ define a cut-off
$\varphi(x)=1\wedge\frac{1}{\varepsilon}\operatorname{dist}(x,B_s)$ and $v_j(x)=u_j(x)(1-\varphi(x))+u(x)\varphi(x)$.
By quasiconvexity and growth conditions we get

$$F(u)=|\Omega|f(A)\le\int f(A+(\nabla v_j(x)-A))\,dx=\int f(\nabla v_j(x))\,dx,$$

If we prove that $F(v_j)$ (i.e. the right-hand side above) is "close" to $F(u_j)$ we are done.
By Leibniz rule, $\nabla v_j(x) = \nabla u_j(x) + (A - \nabla u_j(x))\varphi(x) + Ax - u_j(x) \otimes \nabla\varphi(x)$. In particular $\nabla v_j(x) = \nabla u_j(x)$ if $x \in B_s$ and $\nabla v_j(x) = A$ if $x \in B_r \setminus B_{s+\varepsilon}$. So

$$\int_{B_r} f(\nabla v_j(x))dx = \int_{B_s} f(\nabla u_j(x))\,dx + \int_{B_r \setminus B_{s+\varepsilon}} f(A)\,dx +$$

$$+ \int_{B_{s+\varepsilon}\setminus B_s} f(\nabla u_j(x) + (A-\nabla u_j(x))\varphi(x) + Ax - u_j(x) \otimes \nabla\varphi(x))dx.$$

By the growth conditions, and since $|\nabla\varphi(x)| \equiv \varepsilon^{-1}$ on $B_{s+\varepsilon}\setminus B_s$ there exists a constant C_1 s.t.

$$\int_{B_r} f(\nabla v_j(x))\,dx \le \int_{B_r} f(\nabla u_j(x))\,dx + \int_{B_r\setminus B_s} f(A)\,dx +$$

$$+ \int_{B_{s+\varepsilon}\setminus B_s} C_1(1 + |\nabla u_j(x)|^p + |A|^p)\,dx + \frac{C_1}{\varepsilon^p}|Ax - u_j(x)|^p dx.$$

The most delicate term is the third. We want that term to disappear as $j \to +\infty$ and $\varepsilon \to 0$. Unfortunately, since $|\nabla u_j|^p$ is a bounded sequence in $L^1(\Omega)$ it may not have a weak limit. To control this term we need to use compactness of Borel measures. We recall the following facts about Borel measures:

(i) Let μ_j, μ be Borel measures on Ω. $\mu_j \xrightarrow{*} \mu$ iff
(ii) $\int_\Omega \varphi(x)\,d\mu_j(x) \xrightarrow{j\to+\infty} \int_\Omega \varphi(x)\,d\mu(x)$ $\forall \varphi \in C(\overline{\Omega})$.
(iii) if $\sup_j \mu_j(\Omega) < +\infty \Rightarrow \exists \{j_k\}_k, \mu$ s.t. $\mu_{j_k} \xrightarrow{*} \mu$
(iv) if $\mu_j \xrightarrow{*} \mu \Rightarrow \overline{\lim_{j\to+\infty}}\, \mu_j(K) \le \mu(K)$ $\forall K \subset\subset \Omega$
(v) $\mu(\partial B_s) = 0$ for a.e. $s > 0$.

Let $\mu_j : \Omega \to [0, +\infty)$ be the Borel measure defined as $\mu_j(E) := \int_E C_1(1 + |A| + |\nabla u_j(x)|^p\,dx$. Since $u_j \xrightarrow{W^{1,p}} u$, $\sup_j \mu_j(\Omega) < +\infty$, then by (iii) $\mu_j \xrightarrow{*} \mu$ uts, for some Borel measure μ.
Taking the limit as $j \to +\infty$, since $u_j \longrightarrow u$ in L^p and by (iv)

$$\underline{\lim_{j\to+\infty}} \int_{B_r} f(\nabla v_j(x))\,dx \le \underline{\lim_{j\to+\infty}} \int_{B_r} f(\nabla u_j(x))\,dx + \int_{B_r\setminus B_s} f(A)\,dx +$$

$$+ \mu(\overline{B_{s+\varepsilon}\setminus B_s})$$

By (v) we can assume that $\mu(\partial B_s)=0$, so by Monotone convergence $\mu(\overline{B_{s+\varepsilon}\setminus B_{s-\varepsilon}})\to 0$ as $\varepsilon\to 0$.
Taking then first the limit as $\varepsilon\to 0$ and the as $s\to r$ we finally obtain

$$F(u)\leq \varliminf_{j\to+\infty}\int_\Omega f(\nabla u_j(x))\,dx=\varliminf_{j\to+\infty} F(u_j).$$

Step 2 (the general case): $u\in W^{1,p}(\Omega;\mathbb{R}^m)$, $u_j\rightharpoonup u$. Let $\varepsilon>0$, define

$$\mathcal{U}^\varepsilon:=\left\{\overline{B_r(x_0)}\subset\Omega:\int_{B_r(x_0)}|\nabla u-\nabla u(x_0)|^p\,dx\leq\varepsilon^p|B_r|\right\}.$$

For every Lebesgue point x_0 of ∇u $B_r(x_0)\in\mathcal{U}^\varepsilon$ $\forall r<r_0$ for some r_0 sufficiently small.
By Lemma 2, $\exists\{B_k\}\subseteq\mathcal{U}^\varepsilon$ disjoint s.t. $|\Omega\setminus\bigcup_k B_k|=0$ and denote $x_k\in\Omega$ the center of B_k.
We define $\psi\in L^p(\Omega;\mathbb{R}^{m\times n})$ with $\psi(x)=\nabla u(x_k)=:A_k\in\mathbb{R}^{m\times n}$ if $x\in B_k$. Then

$$(*)\quad\int_\Omega|\nabla u(x)-\psi(x)|^p=\sum_k\int_{B_k}|\nabla u(x)-A_k|^p\,dx\leq\varepsilon^p\sum_k|B_k|=\varepsilon^p|\Omega|.$$

We approximated u with affine functions in balls. In each ball we can use the result from step 1: $\forall N\in\mathbb{N}$

$$\sum_{k=1}^N\int_{B_k}f(\nabla u_j(x))\,dx=\sum_{k=1}^N\int_{B_k}f(\nabla u_j(x))-f(\nabla u_j(x)-\nabla u(x)+A_k)\,dx+$$

$$(*_1)\qquad +\sum_{k=1}^N\int_{B_k}f(\nabla u_j(x)-\nabla u(x)+A_k)-f(A_k)\,dx+$$

$$+\sum_{k=1}^N\int_{B_k}f(A_k)-f(\nabla u(x))\,dx+\sum_{k=1}^N\int_{B_k}f(\nabla u(x))\,dx.$$

We control the first three terms of the right-hand side:
- Since $u_j-u+A_kx\rightharpoonup A_kx$ in $W^{1,p}(B_k;\mathbb{R}^m)$, by step 1

$$\varliminf_{j\to+\infty}\sum_{k=1}^N\int_{B_k}f(\nabla u_j(x)-\nabla u(x)+A_k)-f(A_k)\,dx\geq 0.$$

• by quasiconvexity and growth conditions, f is rank-1 convex then $|f(A)-f(B)| \leq C_1(1+|A|^{p-1}+|B|^{p-1})|A-B| \quad \forall A,B \in \mathbb{R}^{m\times n}$ so

$$\left|\int_\Omega f(\nabla u_j(x)) - f(\nabla u_j(x) - \nabla u(x) + \psi(x))\, dx\right| \leq$$

$$\leq C_1 \int_\Omega (1+|\nabla u_j(x) - \nabla u(x) + \psi(x)|^{p-1} + |\nabla u_j|^{p-1})|\nabla u(x) - \psi(x)|\, dx$$

$$\overset{\text{Hölder}}{\leq} \tilde{C}_1 \left(\int_\Omega 1+|\nabla u_j(x)|^p + |\nabla u(x)|^p + |\psi(x)|^p\, dx\right)^{1/p} \|\nabla u - \psi(x)\|_{L^p(B_k)}$$

$$\overset{(*)}{\leq} \tilde{\tilde{C}}_1 M \varepsilon |\Omega|$$

where $\tilde{\tilde{C}}_1$ is some constant and $M = \sup_j \left(\int_\Omega 1+|\nabla u_j|^p + |\nabla u|^p + |\psi|^p dx\right)$.

• Working identically we get

$$\left|\int_\Omega f(\psi(x)) - f(\nabla u(x))\, dx\right| \leq \hat{C}\varepsilon|\Omega|.$$

Taking the limit in $(*_1)$ as $j \to +\infty$ we then get

$$\varliminf_{j\to+\infty} \int_\Omega f(\nabla u_j(x))dx \geq \varliminf_{j\to+\infty} \sum_{k=1}^N \int_{B_k} f(\nabla u_j(x))\, dx$$

$$\geq \varliminf_{j\to+\infty} \sum_{k=1} \int_{B_k} f(\nabla u(x))\, dx +$$

$$- \varlimsup_{j\to+\infty} \left|\int_\Omega f(\nabla u_j(x)) - f(\nabla u_j(x) - \nabla u(x) + \psi(x))\, dx\right| +$$

$$+ \varliminf_{j\to+\infty} \sum_{k=1}^N \int_{B_k} f(\nabla u_j(x) - \nabla u(x) + A_k) - f(A_k)\, dx +$$

$$- \varlimsup_{j\to+\infty} \left|\int_\Omega f(\psi(x)) - f(\nabla u(x))\, dx\right|$$

$$\geq \varliminf_{j\to+\infty} \sum_{k=1}^N \int_{B_k} f(\nabla u(x))\, dx - (\tilde{\tilde{C}}_1 M + \hat{C})\varepsilon|\Omega|,$$

where we also used that $\varliminf_{j\to+\infty} a_j = -\varlimsup_{j\to+\infty} -a_j \geq -\varlimsup_{j\to+\infty} |a_j|$ for any sequence a_j.

So we have that

$$\sum_{k=1}^{N} \int_{B_k} f(\nabla u(x))\,dx \leq \int_\Omega f(\nabla u_j(x))\,dx + (\tilde{C}M + \hat{C})\,\varepsilon\,|\Omega|,$$

and by monotone convergence sending $N \to +\infty$ and by arbitrariness of ε we conclude the proof. $\square$

In the end, as in the Lebesgue case, if we have also growth conditions from below our functionals are weakly coercive and (seq.) weak lsc coincides with strong-L^p lsc.

Theorem: let $f: \mathbb{R}^{m\times n} \to \mathbb{R}$ be Borel s.t. $c_1|A|^p \leq f(A) \leq c_2(|A|^p+1)$ $\forall A \in \mathbb{R}^{m\times n}$ and let $F(u) = \int_\Omega f(\nabla u(x))\,dx$.
F is lsc (wrt strong L^p) $\iff$ f is quasiconvex.

proof: ($\Rightarrow$) immediate by Rellich. Indeed, $u_j \xrightarrow{W^{1,p}} u \Rightarrow u_j \xrightarrow{L^p} u$ so $F(u) \leq \liminf_{j\to+\infty} F(u_j) \Rightarrow$ by Theorem 3 f is quasiconvex.

($\Leftarrow$) let $u_j \to u$ in L^p s.t. $\liminf_j F(u_j) < +\infty$.
Then $\exists\, u_{j_k} \to u$ s.t. $\lim_k F(u_{j_k}) = \liminf_j F(u_j) \Rightarrow F(u_{j_k}) \leq C_1$.
By growth condition

$$c_1 \int_\Omega |\nabla u_{j_k}(x)|^p\,dx \leq F(u_{j_k}) \leq C_1,$$

so $u_{j_k} \underset{W^{1,p}}{\rightharpoonup} u$. Since f has growth conditions and is quasiconvex

$$F(u) \leq \liminf_{k\to+\infty} F(u_{j_k}) = \liminf_{j\to+\infty} F(u_j).$$

If $\liminf_{j\to+\infty} F(u_j) = +\infty$ there is nothing to prove. $\square$

Lecture 9: 10.06.2024

Localization method

Main reference for this part:

- Ch. 9 [BD, 1998], integral representation
- Ch. 10, 11, 12 [BD, 1998], localization method for integra functionals
- Ch. 16 [B, 2000], localization methods in a nutshell

We recall that we went to study $F_j : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ as

$$F_j(u) = \int_\Omega f_j(x, \nabla u(x))\, dx$$

with $f_j : \Omega \times \mathbb{R}^{m\times n} \to [0, +\infty)$ Borel and s.t. $c_1 |A|^p \leq f_j(x, A) \leq c_2(|A|^p + 1)$.
We know that (uts) $\exists\ \Gamma\text{-}\lim\limits_{j\to+\infty} F_j =: F$, but we don't know whether F has an integral representation, i.e. whether $\exists\ f$ s.t.

$$F(u) = \int_\Omega f(x, \nabla u(x))\, dx.$$

__Strategy__: we extend F_j as a functional on $(u, U) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega)$ where $\mathcal{A}(\Omega) = \{U \subset \Omega : \text{open}\}$, as

$$F_j(u, U) = \int_U f_j(x, \nabla u(x))\, dx.$$

(1) proving that (uts) $\exists\ \Gamma\text{-}\lim\limits_{j\to+\infty} F_j(\cdot, U) =: F(\cdot, U)\ \forall\, U \in \mathcal{A}(\Omega)$

(2) proving that $F(u, \cdot)$ is (the restriction of) a Borel measure (coupling with some other properties) $\Rightarrow$ $F(u, \cdot)$ has a density, so has an integral form.

__Rmk__: when needed, we will extend $F_j(\cdot, U)$ to functionals on the whole $L^p(\Omega; \mathbb{R}^m)$ by putting $F_j(u, U) = +\infty$ if $u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(\Omega; \mathbb{R}^m)$.

Integral representation

Point (1) in the strategy above can be achieved by exploiting the compactness property of Γ-convergence, as we will see in the sequel.

To obtain point (2) we need the following result, which we will apply to our Γ-limit, once we proved it exists (at least along a subsequence).

Theorem 1: Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open and let $1 \le p < \infty$. Let $F : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty)$ satisfy:

(i) $F(u,U) = F(v,U)$ $\forall\, u = v$ a.e. on U, $\forall\, U \in \mathcal{A}(\Omega)$;

(ii) $\forall\, u \in W^{1,p}(\Omega;\mathbb{R}^m)$, the set map $U \mapsto F(u,U)$ is the restriction of a Borel measure on $\mathcal{A}(\Omega)$;

(iii) $\exists\, c > 0,\ a \in L^1(\Omega)$ s.t. $F(u,U) \le c \int_U a(x) + |\nabla u(x)|^p \, dx$ $\forall\, u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $U \in \mathcal{A}(\Omega)$;

(iv) $F(u+z,U) = F(u,U)$ $\forall\, u \in W^{1,p}(\Omega;\mathbb{R}^m), U \in \mathcal{A}(\Omega), z \in \mathbb{R}^m$;

(v) $\forall\, U \in \mathcal{A}(\Omega)$, $F(\cdot,U)$ is (seq.) weakly lsc.

Then $\exists\, f : \Omega \times \mathbb{R}^m \to [0,+\infty)$ a Carathéodory function s.t. $0 \le f(x,A) \le c(a(x) + |A|^p)$ $\forall\, A \in \mathbb{R}^{m\times n}$, a.e. $x \in \Omega$ and

$$F(u,U) = \int_U f(x,\nabla u(x))\, dx,$$

$\forall\, u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $U \in \mathcal{A}(\Omega)$.

proof: We work in several steps.

Step 1 (defining f): let $A \in \mathbb{R}^{m\times n}$. By (ii) and (iii), $F(Ax,\cdot)$ extends to an absolutely continuous Borel measure. By Radon-Nykodim $\exists\, g_A \in L^1(\Omega)$ s.t.

$$F(Ax,U) = \int_U g_A(x)\,dx, \quad \forall\, U \in \mathcal{A}(\Omega).$$

We write $f(x,A) := g_A(x)$. By (iii), $\forall y \in \Omega, \rho>0$ s.t. $B_\rho(y) \subset \Omega$

$$\fint_{B_\rho(y)} f(x,A)\,dx = \frac{1}{|B_\rho|} F(Ax, B_\rho(y)) \leq c \fint_{B_\rho(y)} a(x) + |A|^p\,dx.$$

Taking the $\overline{\lim}_{\rho\to 0^+}$ for every Lebesgue point y of $f(\cdot,A)$ and a we get $0 \leq f(y,A) \leq c(a(y) + |A|^p)$ for a.e. $y \in \Omega$.

Step 2 (integral representation on piecewise affine functions): let $u|_U(x) = \sum_{j=1}^N \chi_{U_j}(x)(A_j x + z_j)$, with $A_j \in \mathbb{R}^{m\times n}$, $z_j \in \mathbb{R}^m$, $U_j \in \mathcal{A}(\Omega)$ disjoint and $|U \setminus \bigcup_{j=1}^N U_j| = 0$, where $N \in \mathbb{N}$. Then

$$F(u,U) \overset{(ii)}{=} \sum_{j=1}^N F(u,U_j) = \sum_{j=1}^N F(A_j x + z_j, U_j)$$

$$\overset{(iv)}{=} \sum_{j=1}^N F(A_j x, U_j)$$

$$\overset{\text{Step 1}}{=} \sum_{j=1}^N \int_{U_j} f(x,A_j)\,dx = \int_U f(x,\nabla u(x))\,dx.$$

Step 3 (rank-1 convexity of $f(x,\cdot)$): it is sufficient to prove

$$F((tA+(1-t)B)x, B_\rho(y)) \leq t\,F(Ax, B_\rho(y)) + (1-t)\,F(Bx, B_\rho(y))$$

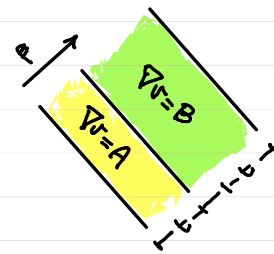
$\forall\, A,B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A-B)=1$, $t\in(0,1)$, $y\in\Omega$, $\rho>0$: $B_\rho(y)\subset\Omega$. Indeed, by taking the $\overline{\lim}_{\rho\to 0^+}$ for every Lebesgue point y of $f(\cdot, tA+(1-t)B)$, $f(\cdot,A)$ and $f(\cdot,B)$ we get that

$$f(y, tA+(1-t)B) \leq t\,f(y,A) + (1-t)\,f(y,B) \quad \text{a.e. } y\in\Omega.$$

Let $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}^n$ s.t. $A-B = \alpha \otimes \beta$ and $v \in W^{1,\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ defined as

$$v(x) = \begin{cases} Bx + (\beta\cdot x)\alpha - (1-t)j\alpha, & j \le \beta\cdot x < j+t,\ j\in\mathbb{Z} \\ Bx + (1+j)t\alpha, & j+t \le \beta\cdot x < j+1,\ j \in \mathbb{Z} \end{cases}$$

and let $E_A := \{x \in \mathbb{R}^n : \nabla v(x) = A\}$ and $E_B := \{x \in \mathbb{R}^n : \nabla v(x) = B\}$.



We define $u_j(x) := \frac{1}{j} v(jx)$ and notice that (by the Remark on oscillating sequences of lecture 7) we have $u_j \xrightarrow{w^{1,p}_{loc}} (tA+(1-t)B)x$.

Notice also that $x \in \frac{1}{j}E_A \Leftrightarrow \nabla u_j(x) = A$ and analogously $x \in \frac{1}{j}E_B \Leftrightarrow \nabla u_j(x) = B$. Moreover by Riemann-Lebesgue lemma $\chi_{\frac{1}{j}E_A} \xrightarrow{*} t$ and $\chi_{\frac{1}{j}E_B} \xrightarrow{*} 1-t$. So

$$F((tA+(1-t)B)x, B_\rho(y)) \overset{(v)}{\le} \lim_{j\to+\infty} F(u_j, B_\rho(y))$$

$$\overset{(ii)}{=} \lim_{j\to+\infty} \left(F(u_j, B_\rho(y) \cap \tfrac{1}{j}E_A) + F(u_j, B_\rho(y) \cap \tfrac{1}{j}E_B) \right)$$

$$\overset{\text{Step 2}}{=} \lim_{j\to+\infty} \left(\int_{B_\rho(y)} \chi_{\frac{1}{j}E_A}(x) f(x,A)\,dx + \int_{B_\rho(y)} \chi_{\frac{1}{j}E_A}(x) f(x,A)\,dx \right)$$

$$= t\int_{B_\rho(y)} f(x,A)\,dx + (1-t)\int_{B_\rho(y)} f(x,B)\,dx$$

which yields the claims. Rank-1 convexity and growth conditions implies (in particular) that $f(x,\cdot)$ is continuous.

Step 4 (inequality): by step 3 and since $f \ge 0$ and with p-growth conditions, the functional $\int_U f(x,\cdot)\,dx$ is strong-L^p lsc (see for instance the Remark about strong lsc of lecture 5). Let $U \subset\subset \Omega$, $U \in \mathcal{A}(\Omega)$ and u_j be piecewise affine in U and s.t. $u_j \xrightarrow{w^{1,p}} u$ (e.g. take $\tilde{u}_j \in C^\infty(\bar{\Omega};\mathbb{R}^m)$, consider a triangulation of amplitude $\frac{1}{j}$ and take u_j the piecewise affine interpolation of $\tilde{u}_j$ in this triangulation). Then

$$F(u,U) \overset{(v)}{\leq} \lim_{j\to+\infty} F(u_j,U) \overset{\text{Step 2}}{=} \lim_{j\to+\infty} \int_U f(x,\nabla u_j(x))\,dx$$

$$= \int_U f(x,\nabla u(x))\,dx$$

where the last step is consequence of strong-L^1 continuity.

Step 5 (final): the inequality proved in Step 4 and the equality for piecewise affine functions is sufficient to prove the claim.
Indeed, define $G(v,U) := F(u+v,U)$. Since G complies with the hypotheses of the Theorem, the results proved in Steps 1–4 hold also for G. That is, $\exists\, \psi$ Carathéodory s.t. $G(v,U) \leq \int_U \psi(x,\nabla v(x))\,dx$ with equality holding true for piecewise affine v.
Let $u_j \overset{w^*}{\to} u$ a sequence of piecewise affine functions as in Step 4, then

$$\int_U \psi(x,0)\,dx \overset{\text{Step 1}}{=} G(0,U) = F(u,U) \overset{\text{Step 4}}{\leq} \int_U f(x,\nabla u(x))\,dx =$$

$$\overset{\text{str. cont.}}{=} \lim_{j\to+\infty} \int_U f(x,\nabla u_j(x))\,dx \overset{\text{Step 2}}{=} \lim_{j\to+\infty} F(u_j,U) =$$

$$= \lim_{j\to+\infty} G(u_j-u,U) \overset{\text{Step 4}}{\leq} \lim_{j\to+\infty} \int_U \psi(x,\nabla u_j(x)-\nabla u(x))\,dx =$$

$$= \int_U \psi(x,0)\,dx.$$

This implies the claim for every $U \in \mathcal{A}(\Omega)$, $U \subset\subset \Omega$. By inner regularity of Borel measure we get that, $\forall\, U \in \mathcal{A}(\Omega)$

$$F(u,U) = \sup\left\{ F(u,U') : U' \subset\subset \Omega,\ U' \in \mathcal{A}(\Omega) \right\}$$

$$= \sup\left\{ \int_{U'} f(x,\nabla u(x))\,dx : U' \subset\subset \Omega,\ U' \in \mathcal{A}(\Omega) \right\}$$

$$= \int_U f(x,\nabla u(x))\,dx. \qquad \square$$

Rmk: in the course we consider Ω Lipschitz and $1<p<\infty$. These two things are not needed in this result.

In the proof we also obtained something more that what it is claimed in the statement, since in Step 3 we proved that $f(x,\cdot)$ is rank1-convex.

An immediate corollary of Theorem 1 can be obtained when deling with homogeneous functionals.

Corollary 2: let all the hypotheses of Theorem 1 hold and additionally assume that

$$F(Ax, B_\rho(y)) = F(Ax, B_\rho(z))$$

$\forall A \in \mathbb{R}^{m\times n}$, $y,z \in \Omega$, $\rho>0$: $B_\rho(y), B_\rho(z) \subseteq \Omega$. Then the result of Theorem 1 holds and $f(x,A) \equiv f(A)$ and f is quasiconvex.

Exercise: try to prove Corollary 2. Hint: send $\rho \to 0$ for every y,z Lebesgue points of $f(\cdot, A)$.

Compactness of the Γ-limit

Now that we have a tool to claim that functionals (complying with some assumptions) are actually integral functionals, our next objective is to prove that (wts) $\exists\ \Gamma\text{-}\lim_{j\to+\infty} F_j(\cdot,U) =: F(\cdot,U)\ \forall\, U\in\mathcal{A}(\Omega)$.

This is not an immediate consequence of compactness of Γ-convergence since the subsequence (along which the Γ-limit exists) may depend on the set U.
To prove it, we work with Γ-liminf and Γ-limsup (that always exist) showing that they are inner regular as set maps.

Once the existence of F is proved, we will check that it satisfies conditions (i)–(v) of Theorem 1.
For both of these goals, the next result is crucial.

Lemma 3 (fundamental estimate): let $f_j : \Omega\times\mathbb{R}^{m\times n}\to[0,+\infty)$ be Borel functions s.t. $\exists\, c_1, c_2>0$ with

$$c_1|A|^p \le f_j(x,A)\le c_2(|A|^p+1),\quad \forall A\in\mathbb{R}^{m\times n},\ \text{a.e. } x\in\Omega,\ \forall j$$

and let $F_j : W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega)\to[0,+\infty)$ defined as

$$F_j(u,U) = \int_U f_j(x,\nabla u(x))\,dx.$$

Denote

$$F'(u,U) = \Gamma(L^p)\text{-}\liminf_{j\to+\infty} F_j(u,U),\quad F''(u,U) = \Gamma(L^p)\text{-}\limsup_{j\to+\infty} F_j(u,U).$$

Then $\forall\, u\in W^{1,p}(\Omega;\mathbb{R}^m)$, $U', U, V\in\mathcal{A}(\Omega)$ with $U'\subset\subset U$, it holds

$$F'(u,U'\cup V)\le F'(u,U)+F''(u,V),\quad F''(u,U'\cup V)\le F''(u,U)+F''(u,V).$$

proof: we prove the statement for F'', the one for F' is completely analogous.

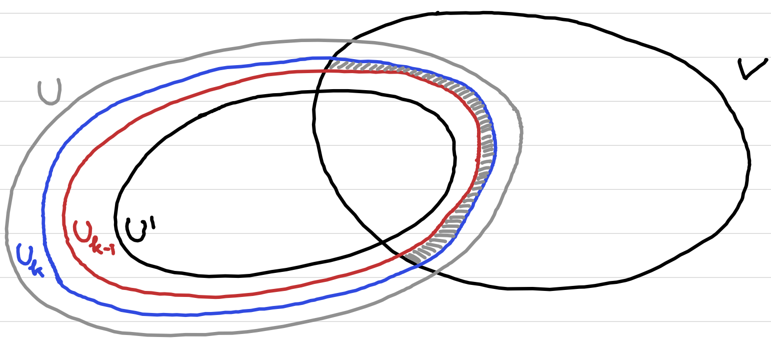
Let $u_j, v_j \in W^{1,p}(\Omega;\mathbb{R}^m)$ s.t. $u_j \xrightarrow{L^p} u$, $v_j \xrightarrow{L^p} u$ and

$$\lim_{j\to+\infty} F_j(u_j,U) = F''(u,U), \qquad \lim_{j\to+\infty} F_j(v_j,V) = F''(u,V).$$

The fact that "recovery sequences" such as u_j and v_j above exist is a direct consequence of the definition of Γ-$\overline{\lim}$.
We want to "glue together" the recovery sequence in U and that in V in an energy convenient way.
Let $\delta := \mathrm{dist}(U', \partial U) > 0$, $N \in \mathbb{N}$ and define for $k=1,\dots,N$

$$U_k := \left\{ x \in U : \mathrm{dist}(U', \partial U) < \frac{k\delta}{N} \right\} \in \mathcal{A}(\Omega).$$

Let φ_k be a cut-off between U_{k-1} and U_k, i.e. $\varphi_k \in$ $\in \mathcal{C}_c^\infty(U_k)$, $0 \le \varphi_k \le 1$, $\varphi_k \equiv 1$ on U_{k-1}.



We define $w_j^{(k)} := u_j \varphi_k + (1-\varphi_k) v_j \in W^{1,p}(\Omega;\mathbb{R}^m)$ and notice that $w_j^{(k)} \xrightarrow{L^p} u$, indeed

$$\int_\Omega |w_j^{(k)}(x) - u(x)|^p dx =$$

$$= \int_\Omega |\varphi_k(x)(u_j(x) - u(x)) + (1-\varphi_k(x))(v_j(x) - u(x))|^p dx \le$$

$$\le \int_{U_k} |u_j(x) - u(x)|^p dx + \int_{\Omega\setminus U_{k-1}} |v_j(x) - u(x)|^p dx \longrightarrow 0.$$

Also, by chain rule, $\nabla w_j^{(k)} = \nabla u_j \varphi_k + \nabla v_j (1-\varphi_k) + (u_j - v_j) \otimes \nabla \varphi_k$, in particular

$$\nabla w_j^{(k)} = \nabla u_j \quad \text{in } U_{k-1}, \qquad \nabla w_j^{(k)} = \nabla v_j \quad \text{in } \Omega \setminus U_k .$$

So we have

$$F_j(w_j^{(k)}, U' \cup V) = \int_{U' \cup V} f_j(x, \nabla w_j^{(k)}(x))\,dx$$

(*₁)

$$= \int_{U' \cup (U_{k-1} \cap V)} f_j(x, \nabla u_j(x))\,dx + \int_{V \setminus U_k} f_j(x, \nabla v_j(x))\,dx +$$

$$+ \underbrace{\int_{(U_k \setminus U_{k-1}) \cap V} f_j(x, \nabla w_j^{(k)}(x))\,dx}_{=: I_j^{(k)}}$$

$$\leq F_j(u_j, U) + F_j(v_j, V) + I_j^{(k)}.$$

We now control the quantity $I_j^{(k)}$. By the growth conditions

$$I_j^{(k)} \leq C_1 \int_{(U_k \setminus U_{k-1}) \cap V} (1 + |\nabla u_j(x)|^p + |\nabla v_j(x)|^p + |\nabla \varphi_k(x)|^p |u_j(x) - v_j(x)|^p\,dx$$

for some $C_1 > 0$. Summing over k we get

$$\sum_{k=1}^{N} I_j^{(k)} \leq \tilde{C}_1 \int_{(U \setminus U') \cap V} (1 + |\nabla u_j(x)|^p + |\nabla v_j(x)|^p + \frac{N^p}{\delta^p k^p} |u_j(x) - v_j(x)|^p\,dx$$

where we used the fact that $|\nabla \varphi_k(x)| \leq c \frac{\delta k}{N}$ for some $c > 1$ and $\tilde{C}_1 = c C_1$. Then $\exists\, k_0 \in \{1, \dots, N\}$ s.t.

$$I_j^{(k_0)} \leq 1/N \sum_{k=1}^{N} I_j^{(k)} \leq \frac{\tilde{C}_1}{N} \int_{(U \setminus U') \cap V} (1 + |\nabla u_j(x)|^p + |\nabla v_j(x)|^p + \frac{N^p}{\delta^p k^p} |u_j(x) - v_j(x)|^p\,dx$$

(*₂)

$$\leq \frac{\tilde{C}_1}{N} |\Omega| + \frac{\tilde{C}_1 c_1^{-1}}{N} (F_j(u_j, U) + F_j(v_j, V)) +$$

$$+ \frac{\tilde{C}_1 N^{p-1}}{\delta^p k^p} \int_\Omega |u_j(x) - v_j(x)|^p\,dx,$$

where we have used the growth conditions from below to say that $\int_{(U \setminus U') \cap V} |\nabla u_j(x)|^p\,dx \leq c_1^{-1} F_j(u_j, U)$ and the same for v_j.

So, defining $v_j := v_j^{(k_0)}$ we have from $(*_1)$ and $(*_2)$

$$\overline{\lim_{j\to+\infty}} F_j(w_j, U'\cup V) \leq \lim_{j\to+\infty} \left(1+\frac{\tilde{C}_1 c_1^{-1}}{N}\right)\left(F_j(u_j,U)+F_j(v_j,V)\right)+$$

$$+\lim_{j\to+\infty} \frac{\tilde{C}_1 N^{p-1}}{\delta^p k^p}\int_\Omega |u_j(x)-v_j(x)|^p\,dx+\frac{\tilde{C}_1}{N}|\Omega|,$$

$$=\left(1+\frac{\tilde{C}_1 c_1^{-1}}{N}\right)\left(F''(u,U)+F''(u,V)\right)+\frac{\tilde{C}_1}{N}|\Omega|$$

$\forall N\in\mathbb{N}$. Since, by definition $F''(u,U'\cup V)\leq \overline{\lim_{j\to+\infty}} F_j(w_j,U'\cup V)$, by taking the limit as $N\to+\infty$ we conclude. □

<u>Rmk</u>: denoting F_j the functional extended on the whole $L^p(\Omega;\mathbb{R}^m)$, the result above holds $\forall u\in L^p(\Omega;\mathbb{R}^m)$.
The prove is completely identical and it is sufficient to consider $u\in W^{1,p}(U\cup V;\mathbb{R}^m)$ otherwise, by growth conditions, $F''(u,U)+$ $+F''(u,V)=+\infty$. Indeed, if not there would exist a sequence u_j s.t. $\sup_j \|\nabla u_j\|_{L^p(U\cup V)}<+\infty$, $u_j \xrightarrow{L^p} u$, but this implies that $u\in W^{1,p}(U\cup V;\mathbb{R}^m)$ which is a contradiction.

The fundamental estimate, proved in Lemma 3 of last lecture, is the key ingredient to prove inner regularity of $F'(u,\cdot)$ and $F''(u,\cdot)$. Thanks to this we will be able to prove the compactness of the Γ-limit of $F_j(\cdot,U)$ $\forall\, U\in\mathcal{A}(\Omega)$ and the measure property of the Γ-limit.

<u>Proposition</u> 1 (inner regularity): let F_j, F' and F'' be as in the statement of Lemma 3 of last lecture. Then, $\forall\, u\in W^{1,p}(\Omega;\mathbb{R}^m)$, $F''(u,\cdot)$ is inner regular, i.e.

$$F''(u,U)=\sup\{F''(u,V): V\in\mathcal{A}(\Omega),\ V\subset\subset U\}$$

and the same for F'.
In particular (by fund. est.) F'' is subadditive.

<u>proof</u>: we prove the statement only for F''. The argument is completely analogous for F'.
Fix $W\in\mathcal{A}(\Omega)$, $K\subset W$ compact. Take $U',U\in\mathcal{A}(\Omega)$ such that $K\subset U'\subset\subset U\subset\subset W$ and define $V:=W\setminus K$.
By the fundamental estimate

$$F''(u,W)\le F''(u,U)+F''(u,W\setminus K).$$

We notice that, $\forall\, E\in\mathcal{A}(\Omega)$ the functional $\int_E 1+|\nabla u(x)|^p\,dx$ is lsc wrt strong-L^p topology.
Thus, since by growth conditions from above $F_j(u,E)\le c_2\int_E 1+|\nabla u(x)|^p\,dx$ taking the $\Gamma(L^p)$-$\overline{\lim}$ we get $F''(u,E)\le c_2\int_E 1+|\nabla u(x)|^p\,dx$.
So, $\forall\, K\subset W$ compact we have

$$F''(u,W)\le \sup\{F''(u,U): U\subset\subset W,\ U\supset K,\ U\in\mathcal{A}(\Omega)\}+c_2\int_{W\setminus K}1+|\nabla u(x)|^p\,dx$$

$$\le \sup\{F''(u,U): U\subset\subset W,\ U\in\mathcal{A}(\Omega)\}+c_2\int_{W\setminus K}1+|\nabla u(x)|^p\,dx.$$

Taking the limit as $|W \setminus K| \to 0$ we get

$$F''(u, W) \leq \sup\{F''(u, U) : U \subset\subset W\}.$$

Notice also that $F''(u, \cdot)$ is increasing. Indeed, $\forall\, U, V \in \mathcal{A}(\Omega)$ s.t. $V \subseteq U$, then $F_j(u, V) \leq F_j(u, U)$ and taking the $\Gamma(L^p)$-$\overline{\lim}$ we obtain it for F''. Hence, since $F''(u, \cdot)$ is increasing

$$F''(u, W) \geq F''(u, U) \quad \forall\, U \subset\subset W$$

and the result is proved.

The subadditivity is an immediate consequence, indeed noting that $\forall\, W \subset\subset U \cup V$, $W \subseteq U' \cup V$ with $U' := W \cap U \subset\subset U$ then

$$F''(u, U \cup V) = \sup\{F''(u, W) : W \subset\subset U \cup V,\ W \in \mathcal{A}(\Omega)\}$$

$$\leq \sup\{F''(u, U' \cup V) : U' \subset\subset U,\ U' \in \mathcal{A}(\Omega)\}$$

$$\leq F''(u, U) + F''(u, V). \qquad \square$$

Rmk: two side result that we prove in the proof above are the following

(1.1) $F''(u, U) \leq c_2 \int_U 1 + |\nabla u(x)|^p \, dx, \quad \forall\, U \in \mathcal{A}(\Omega),\ u \in W^{1,p}(\Omega; \mathbb{R}^m)$;

(1.2) $F''(u, V) \leq F''(u, U), \quad \forall\, V, U \in \mathcal{A}(\Omega),\ V \subseteq U,\ u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

We now mention a result which allows us to conclude that, if the Γ-limit exists $\forall u$, then it is (the restriction of) a Borel measure. We omit the proof since it is a classical result in measure theory.

Theorem 2 (De Giorgi–Letta criterion): let $\alpha : \mathcal{A}(\Omega) \to [0,+\infty]$ be s.t. $\alpha(\emptyset)=0$ and $\alpha(V) \le \alpha(U)$ $\forall\, V \subseteq U$. The following are equivalent:

(1) α is the restriction of Borel measure on Ω;

(2) α is subadditive, i.e. $\alpha(U \cup V) \le \alpha(U) + \alpha(V)$
superadditive, i.e. $\alpha(U \cup V) \ge \alpha(U) + \alpha(V)$, $\forall\, U \cap V = \emptyset$
inner regular, i.e. $\alpha(U) = \sup\{\alpha(V) : V \subset\subset U,\ V \in \mathcal{A}(\Omega)\}$

(3) $\beta(E) = \inf\{\alpha(U) : U \supseteq E,\ U \in \mathcal{A}(\Omega)\}$ defines a Borel measure.

We are now ready to state our main result.

Theorem 3 (compactness and integral representation of Γ-limit): let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open, Lipschitz set and $1<p<\infty$. Let $f_j : \Omega \times \mathbb{R}^{m\times n} \to [0,+\infty)$ Borel s.t. $\exists\, c_1, c_2 > 0$ s.t.

(G) $c_1 |A|^p \le f_j(x,A) \le c_2(|A|^p + 1)$ $\quad \forall A \in \mathbb{R}^{m \times n}$, $x \in \Omega$ a.e.

and let $F_j : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined as

$$F_j(u,U) = \int_U f_j(x, \nabla u(x))\, dx.$$

Then $\exists \{j_k\}$, $f : \Omega \times \mathbb{R}^{m\times n} \to [0,+\infty]$ Caratheodory, complying with (G) s.t. $\forall\, U \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega;\mathbb{R}^m)$

$$\exists\ \Gamma(L^p)-\lim_{k\to+\infty} F_{j_k}(u,U) = \int_U f(x, \nabla u(x))\, dx.$$

proof: we work in several steps.

Step 1 (compactness of Γ limit): let $\mathcal{R}$ be the family of

sets that are finite unions of (n-th dimensional) rectangles with rational vertices. Notice that $\mathcal{R}$ is countable.
Since $L^p(\Omega;\mathbb{R}^m)$ is separable, by compactness of Γ-convergence, by a diagonal argument, $\exists \{j_k\}$ s.t. $\forall\, R\in\mathcal{R}$

$$\exists\ \Gamma(L^p)\text{-}\lim_{k\to+\infty} F_{j_k}(u,R) =: F(u,R).$$

Denote $F'(\cdot,U)=\Gamma(L^p)\text{-}\liminf_{k\to+\infty} F_{j_k}(\cdot,U)$, $F''(\cdot,U)=\Gamma(L^p)\text{-}\limsup_{k\to+\infty} F_{j_k}(\cdot,U)$ $\forall\, U\in\mathcal{A}(\Omega)$. By Proposition 1, $F'(u,\cdot)$ and $F''(u,\cdot)$ are inner regular $\forall\, u\in W^{1,p}(\Omega;\mathbb{R}^m)$, so

$$F'(u,U) = \sup\{F'(u,V) : V\subset\subset U,\ V\in\mathcal{A}(\Omega)\}$$

$$(*_1)\qquad \geq \sup\{F'(u,R) : R\subset\subset U,\ R\in\mathcal{R}\}$$

$$= \sup\{F''(u,R) : R\subset\subset U,\ R\in\mathcal{R}\}.$$

Since $\forall\, V\subset\subset U\ \exists\, \tilde{R}\in\mathcal{R}$ s.t. $V\subset\subset\tilde{R}\subset\subset U$, by characterization of sup, $\forall\,\varepsilon\ \exists\, V_\varepsilon$ s.t.

$$\sup\{F''(u,V) : V\subset\subset U,\ V\in\mathcal{A}(\Omega)\} < F''(u,V_\varepsilon)+\varepsilon$$

$$\leq F''(u,\tilde{R}_\varepsilon)+\varepsilon$$

$$\leq \sup\{F''(u,R) : R\subset\subset U,\ R\in\mathcal{R}\}+\varepsilon,$$

so by arbitrariness of ε

$$(*_2)\quad \sup\{F''(u,V) : V\subset\subset U,\ V\in\mathcal{A}(\Omega)\} \leq \sup\{F''(u,R) : R\subset\subset U,\ R\in\mathcal{R}\}.$$

Gathering $(*_1)$ and $(*_2)$ we get

$$F'(u,U) \geq \sup\{F''(u,V) : V\subset\subset U,\ V\in\mathcal{A}(\Omega)\} = F''(u,U).$$

Since $F'(u,U) \leq F''(u,U)$, we have that $F'(u,U) = F''(u,U)$ $\forall$ $U \in \mathcal{A}(\Omega)$ which is equivalent to the $\exists$ of the Γ-limit.

Step 2 (measure property): since $F_{j_k}(u,\emptyset) \equiv 0$ then $F(u,\emptyset) = 0$. By Proposition 1 and (1.2) $F''(u,\cdot)$ is increasing, inner regular and subadditive.
It remains to show it is superadditive. Let $U, V \in \mathcal{A}(\Omega)$ two disjoint sets and let $u_k \to u$ a recovery sequence in $U \cup V$

$$F(u, U\cup V) = \lim_{k\to+\infty} F_{j_k}(u_k, U\cup V) = \lim_{k\to+\infty} \left(F_{j_k}(u_k, U) + F_{j_k}(u_k, V)\right)$$

$$\geq \liminf_{k\to+\infty} F_{j_k}(u_k, U) + \liminf_{k\to+\infty} F_{j_k}(u_k, V)$$

$$\geq F(u,U) + F(u,V).$$

By De Giorgi–Letta criterion, $F(u,\cdot)$ is the restriction of a Borel measure.

Step 3 (integral representation): we want to apply the integral representation Theorem (i.e., Theorem 1 of last lecture). Condition (i) and (iv) are obvious. Condition (iii) is satisfied by (1.1). Condition (v) is also satisfied.
Indeed, since F is the $\Gamma(L^p)$-limit of F_{j_k}, it is strong-L^p lsc, so $\forall$ $u_j \xrightarrow{W^{1,p}} u$ (by Rellich) $\Rightarrow$ $u_j \xrightarrow{L^p} u$ so

$$F(u,U) \leq \liminf_{j\to+\infty} F(u_j, U).$$

It remains to prove only that $f(x,A) \geq c_1|A|^p$, but by (c_1) $F_j(u,U) \geq c_1 \int_U |\nabla u(x)|^p\,dx$. By taking the Γ-limit we obtain (since $\int_U |\nabla u(x)|^p dx$ is strong-L^p lsc) $F(u,U) \geq c_1 \int_U |\nabla u(x)|^p\,dx$.
Then $\forall$ $A \in \mathbb{R}^{m\times n}$, $y \in \Omega$ Lebesgue point of $f(\cdot, A)$, choosing $u(x) = Ax$, $U = B_\rho(y)$ and taking the limsup as $\rho \to 0^+$ we have the claim. $\square$

Lecture 11: 24.05.2024

Last lecture we saw that the class of integral functionals whose densities complies with p-growth conditions from above and below is compact <u>in their class</u> wrt Γ convergence.

This is a nice result (which required some effort) but, alone, it gives no explicit information about the limit density f.
For this, we need some structure of the energies F_j.
We will see that, in some cases, we can give explicit formulas for the limit densities f.

We now mention a useful result in this direction, that is that if we have Γ-converging energies, we "authomatically" have Γ-convergence subject to boundary conditions.
While this result is interesting on its own, we will also use it to give an explicit formula for the limit density f in some cases.

<u>Corollary</u> 1 (boundary conditions): Let F_j be as in the statement of Theorem 3 of last lecture, let $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let

$$F_j^{\phi}(u) = \begin{cases} F_j(u) & u-\phi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise in } W^{1,p}(\Omega;\mathbb{R}^m). \end{cases}$$

If $F_j \xrightarrow{\Gamma(L^p)} F \Rightarrow F_j^{\phi} \xrightarrow{\Gamma(L^p)} F^{\phi}$ where

$$F^{\phi}(u) = \begin{cases} F(u) & u-\phi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise in } W^{1,p}(\Omega;\mathbb{R}^m). \end{cases}$$

<u>Notes</u> (ideas of proof): We don't see the proof of this result (you find it anyway in [BD, 1998] Proposition 11.7] but we give some intuition about it.
Since $F_j^{\phi} \geq F_j$ then obviously $\Gamma(L^p)\text{-}\lim_{j\to+\infty} F_j^{\phi} \geq F$, moreover since $\phi + W_0^{1,p}(\Omega;\mathbb{R}^m)$ is strongly (and since convex also weakly) closed $\Rightarrow \Gamma(L^p)\text{-}\lim F_j^{\phi} \geq F^{\phi}$.

Applying the fundamental estimate to attach to any recovery sequence the boundary condition ϕ we also get that $\Gamma(L^p)-\varlimsup_{j\to+\infty} F_j^{\phi}(u) \leq F(u) \quad \forall\ u-\phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

<u>Rmk</u> (equi-coercivity): functionals F_j as above are equi-L^p-coercive. Indeed, $u \in \{F_j \leq t\} \Rightarrow u-\phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\|\nabla u\|_{L^p(\Omega)} \leq \left(\frac{t}{c_1}\right)^{1/p}$.
By Poincaré's inequality this implies also that

$$\|u\|_{L^p(\Omega)} \leq \|u-\phi\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)}$$

$$\overset{\text{Poincaré}}{\leq} C_1(\Omega)\,\|\nabla u - \nabla\phi\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)}$$

$$\leq C_1(\Omega)\|\nabla u\|_{L^p(\Omega)} + (1+C_1(\Omega))\|\phi\|_{W^{1,p}(\Omega)} \leq C_{t,\Omega,\phi}$$

for some constant $C_{t,\Omega,\phi}>0$ depending on t,Ω and ϕ.
So $\{F_j \leq t\} \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ is bounded, so precompact in $L^p(\Omega;\mathbb{R}^m)$ by Rellich.

<u>Exercise</u>: prove that F_j as in the Remark above are not equi-coercive (without boundary conditions) but are equi-mildly coercive.

<u>Rmk</u> (convergence of minimum problems): let $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$ a boundary conditions, and consider F_j a sequence as in Corollary 1. Then

$$\lim_{j\to+\infty} \inf\{F_j(u) : u-\phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\} = \min\{F(u) : u-\phi\in W_0^{1,p}(\Omega;\mathbb{R}^m)\}$$

This is a direct application of Corollary 1, the previous Remark and the Fundamental Theorem of Γ-convergence.

Γ-convergence of homogeneous functionals

One case in which the density f has an explicit form is when F_j are homogeneous functionals, i.e. $f_j(x,A) = f_j(A)$.

Before stating the main result we introduce the notion of quasi-convexification of a function.

<u>Def</u> (quasiconvexification): let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ a Borel function.
We define the quasiconvexification Qf of f by

$$Qf(A) := \inf \left\{ \frac{1}{|E|} \int_E f(A + \nabla\varphi(x))\, dx : \varphi \in C_c^\infty(E;\mathbb{R}^m) \right\}$$

for any $E \subseteq \mathbb{R}^n$ open, bounded with $|\partial E| = 0$.

<u>Rmk</u>: one can prove the following (nontrivial) properties:

(1.1) the definition of Qf does not depend on E;

(1.2) if f is locally bounded then Qf is quasiconvex and it coincides with the quasiconvex envelope of f, i.e.

$$Qf = \max \{ g : \mathbb{R}^{m\times n} \to \mathbb{R} : g \leq f,\ g \text{ is quasiconvex} \};$$

(1.3) if $0 \leq f(A) \leq c(|A|^p + 1)$ then $Qf(A) = \inf_{v \in W_0^{1,p}(E;\mathbb{R}^m)} \fint_E f(A + \nabla v)$.

<u>Proposition 2</u>: let $f_j : \mathbb{R}^{m\times n} \to [0,+\infty)$ continuous s.t. $\exists c_1, c_2 > 0 : c_1|A|^p \leq f_j(A) \leq c_2(|A|^p + 1)$ $\forall A \in \mathbb{R}^{m\times n}$, $\forall j$ and let $F_j(u) := \int_\Omega f_j(\nabla u(x))\, dx$.
Then $F_j \xrightarrow{\Gamma(L^p)} F \iff Qf_j \to f$ pointwise, where we denoted $F(u) := \int_\Omega f(\nabla u(x))\, dx$.

proof: ($\Leftarrow$) by Theorem 3 and Corollary 2 of last lecture $\exists \{j_k\}$ and φ quasiconvex s.t. $c_1|A|^p \leq \varphi(A) \leq c_2(|A|^p+1)$ $\forall A \in \mathbb{R}^{m\times n}$ s.t.

$$\Gamma(L^p)-\lim_{k\to+\infty} F_{j_k}(u) = \int_\Omega \varphi(\nabla u(x))\,dx.$$

By (1.2) of previous Remark, $\varphi = Q\varphi$, $\forall A \in \mathbb{R}^{m\times n}$ we have

$$|\Omega|\varphi(A) \overset{(1.3)}{=} \min\left\{\int_\Omega \varphi(A+\nabla v(x))\,dx \;:\; v \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

$$= \min\left\{\int_\Omega \varphi(\nabla u(x))\,dx \;:\; u - Ax \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

$$\overset{\text{Fund. Thm}}{=} \lim_{k\to+\infty} \inf\left\{F_{j_k}(u) \;:\; u - Ax \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

$$= \lim_{k\to+\infty} \inf\left\{\int_\Omega f_{j_k}(\nabla u(x))\,dx \;:\; u - Ax \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

$$\overset{\text{def}}{=} \lim_{k\to+\infty} |\Omega|\, Qf_{j_k}(A) = |\Omega| f(A).$$

Notice that the minimum above is attained since $\int_\Omega \varphi(A+\nabla v)$ is lsc.

So $\varphi = f$, in particular does not depend on the subsequence, thus by Urysohn property of Γ-convergence $F_j \xrightarrow{\Gamma(L^p)} F$.

($\Rightarrow$) proceeding exactly as above with f in place of φ we get

$$|\Omega| f(A) = \lim_{j\to+\infty} |\Omega|\, Qf_j(A)$$

and we immediately obtain the result. $\square$

Corollary 3 (relaxation): let $f: \mathbb{R}^{m\times n} \to [0,+\infty)$ continuous s.t. $\exists$ $c_1, c_2 > 0$: $c_1|A|^p \leq f(A) \leq c_2(|A|^p+1)$ $\forall A \in \mathbb{R}^{m\times n}$ and let $F(u) := \int_\Omega f(\nabla u(x))\,dx$. Then

$$sc_{L^p}(F)(u) = \int_\Omega Qf(\nabla u(x))\,dx,$$

where $sc_{L^p}(F)$ denotes the lsc envelope of F wrt the strong-L^p topology.

The results above give a way to find explicit formula for the limit densities f.

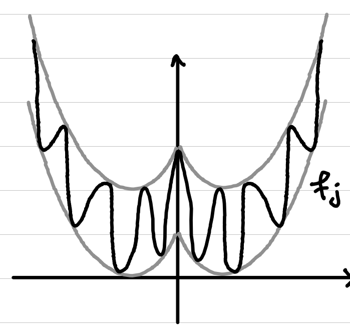
Let us remind though that quasiconvexity is yet not fully understood and it is, in general, very difficult to compute quasiconvex envelopes of functions.
When we consider scalar functions, quasiconvex envelopes are convex envelopes, so these cases are a bit easier to treat.

We see an explanatory example in 1D (it can be easily generalized to any dimensions n, still with scalar functions, i.e. $m=1$).

<u>Ex</u>: consider $F_j : W^{1,2}(0,2\pi) \longrightarrow [0,+\infty)$ defined as

$$F_j(u) = \int_0^{2\pi} f_j(u'(x)) + (|u(x)| - \cos(x))^2 \, dx$$

where $f_j(z) = (|z|-1)^2 + 1 + \cos(jx)$.



Say that we are interested in studying

$$\inf\{F_j(u) : u \in W^{1,2}(0,2\pi)\} =: M_j$$

but this minimum problem might be not easy since f_j is highly oscillating so F_j is not lsc. We obtain information on M_j thanks to Γ-convergence.

- F_j is strong-L^2 coercive, i.e. $\{F_j \leq t\}$ is L^2-precompact $\forall t>0$.
  Indeed, if $u \in \{F_j \leq t\}$ then

$$t \geq F_j(u) \geq \int_0^{2\pi} f_j(u'(x))\,dx > \int_0^{2\pi} (|u'(x)|-1)^2\,dx =$$

$$= \int_0^{2\pi} |u'(x)|^2 - 2|u'(x)| + 1 \, dx \geq$$

$$\geq \|u'\|^2_{L^2(0,2\pi)} - 2(2\pi)^{1/2} \|u'\|_{L^2(0,2\pi)} + 2\pi =$$

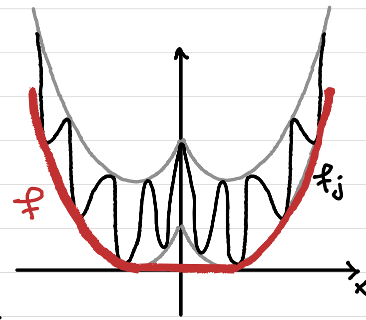
$$= (\|u'\|_{L^2(0,2\pi)} - 2\pi)^2$$

which implies $\|u'\|_{L^2(0,2\pi)} \leq \sqrt{t} + 2\pi$.
Working analogously for the other term we also obtain $\|u\|_{L^2(0,2\pi)} \leq C_t$ for some constant $C_t > 0$ depending on t, so by Rellich $\{F_j < t\}$ is strong-L^2 precompact.

- $Qf_j = f_j^{**} \longrightarrow f$, where $f(z) = \begin{cases} z^2 - 2|z| & |z| > 1 \\ 0 & |z| \leq 1 \end{cases}$. Then, by Proposition 2

$$\Gamma(L^2)\text{-}\lim_{j \to +\infty} \int_0^{2\pi} f_j(u'(x))\,dx =$$

$$= \int_0^{2\pi} f(u'(x))\,dx.$$



Since $(|z| - \cos(x))^2$ is continuous, positive and $\leq |z|^2$, by Fatou's Lemma

$$\int_0^{2\pi} (|u(x)| - \cos(x))^2\,dx$$

is continuous wrt the strong L^2 topology. By stability of Γ-convergence under continuous perturbations we have

$$\Gamma(L^2)\text{-}\lim_{j \to +\infty} F_j(u) = \int_0^{2\pi} f(u'(x)) + (|u(x)| - \cos(x))^2\,dx =: F(u).$$

Since F_j are equi-coercive, by Fundamental Theorem of Γ-convergence we have

$$\lim_{j \to +\infty} M_j = \min\left\{ F(u) : u \in W^{1,2}(0,2\pi) \right\}.$$

- We are reduced to compute the minimum of F. For this, we notice that, $\forall\, u \in W^{1,2}(0,2\pi)$

$$\int_0^{2\pi} (|u(x)| - \cos(x))^2\,dx \geq \int_{\pi/2}^{3\pi/2} (|u(x)| - \cos(x))^2\,dx.$$

(*)

$$\geq \int_{\pi/2}^{3\pi/2} \cos^2(x)\,dx$$

where we used that, for $x \in (\pi/2, 3\pi/2)$, $\cos(x) < 0$.
If we want to minimize F we would like to make both the terms $f(u')$ and $(|u| - \cos(x))^2$ as small as pos-sible.
To make $f(u')$ small we should choose $|u'| \leq 1$ and to make the other small (from the previous comment) we want $u = \cos(x)$ on $(\pi/2, 3\pi/2)$.
For this, our candidate minimizer is $u_0(x) = (\cos(x))^+$.
Indeed $u_0'(x) = -\sin(x)\, \chi_{(0,\pi/2)\cup(3\pi/2, 2\pi)}(x)$. Then $|u_0'| \leq 1$.
Moreover

$$(|u_0(x)| - \cos(x))^2 = \cos^2(x)\, \chi_{(\pi/2, 3\pi/2)}(x).$$

So by $(*)$

$$F(u_0) = \int_{\pi/2}^{3\pi/2} \cos^2(x)\, dx \leq F(u), \quad \forall u \in W^{1,2}(0, 2\pi).$$

Notice that we could find u_0 also solving the EL-equation of F, i.e.

$$f''(u_0'(x)) = 2(|u_0(x)| - \cos(x))\operatorname{sgn}(u_0(x)).$$

So we conclude that $\lim_{j\to+\infty} M_j = \pi/2$.

<u>Exercise</u> 1: try to generalize this example for functionals in $W^{1,2}(\Omega)$ with $\Omega = B_{2\pi} \subseteq \mathbb{R}^n$, $n \geq 2$.

<u>Exercise</u> 2: find a recovery sequence for $F_j \xrightarrow{\Gamma} F$ at u_0.

Periodic Homogenization

Main reference:
- Ch. 14 of [BD, 1998]

The main case in which the limit density f has an explicit formula is the case of (periodic) homogenization.
Given energies F_j whose density f_j has some "structure" in the space variable, e.g. periodicity with period which is going to zero, then the limit density f is homogeneous and can be explicitely determined.

The setting is the following: let $f : \mathbb{R}^n \times \mathbb{R}^{m\times n} \longrightarrow [0,+\infty)$ be a a Borel function satisfying

(P) $\qquad f(\cdot, A)$ is $[0,1]^n$- periodic, $\forall A \in \mathbb{R}^{m\times n}$,

(G_p) $\exists c_1, c_2 > 0$ s.t. $c_1|A|^p \leq f(x,A) \leq c_2(|A|^p+1)$, $\forall A \in \mathbb{R}^{m\times n}$, a.e. $x \in \mathbb{R}^n$.

Given $\Omega \subseteq \mathbb{R}^n$ bounded, open, Lipschitz domain, for every $\varepsilon > 0$, we consider the energies $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \longrightarrow [0,+\infty)$ as

(E) $$F_\varepsilon(u) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

possibly extended to $+\infty$ on $L^p(\Omega;\mathbb{R}^m) \setminus W^{1,p}(\Omega;\mathbb{R}^m)$ if needed.

Rmk: by Theorem 3 of last lecture $\forall \varepsilon_j \to 0$ $\exists \{\varepsilon_{j_k}\} \subseteq \{\varepsilon_j\}$ and $\varphi : \Omega \times \mathbb{R}^{m\times n} \longrightarrow [0,+\infty)$ Carathéodory, complying with (G_p) s.t. $\forall U \in \mathcal{A}(\Omega)$

(*) $$\Gamma(L^p)\text{-}\lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u,U) = \int_U \varphi(x, \nabla u(x))\, dx,$$

where $F_\varepsilon(u,U)$ denotes the localized functional, as defined in the last lecture.

So far, the function φ (defined in the Remark above) may depend on the subsequence and on the space variable.
We will prove that, $\forall$ subsequence φ does not depend on x.
We will also prove that φ is independent of the subsequence, i.e. the whole family F_ε $\Gamma(L^p)$-converges.

<u>Remind</u>: when we have a family $\{F_\varepsilon\}_{\varepsilon>0}$, we say that $F_\varepsilon \xrightarrow{\Gamma} F$ as $\varepsilon \to 0$ if $\forall\, \varepsilon_j \to 0$ $F_{\varepsilon_j} \xrightarrow{\Gamma} F$ as $j \to +\infty$.

At the end of last lecture we introduced the topic of periodic homogenization. We recall the setting:
let $f:\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,+\infty)$ be a Borel function satisfying

(P) $\quad f(\cdot,A)$ is $[0,1]^n$-periodic, $\forall A\in\mathbb{R}^{m\times n}$,

(G_p) $\quad\exists c_1,c_2>0$ s.t. $c_1|A|^p\leq f(x,A)\leq c_2(|A|^p+1)$, $\forall A\in\mathbb{R}^{m\times n}$, a.e. $x\in\mathbb{R}^n$.

For every $\varepsilon>0$, we consider the energies $F_\varepsilon: W^{1,p}(\Omega;\mathbb{R}^m)\to[0,+\infty)$ as

(E) $$F_\varepsilon(u):=\int_\Omega f\left(\frac{x}{\varepsilon},\nabla u(x)\right)dx.$$

By Theorem 3 of lecture 10, $\forall\,\varepsilon_j\to0\ \exists\{\varepsilon_{j_k}\}\subseteq\{\varepsilon_j\}$ and $\varphi:\Omega\times\mathbb{R}^{m\times n}\to[0,+\infty)$ Carathéodory, satisfying (G_p) s.t. $\forall\,U\in\mathcal{A}(\Omega)$

(*) $$\Gamma(L^p)\text{-}\lim_{k\to+\infty}F_{\varepsilon_{j_k}}(u,U)=\int_U\varphi(x,\nabla u(x))\,dx.$$

Our goals are the following:
- prove that φ does not depend on x (i.e. is homogeneous);
- prove that φ is independent of $\{\varepsilon_{j_k}\}$, which implies Γ-convergence of the whole family F_ε (as $\varepsilon\to0$);
- find an "explicit" formula for φ.

With the next result we prove the first point, that is that any cluster point (wrt Γ-convergence) of $\{F_\varepsilon\}_{\varepsilon>0}$ are homogeneous integral functionals.

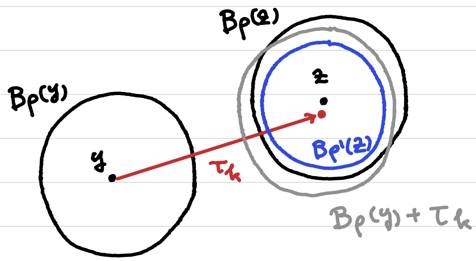
<u>Proposition</u> 1: Let F_ε as in (E) with f satisfying (P) and (G_p).
Let $\{\varepsilon_{j_k}\}$ and φ such that (*) holds true.
Then $\varphi(x,A)=\varphi(A)$ $\forall\,A\in\mathbb{R}^{m\times n}$ and it is quasiconvex.

proof: by Corollary 2 of lecture 9, it is sufficient to prove that, denoting as $\Phi(u,U) = \int_U \varphi(x, \nabla u(x))\,dx$, it holds

$$\Phi(Ax, B_\rho(y)) = \Phi(Ax, B_\rho(z))$$

$\forall A \in \mathbb{R}^{m\times n}$, $y, z \in \Omega$, $\rho > 0$ s.t. $B_\rho(y), B_\rho(z) \subseteq \Omega$.
Let $u_k \in W^{1,p}(B_\rho(y);\mathbb{R}^m)$ be a recovery sequence in $B_\rho(y)$, i.e. $u_k \to Ax$ in $L^p(B_\rho(y);\mathbb{R}^m)$

$$\lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u_k, B_\rho(y)) = \Phi(Ax, B_\rho(y)).$$



We want to translate u_k on $B_\rho(z)$ without create "too much" energy. To do this, we translate by a multiple of ε_{j_k}. Due to possible mismatch we need to consider a slightly smaller ball.

So let $\tau_k \in \mathbb{R}^n$: $(\tau_k)_i := \varepsilon_{j_k}\left[\frac{z_i - y_i}{\varepsilon_{j_k}}\right]$ and define

$$v_k(x) := u_k(x-\tau_k) + A\tau_k.$$

Notice that $\tau_k \to z-y$, so $\forall \rho' < \rho$, $B_{\rho'}(z) - \tau_k \subseteq B_\rho(y)$ for k large enough.
This yields also that $v_k \to Ax$ in $L^p(B_{\rho'}(z);\mathbb{R}^m)$.
By periodicity of f we have

$$
\begin{aligned}
F_{\varepsilon_{j_k}}(v_k, B_{\rho'}(z)) &= \int_{B_{\rho'}(z)} f\left(\frac{x}{\varepsilon_{j_k}}, \nabla v_k(x)\right) dx \\
&= \int_{B_{\rho'}(z)} f\left(\frac{x}{\varepsilon_{j_k}}, \nabla u_k(x-\tau_k)\right) dx \\
&\overset{x'=x-\tau_k}{=} \int_{B_{\rho'}(z)-\tau_k} f\left(\frac{x'}{\varepsilon_{j_k}} + \frac{\tau_k}{\varepsilon_{j_k}}, \nabla u_k(x')\right) dx' \\
&\leq F_{\varepsilon_{j_k}}(u_k, B_\rho(y)),
\end{aligned}
$$

where in the last step we used that $\frac{\tau_k}{\varepsilon_{j_k}} \in \mathbb{Z}^n$, the $[0,1]^n$-
- periodicity of f in the first variable and the fact that $B_{\rho'}(z) - \tau_k \subseteq B_\rho(y)$.

Taking the liminf as $k \to +\infty$ above we get

$$\Phi(Ax, B_{\rho'}(z)) \leq \varliminf_{k\to+\infty} F_{\varepsilon_{j_k}}(v_k, B_{\rho'}(z))$$

$$\leq \lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u_k, B_\rho(z)) = \Phi(Ax, B_\rho(z)).$$

Finally, by inner regularity of $F(Ax,\cdot)$ we obtain that

$$\Phi(Ax, B_\rho(z)) \leq \Phi(Ax, B_\rho(y)),$$

and the result is proved by exchanging the roles of z and y. $\square$

We are now ready to prove our main homogenization result.

<u>Theorem</u> 2 (homogenization): Let Ω open, bounded and Lipschitz and let $1<p<\infty$. Let $f : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0,+\infty)$ be a Borel function satisfying (P) and (G_p) and let F_ε be as in (E) $\forall \varepsilon > 0$. Then

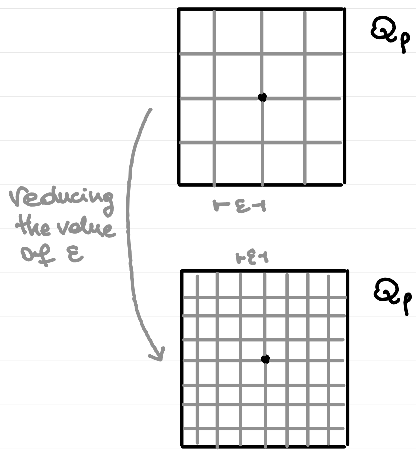
$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega f_{hom}(\nabla u(x))\,dx,$$

where $f_{hom} : \mathbb{R}^{m\times n} \to [0,+\infty)$ is a quasiconvex function satisfying (G_p) and defined as

(HF) $$f_{hom}(A) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\left\{ \int_{(0,t)^n} f(x, A+\nabla u(x))\,dx : u \in W_0^{1,p}((0,t)^n;\mathbb{R}^m)\right\}$$

$\forall A \in \mathbb{R}^{m\times n}$.

Rmk (intuition for the homogenization formula): we give a quick intuition/explanation for the formula (HF).



We consider a small cube around a point $z \in \Omega$ (that for simplicity we assume to be 0) and see how F_ε acts on u, i.e. how $f(\cdot/\varepsilon, \cdot)$ acts on ∇u.
Since we are reasoning locally we can assume $\nabla u(0) \sim A \in \mathbb{R}^{m\times n}$.

A way to <u>naïvely</u> interpret Γ-convergence is that F "minimizes" the energy at level ε. Let us write then (the <u>unprecise</u> formula)

$$F(u, Q_\rho) \sim \inf_{v \in W_0^{1,p}(Q_\rho)} \int_{Q_\rho} f\left(\tfrac{x}{\varepsilon}, A + \nabla v(x)\right) dx.$$

By the structure of f, we are tempted to change variable $x' = x/\varepsilon$, so that we obtain

$$f_{hom}(A) \sim \frac{1}{\rho^n} F(u, Q_\rho) \sim \left(\frac{\varepsilon}{\rho}\right)^n \inf_{u \in W_0^{1,p}(Q_{\rho/\varepsilon})} \int_{Q_{\rho/\varepsilon}} f(x', A + \nabla u(x'))\, dx'.$$

We see now that, denoting $t := \rho/\varepsilon \to +\infty$ as $\varepsilon \to 0$, we obtain the limit on the right-hand side of (HF).

This naïve argument is made rigorous in the next proof: while the explanation above is a consequence of the structure of the density at level ε (and it is proved rigorously in Step 2), the fact that this limit exists is a consequence of the periodicity of f (see Step 1).

Proof: we work in two steps.

Step 1 (asymptotic homogenization formula): for any $A \in \mathbb{R}^{m\times n}$ and $t>0$, let

$$g_t(A) := \frac{1}{t^n} \inf\left\{ \int_{(0,t)^n} f(x, A+\nabla u(x))\,dx : u \in W_0^{1,p}((0,t)^n;\mathbb{R}^m)\right\}$$

and let $u_t \in W_0^{1,p}((0,t)^n;\mathbb{R}^m)$ s.t.

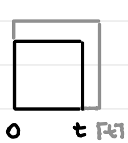
$$\frac{1}{t^n}\int_{(0,t)^n} f(x, A+\nabla u_t(x))\,dx \le g_t(A) + \frac{1}{t}.$$

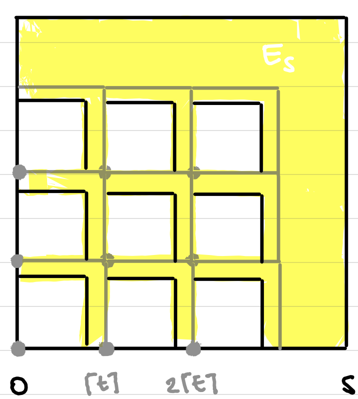
Let $s>t$ and construct $u_s \in W_0^{1,p}((0,s)^n;\mathbb{R}^m)$ as follows: we denote $\mathcal{J} := \lceil t\rceil \mathbb{Z}^n \cap [0, s-\lceil t\rceil]^n$ and notice that $\#\mathcal{J} = \lfloor s/\lceil t\rceil \rfloor^n$. We also write $E_s := \bigcup_{i\in\mathcal{J}} (i+(0,t)^n)$ and notice that

$$\begin{aligned}|(0,s)^n \setminus E_s| &= (\lceil t\rceil^n - t^n)\cdot \#\mathcal{J} + s^n - \left(\lceil t\rceil \lfloor s/\lceil t\rceil \rfloor\right)^n \\ &\le n\frac{s^n}{\lceil t\rceil} + n\lceil t\rceil s^{n-1} \le n\frac{s^n}{t} + n(t+1)s^{n-1}\end{aligned}$$

We define

$$u_s(x) := \begin{cases} u_t(x-i) & x\in i+(0,t)^n,\ i\in\mathcal{J} \\ 0 & \text{otherwise}\end{cases}$$





We estimate $g_s(A)$ by using u_s as a test function.

$$g_s(A) \leq \frac{1}{s^n}\int_{(0,s)^n} f(x, A+\nabla u_s(x))\,dx$$

$$= \tfrac{1}{s^n}\sum_{i\in J}\int_{i+(0,t)^n} f(x, A+\nabla u_t(x-i))\,dx + \tfrac{1}{s^n}\int_{(0,s)^n\setminus E_s} f(x,A)\,dx$$

$$\overset{y=x-i}{=} \tfrac{1}{s^n}\sum_{i\in J}\int_{(0,t)^n} f(y+i, A+\nabla u_t(y))\,dy + \tfrac{1}{s^n}\int_{(0,s)^n\setminus E_s} f(x,A)\,dx$$

$$\overset{(P),(G_p)}{\leq} \frac{\# J}{s^n}\int_{(0,t)^n} f(y, A+\nabla u_t(y))\,dy + \frac{|(0,s)^n\setminus E_s|}{s^n}\, c_2(|A|^p+1)$$

$$\leq \tfrac{1}{t^n}\int_{(0,t)^n} f(y, A+\nabla u_t(y))\,dy + \left(\frac{n}{t}+\frac{n(t+1)}{s}\right)c_2(|A|^p+1)$$

By definition of u_t we get

$$g_s(A) \leq g_t(A) + \tfrac{1}{t} + \left(\frac{n}{t}+\frac{n(t+1)}{s}\right)c_2(|A|^p+1)$$

which holds true $\forall\, s>t$. Taking the limsup as $s\to+\infty$ we obtain that

$$\overline{\lim_{s\to+\infty}}\, g_s(A) \leq g_t(A) + \frac{1}{t} + \frac{n}{t}\, c_2(|A|^p+1), \quad \forall\, t>0.$$

Taking now the liminf as $t\to+\infty$ we finally have

$$\overline{\lim_{s\to+\infty}}\, g_s(A) \leq \underline{\lim_{t\to+\infty}}\, g_t(A)$$

which implies that the limit of the right-hand side of (HF) exists for every $A\in\mathbb{R}^{m\times n}$.

Step 2 (homogenization result): let $\{\varepsilon_{j_k}\}$ and φ as in $(*)$. By quasi-convexity and (G_p) of φ (e.g. by (1.3) of Remark in the last lecture) and by convergence of minimum problems subject to boundary conditions (again as in the Remark of last lecture) we obtain $\forall\, A\in\mathbb{R}^{m\times n}$

$$\varphi(A) = \min\left\{\int_{(0,1)^n} \varphi(A+\nabla v(y))\,dy : v \in W_0^{1,p}((0,1)^n;\mathbb{R}^m)\right\}$$

$$= \lim_{k\to+\infty} \min\left\{\int_{(0,1)^n} f\left(\tfrac{y}{\varepsilon_{j_k}}, A+\nabla v(y)\right)dy : v \in W_0^{1,p}((0,1)^n;\mathbb{R}^m)\right\}$$

$$\overset{x=y/\varepsilon_{j_k}}{=} \lim_{k\to+\infty} \min\left\{\frac{1}{t_k^n}\int_{(0,t_k)^n} f(x, A+\nabla u(x))\,dx : u \in W_0^{1,p}((0,t_k)^n;\mathbb{R}^m)\right\}$$

where we denoted $t_k := \varepsilon_{j_k}^{-1}$ and $u(x) := \varepsilon_{j_k} v(x/\varepsilon_{j_k})$. By Step 1 $\varphi(A) = f_{hom}(A)$.

Since f_{hom} does not depend on the subsequence, by Urysohn property of Γ-convergence we get the result. □

The previous result is (essentially) a consequence of the convergence of minimum problems and the periodicity of f and comes from the characterization of quasiconvex functions.

Since quasiconvexity (in certain cases) can also be characterized by minimizing over periodic funtions, the asymptotic homogenization formula (HF) can be written also in a different way.

Rmk (asymtotic formula on periodic functions): working similarly as in Step 1 of the proof above, we can obtain

$$\text{(PF)}\quad f_{hom}(A) = \lim_{j\to+\infty} \inf\left\{\frac{1}{j^n}\int_{(0,j)^n} f(x, A+\nabla u(x))\,dx : u \in W_{\#}^{1,p}((0,j)^n;\mathbb{R}^m)\right\},$$

where $W_{\#}^{1,p}((0,j)^n;\mathbb{R}^m) := \{u \in W_{loc}^{1,p}(\mathbb{R}^n;\mathbb{R}^m) : u \text{ is } (0,j)^n\text{-periodic}\}$.

In the case in which f is convex in the second variable, the formula defining f_{hom} simplifies to a single cell formula.

This is because, in this case, a superposition of "minimizers" of the problem in $(0,j)^n$ is more energetically convenient. For this it is useful the periodic formula (PF).

<u>Corollary</u> 3 (cell formula): assume that all the hypotheses of Theorem 2 hold true and assume additionally that $f(x,\cdot)$ is convex $\forall x \in \Omega$. Then the result of Theorem 2 holds with

$$\text{(CF)} \quad f_{hom}(A) = \inf \left\{ \int_{(0,1)^n} f(y, A + \nabla v(y))\,dy : v \in W^{1,p}_{\#}((0,1)^n; \mathbb{R}^m) \right\}.$$

We will prove this next time.

Lecture 13: 08.07.2024

We saw last lecture that nonhomogeneous functionals as in (E) whose dependence on the space variable is $[0,1]^n$-periodic Γ-converge to an homogeneous integral functional whose limit density f_{hom} can be explicitly characterized by solving minimum problems on larger and larger cubes.

While this is useful, in practical situations might still be difficult to find an expression for f_{hom}.
In the case in which the density $f(x,\cdot)$ is convex in the second variable we can further simplify this formula.

<u>Corollary</u> 1 (cell formula): assume that all the hypotheses of Theorem 2 of last lecture hold and assume additionally that $f(x,\cdot)$ is convex $\forall x \in \mathbb{R}^n$. Then the results of Theorem 2 (of last lecture) hold with

$$\text{(CF)} \quad f_{hom}(A) = \inf\left\{ \int_{(0,1)^n} f(y, A + Du(y))\,dy : u \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m) \right\},$$

$$\forall A \in \mathbb{R}^{m\times n}.$$

<u>proof</u>: by the last Remark of last lecture, f_{hom} is defined by formula (PF). We denote $g(A)$ the right-hand side of (CF). We want to prove that $f_{hom}(A) = g(A)$, $\forall A \in \mathbb{R}^{m\times n}$.

We start noticing that $u \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m) \subseteq W^{1,p}_{\#}((0,j)^n;\mathbb{R}^m)$ for every $j \in \mathbb{N}$, $j \geq 1$. Moreover

$$(*) \quad \int_{(0,j)^n} f(y, A + Du(y))\,dy = \sum_{i \in \mathbb{Z}^n \cap [0,j)^n} \int_{i+(0,1)^n} f(y, A+Du(y))\,dy \overset{[0,1]^n\text{-periodicity}}{=} j^n \int_{(0,1)^n} f(y, A + Du(y))\,dy.$$

By monotonicity of the inf (taking the inf in $W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)$ in (*)) we get

$$\inf\left\{ \frac{1}{j^n}\int_{(0,j)^n} f(y, A+\nabla u(y))\,dy : u\in W^{1,p}_{\#}((0,j)^n;\mathbb{R}^m)\right\} \le g(A)$$

$\forall\, j\in\mathbb{N},\ j\geq 1$ which yields $f_{hom}(A)\le g(A)$ by taking the limit as $j\to+\infty$.

To prove the converse inequality we use the so called su-perposition principle.
Let $v\in W^{1,p}_{\#}((0,j)^n;\mathbb{R}^m)$ we define

$$u(x) := \sum_{i\in\mathbb{Z}^n\cap[0,j)^n} \frac{1}{j^n}\, v(x+i).$$

This function is obviously still $W^{1,p}_{loc}$. It can be easily shown that it is also $[0,1]^n$-periodic, indeed

$$\begin{aligned}
u(x+e_1) &= \sum_{i\in\mathbb{Z}^n\cap[0,j)^n} \frac{1}{j^n}\, v(x+e_1+i)\\
&\overset{i'=i+e_1}{=} \sum_{i'_1\in\mathbb{Z}\cap[1,j+1)}\ \sum_{\substack{i'_h\in\mathbb{Z}\cap[0,j)\\ 1<h\le n}} \frac{1}{j^n}\, v(x+i')\\
&= \sum_{i'_1\in\mathbb{Z}\cap[1,j)}\ \sum_{\substack{i'_h\in\mathbb{Z}\cap[0,j)\\ 1<h\le n}} \frac{1}{j^n}\, v(x+i') +\\
&\qquad + \sum_{\substack{i'_h\in\mathbb{Z}\cap[0,j)\\ 1<h\le n}} \frac{1}{j^n}\, \overset{=\,v(x_1,\dots,x_n+i'_n)}{v(x_1+j,\dots,x_n+i'_n)}\\
&= u(x),
\end{aligned}$$

where in the last equality we used the $[0,j]^n$-periodicity of v. Working analogously in all the other directions we obtain that $u\in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)$.
By (*) and convexity, we have

$$\begin{aligned}
\int_{(0,1)^n} f(x,A+\nabla u(x))\,dx &= \frac{1}{j^n}\int_{(0,j)^n} f(x,A+\nabla u(x))\,dx\\
&= \frac{1}{j^n}\int_{(0,j)^n} f\Big(x, A+\sum_{i\in\mathbb{Z}^n\cap[0,j)^n} \frac{1}{j^n}\nabla v(x+i)\Big)\,dx
\end{aligned}$$

$$\overset{\text{convexity}}{\leq} \frac{1}{j^n}\int_{(0,j)^n} \sum_{i\in\mathbb{Z}^n\cap[0,j)^n} \frac{1}{j^n} f(x, A+\nabla v(x+i))\,dx$$

$$\overset{x'=x+i}{=} \frac{1}{j^n} \sum_{i\in\mathbb{Z}^n\cap[0,j)^n} \frac{1}{j^n}\int_{i+(0,j)^n} \overset{= f(x', A+\nabla v(x'))}{f(x'+i, A+\nabla v(x'))}\,dx'$$

$$= \frac{1}{j^n}\int_{(0,j)^n} f(x, A+\nabla v(x))\,dx$$

where in the last step we used that $[0,j]^n$- periodic functions have the same average in ever cube of length side j.
Since $\forall\, v \in W^{1,p}_{\#}((0,j)^n;\mathbb{R}^m)$ we found a $u\in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)$ such that

$$\int_{(0,1)^n} f(x, A+\nabla u(x))\,dx \leq \frac{1}{j^n}\int_{(0,j)^n} f(x, A+\nabla v(x))\,dx$$

and since $g(A) \leq \int_{(0,1)^n} f(x, A+\nabla u(x))\,dx$, by taking the inf over $v\in W^{1,p}_{\#}((0,j)^n;\mathbb{R}^m)$ and then the limit as $j\to+\infty$ we obtain that $g(A)\leq f_{hom}(A)$. $\square$

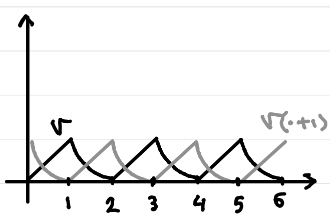
The result above can be interpreted by saying that the Γ-limit is completely understood just by solving a <u>single</u> minimum problem.

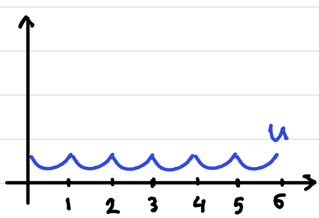
<u>Rmk</u>: to visualize the superposition principle we make an easy one-dimensional example. Let

$$v(x) = \begin{cases} x & 0\leq x<1 \\ (x-2)^2 & 1\leq x<2 \end{cases}, \quad 2\text{- periodically extended.}$$

Then

$$u(x) = \frac{1}{2}\left(v(x)+v(x+1)\right) = \frac{x^2-x+1}{2}, \quad 1\text{- periodically extended.}$$





Some example on convex homogenization

Now we have all the tools to solve some "practical" problems, some of which we introduced in the past lectures.

<u>Ex</u> (limit of nonhomogeneous, nonlinear, variational PDEs): consider $\Omega \subseteq \mathbb{R}^n$ open, bounded and Lipschitz and $1<p<\infty$.
One can be interested in studying the following problem: let $g \in L^{p'}(\Omega)$, $a : \mathbb{R}^n \longrightarrow [\alpha,\beta]$ $[0,1]^n$- periodic, Borel function with $0<\alpha<\beta<+\infty$. For any $\varepsilon>0$, consider

$$(P_\varepsilon) \qquad \begin{cases} -\operatorname{div}\left(a(\cdot/\varepsilon)|\nabla u|^{p-2}\nabla u\right) = g & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

This studies the propagation of electrical potential u, in a body Ω (characterized by a nonlinear propagation of electri-city) in presence of a source term g, with conductivity a that varies $[0,\varepsilon]^n$- periodically.
The problem at level ε presents several difficulties:
- the nonhomogeneity of the operator, i.e. a depends on x;
- the frequency of the conductivity is very high, ε^{-1}.

Thanks to Γ- convergence we can say that solutions of (P_ε) can be approximated by the solutions of an "easier" problem.

Since the operator $-\operatorname{div}(a(\cdot/\varepsilon)|\nabla\cdot|^{p-2}\nabla\cdot)$ is uniformly elli-ptic and coercive $\exists!\, u_\varepsilon \in W_0^{1,p}(\Omega)$ solution to (P_ε).
This correspond to the unique minimizer of the energy functional

$$E_\varepsilon(u) := \int_\Omega \frac{1}{p}|\nabla u(x)|^p - g(x)u(x)\,dx.$$

The existence and uniqueness of a minimizer for E_ε comes from the Direct Method and by strict convexity. This corresponds to the solution of (P_ε) since (P_ε) is the Euler-Lagrange equation of E_ε.

Since the linear part of E_ε can be seen as a continuous perturbation, we study

$$F_\varepsilon(u) := \int_\Omega \frac{1}{p} Q(x/\varepsilon)\, |\nabla u(x)|^p \, dx.$$

By Corollary 1 (applied with $f(x,\xi) = \frac{1}{p} Q(x)\,|\xi|^p$) we have

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega f_{hom}(\nabla u(x))\, dx$$

where, $\forall\, \xi \in \mathbb{R}^n$,

$$f_{hom}(\xi) = \inf\left\{ \int_{(0,1)^n} \frac{1}{p} Q(y)\,|\xi + \nabla u(y)|^p \, dy : u \in W^{1,p}_{\#}((0,1)^n) \right\}.$$

We can exploit the formula above to prove one property of f_{hom} and to find a more transparent expression.
f_{hom} is p-homogeneous, i.e. $f_{hom}(t\xi) = |t|^p f_{hom}(\xi)$ $\forall\, t \in \mathbb{R}\setminus\{0\}$, $\xi \in \mathbb{R}^n$.
By taking $u = \text{const.}$ it is immediate to see that $f_{hom}(0) = 0$. Let $\xi \in \mathbb{R}^n\setminus\{0\}$, then

$$\begin{aligned} f_{hom}(t\xi) &= \inf\left\{ \frac{1}{p}\int_{(0,1)^n} Q(y)\,|t\xi + \nabla u(y)|^p \, dy : u \in W^{1,p}_{\#}((0,1)^n) \right\} \\ &\overset{v = \frac{1}{t}u}{=} \inf\left\{ \frac{1}{p}\int_{(0,1)^n} Q(y)\,|t(\xi + \nabla v(y))|^p \, dy : v \in W^{1,p}_{\#}((0,1)^n) \right\} \\ &= |t|^p f_{hom}(\xi). \end{aligned}$$

Thanks to this property we can write $f_{hom}(\xi) = \frac{1}{p} Q_{hom}(\xi/|\xi|)\,|\xi|^p$ where $Q_{hom} : S^{n-1} \to [\alpha,\beta]$ defined as

$$a_{hom}(v) = \int_{(0,1)^n} a(y)\,|v + \nabla u(y)|^p\, dy$$

with $u_v \in W^{1,p}_{\#}((0,1)^n)$ solution of $\operatorname{div}(a|v+\nabla u_v|^{p-2}(v+\nabla u_v)) = 0$.

By stability under continuous perturbations we have

$$\Gamma(L^p)-\lim_{\varepsilon\to 0} E_\varepsilon(u) = \int_\Omega f_{hom}(\nabla u(x)) - g(x)\,u(x)\, dx.$$

In particular, by stability under boundary conditions and the Fundamental Theorem of Γ-convergence we infer that: given $u_\varepsilon \in W^{1,p}_0(\Omega)$ solutions of (P_ε)

$$u_\varepsilon \xrightarrow{W^{1,p}} u_0 \quad \text{as} \quad \varepsilon \to 0$$

where u_0 is the unique solution of

$$(P_0) \qquad \begin{cases} -\operatorname{div}(\nabla f_{hom}(\nabla u)) = g & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

We sensibly simplified our initial problem since (P_0) is now a homogeneous PDE.
One can also prove (we will not see it here) that f_{hom} is differentiable (and so is a_{hom}), so that (P_0) is well-defined.

To know the value of a_{hom} we need to solve a nonhomogeneous PDE but in the simple domain $(0,1)^n$ and we no appearence of ε.
Moreover, we have (for free) some estimate for a_{hom}, e.g. $\alpha \leq a_{hom} \leq \beta$, that can already help in giving some information of u_ε.

<u>Notice</u>: the p-homogeneity of f_{hom} corresponds to that of the functional $\int_\Omega f_{hom}(\nabla u(x))\,dx$ (one direction is trivial

the other can be easily obtain by testing the functional on affine functions).
It is a general property of Γ-convergence that the Γ-limit of p-homogeneous functionals is p-homogeneous.

The expression for a_{hom} in the previous example simplifies in the one-dimensional case.
It can then be compute without much effort.

Ex: let $a, b \in \mathbb{R}$, $a<b$ and consider

$$F_\varepsilon(u) := \int_a^b a(x/\varepsilon)\, |u'(x)|^p \, dx.$$

In the previous example we saw that

$$\Gamma(L^p) - \lim_{\varepsilon \to 0} F_\varepsilon(u) = \int_a^b a_{hom} |u'(x)|^p \, dx$$

with $a_{hom} = \int_a^b a(y) |1+\tilde{u}'(y)|^p dy$ where $\tilde{u} \in W^{1,p}_{\#}(0,1)$

$$(*_1) \quad \begin{cases} \frac{d}{dy}\left(a(y)|1+\tilde{u}'(y)|^{p-2}(1+\tilde{u}'(y))\right) = 0 & \text{in } (0,1), \\ \tilde{u} \in W^{1,p}_{\#}(0,1). \end{cases}$$

By integrating $(*_1)$ we get

$$a(y)|1+\tilde{u}'(y)|^{p-2}(1+\tilde{u}'(y)) = C_1$$

for some constant $C_1 \in \mathbb{R}$.
Since a is strictly positive, $1+\tilde{u}'$ never changes sign. So, up to chang the of C_1, we can assume that $1+\tilde{u}' > 0$. We then obtain

$$(1+\tilde{u}'(y))^{p-1} = \frac{C_1}{a(y)} \iff \tilde{u}'(y) = \left(\frac{C_1}{a(y)}\right)^{\frac{1}{p-1}} - 1.$$

Since $\tilde{u}$ is periodic, $\tilde{u}'$ has zero average, so

$$0 = c^{\frac{1}{p-1}} \int_0^1 a(y)^{-\frac{1}{p-1}} dy - 1 \iff c^{\frac{1}{p-1}} = \left(\int_0^1 a(y)^{-\frac{1}{p-1}} dy\right)^{-1}.$$

Plugging these into the formula defing a_{hom} we get

$$a_{hom} = \int_0^1 a(y)\, c^{\frac{p}{p-1}}\, a(y)^{-\frac{p}{p-1}} dy$$

$$= \int_0^1 a(y)^{-\frac{1}{p-1}} dy \cdot \left(\int_0^1 a(y)^{-\frac{1}{p-1}} dy\right)^{-p}$$

$$= \left(\int_0^1 a(y)^{-\frac{1}{p-1}} dy\right)^{-(p-1)} =: a^*_p.$$

The value a^*_p is sometimes said the p-harmonic mean of a.

We now consider a problem in higher dimension in which we can give a better characterization of the Γ-limit.
This is the case in which the density $f(x,\cdot)$ is a quadratic form for every $x \in \mathbb{R}^n$.

<u>Ex</u> (homogenization of quadratic forms): let $A: \mathbb{R}^n \to \mathbb{R}^{n\times n}_{sym}$ be Borel, elliptic, i.e. $\exists\, \alpha, \beta > 0$ s.t.

$$\alpha |\xi|^2 \leq (A(x)\xi)\cdot\xi \leq \beta |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \mathbb{R}^n$$

and $[0,1]^n$-periodic, and consider

$$F_\varepsilon(u) = \int_\Omega (A(x/\varepsilon) \nabla u(x)) \cdot \nabla u(x)\, dx.$$

Again by Corollary 1 we know that

$$\Gamma(L^2)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega f_{hom}(\nabla u(x))\, dx,$$

with f_{hom} defined by (CF). Working analogously as in the example before, we can infer that f_{hom} is 2-homogeneous.
We can actually prove that f_{hom} is a quadratic form:
it is sufficient to prove the parallelogram identity, i.e. that
$f_{hom}(\xi_1+\xi_2)+f_{hom}(\xi_1-\xi_2)=2f_{hom}(\xi_1)+2f_{hom}(\xi_2)\ \forall \xi_1,\xi_2\in\mathbb{R}^n$.

By the previous example we know that, $\forall \xi\in\mathbb{R}^n$

$$f_{hom}(\xi)=\int_{(0,1)^n}\left(A(y)(\xi+\nabla u_\xi(y))\right)\cdot(\xi+\nabla u_\xi(y))\,dy$$

where $u_\xi\in W^{1,2}_{\#}((0,1)^n)$ solves $\operatorname{div}(A(\xi+\nabla u_\xi))=0$.
By the linearity of the PDE, we have that $u_{\xi_1\pm\xi_2}=u_{\xi_1}\pm u_{\xi_2}$.
By this, since $(A(y)\xi)\cdot\xi$ is a quadratic form $\forall y\in(0,1)^n$, and hence satisfies the parallelogram identity, we get

$$f_{hom}(\xi_1+\xi_2)+f_{hom}(\xi_1-\xi_2)=$$

$$=\int_{(0,1)^n}\Big(A(y)\big((\xi_1+\nabla u_{\xi_1}(y))+(\xi_2+\nabla u_{\xi_2}(y))\big)\cdot\big((\xi_1+\nabla u_{\xi_1}(y))+(\xi_2+\nabla u_{\xi_2}(y))\big)+$$

$$+\Big(A(y)\big((\xi_1+\nabla u_{\xi_1}(y))-(\xi_2+\nabla u_{\xi_2}(y))\big)\cdot\big((\xi_1+\nabla u_{\xi_1}(y))-(\xi_2+\nabla u_{\xi_2}(y))\big)\,dy$$

$$=\int_{(0,1)^n}2\big(A(y)(\xi_1+\nabla u_{\xi_1}(y))\big)\cdot(\xi_1+\nabla u_{\xi_1}(y))+2\big(A(y)(\xi_2+\nabla u_{\xi_2}(y))\big)\cdot(\xi_2+\nabla u_{\xi_2}(y))\,dy$$

$$=2f_{hom}(\xi_1)+2f_{hom}(\xi_2).$$

Since f_{hom} is a quadratic form $\Rightarrow \exists\, A_{hom}\in\mathbb{R}^{n\times n}_{sym}$ with $\alpha|\xi|^2\leq(A_{hom}\xi)\cdot\xi\leq\beta|\xi|^2$ s.t. $f_{hom}(\xi):=(A_{hom}\xi)\cdot\xi$.
Again by (CF) we also have a formula to compute the entries of A_{hom}.
Indeed, $\forall\, j=1,\dots,n$

$$(A_{hom})_{jj}=f_{hom}(e_j)=\int_{(0,1)^n}\left(A(y)(e_j+\nabla u^{(j)}(y))\right)\cdot(e_j+\nabla u^{(j)}(y))\,dy$$

with $u^{(j)}\in W^{1,2}_{\#}((0,1)^n)$ solving $\operatorname{div}(A(e_j+\nabla u^{(j)}))=0$.

Moreover, $\forall\, i,j=1,\dots,n$ we also have

$$(A_{hom})_{ij} = \frac{1}{2}\left(f_{hom}(e_i+e_j) - (A_{hom})_{ii} - (A_{hom})_{jj}\right)$$

$$= \frac{1}{2}\int_{[0,1)^n} (A(y)(e_i+e_j+\nabla u^{(i)}(y)+\nabla u^{(j)}(y))) \cdot (e_i+e_j+\nabla u^{(i)}(y)+\nabla u^{(j)}(y)) -$$

$$- (A(y)(e_i+\nabla u^{(i)}(y))) \cdot (e_i+\nabla u^{(i)}(y)) -$$

$$- (A(y)(e_j+\nabla u^{(j)}(y))) \cdot (e_j+\nabla u^{(j)}(y))\, dy$$

$$= \int_{[0,1)^n} (A(y)(e_i+\nabla u^{(i)}(y))) \cdot (e_j+\nabla u^{(j)}(y))\, dy.$$

Notice: as for the case of p-homogeneous functionals, Γ-limits of quadratic forms is (in general) a quadratic form.

We can apply the results of the previous two example to under-stand elastic properties of layered materials. These are mate-rials composed by layers of two (or more) different materials which alternate at very small length scales and have different elastic parameters.
It is interesting to understand the macroscopic elastic beha-viour of these materials.

Ex (homogenization of layered materials): consider $n=2$ and $A(y) = a(y)\, I_2$ where $a(y) = \begin{cases} \alpha & 0 \le y_1 < \theta \\ \beta & \theta \le y_1 < 1 \end{cases}$ extended $[0,1]^2$-periodi-cally.
By the previous example

$$\Gamma(L^2)-\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega (A_{hom}\nabla u(x)) \cdot \nabla u(x)\, dx.$$

In this case, the computation of the entries of A_{hom} can be reduced to a one-dimensional problem.

$$(A_{hom})_{11} = \int_{(0,1)^2} a(y)\,|e_1 + \nabla u^{(1)}(y)|^2\,dy$$

where $u^{(1)} \in W^{1,2}_{\#}((0,1)^2)$ solves $\partial_1(a(1+\partial_1 u^{(1)})) + \partial_2(a\,\partial_2 u^{(1)}) = 0$.
By solving $\forall\, y_2 \in (0,1)$ $\partial_1(a(\cdot,y_2)(1+\partial_1 u^{(1)}(\cdot,y_2)) = 0$, working as in the one-dimensional example we obtain

$$(*_1) \qquad \partial_1 u^{(1)}(y_1,y_2) = \frac{a^*}{a(y_1,y_2)} - 1,$$

where a^* is the harmonic mean of a. Integrating in y_1

$$u^{(1)}(y_1,y_2) = a^* \int_0^{y_1} \frac{1}{a(t,y_2)}\,dt - y_1 + c(y_2).$$

Since a is constant in y_2, $\partial_2 u^{(1)}(y_1,y_2) = c'(y_2)$. Then
$\partial_2(a(y_1,y_2)\,\partial_2 u^{(1)}(y_1,y_2)) = \partial_2(a(y_1,y_2)\,c'(y_2)) = a(y_1,y_2)\,c''(y_2) = 0$.
By the periodicity of $u^{(1)}$ we get $c =$ const. This also yields $\partial_2 u^{(1)} = 0$.
Plugging $(*_1)$ into the formula for $(A_{hom})_{11}$ we get

$$(A_{hom})_{11} = \int_{(0,1)^2} a(y)\,\frac{(a^*)^2}{a(y)^2}\,dy = a^*.$$

Whereas

$$(A_{hom})_{22} = \int_{(0,1)^2} a(y)\,|e_2 + \nabla u^{(2)}(y)|^2\,dy$$

with $u^{(2)} \in W^{1,2}_{\#}((0,1)^2)$ solves $\partial_1(a\,\partial_1 u^{(2)}) + \partial_2(a(1+\partial_2 u^{(2)})) = 0$,
This is solved by $u^{(2)} =$ const. So $(A_{hom})_{22} = \bar{a}$, where $\bar{a}$ is the average of a.
Finally

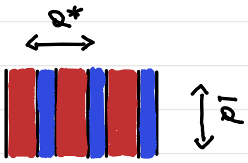
$$(A_{hom})_{12} = \int_{(0,1)} a(y)\left(e_1 + \begin{pmatrix} \partial_1 u^{(1)}(y) \\ 0 \end{pmatrix}\right)\cdot e_2\,dy = 0,$$

so $A_{hom} = \begin{pmatrix} a^* & 0 \\ 0 & \bar{a} \end{pmatrix}$.

It is trivial to generalize this to every dimension.

We can interpret the previous example from a modeling point of view.

A layered material has different responce to deformation in different direnctions. Since $a^* \leq \bar{a}$, it is "easier" to deform the material ortoghonally to the layers while it is "harder" to do it in the directions parallel to the layers.



Lecture 14: 15.07.2024

Main references:

- "A Handbook of Γ-convergence", Braides (2006), Sec. 7.2

Both in the case of Lebesgue and Sobolev spaces, we used Γ-convergence to prove that minimum problems of integral functionals converge to minimum problems of a limit functional of the "same nature", i.e. still of integral type, even though its explicit form was far from the first intuition.

The versatility of Γ-convergence comes from the fact that we can use it to deal with other two intersting phenomena: limit of functionals defined in different spaces; a change of type of the limit energy.

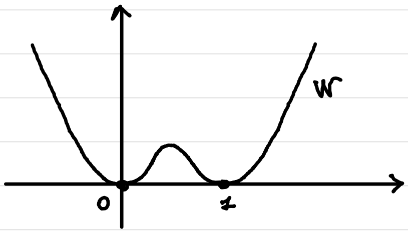
While the former is usually present in discrete-to-continuum problems (that unfortunately we don't have time to treat), the latter is characteristic of problems relate to phase transitions.

Gradient theory of phase transitions

We start by giving some "physical" intuition which motivates the interest in the mathematical model.

Let $\Omega \subseteq \mathbb{R}^n$ open, bounded, Lipschitz represent a container of a fluid. Let $u : \Omega \to \mathbb{R}$ be the concentration of the fluid. We want to model equilibrium configuration of two fluids with different concentration (e.g. 0 and 1) inside Ω.

We consider a double-well energy density $W: \mathbb{R} \to [0,+\infty)$ sometimes called the Van der Waals energy density.



The equilibrium configurations subject to a volume constraint (of one of the two fluids, e.g. the of concentration 1) are given by minimizing

$$F_\varepsilon(u) = \int_\Omega \frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \, dx$$

among $u \in W^{1,2}(\Omega)$, for instance with $\int_\Omega u(x)\,dx = V$.
Here the term $\varepsilon^2 |\nabla u|^2$ represent the surface tension between the two fluids.
Minimizing the term $\varepsilon^{-1} W(u)$ we expect that the two fluids do not mix, by minimizing $\varepsilon |\nabla u|^2$ we expect that the region in which we observe a phase transition is "as small as possible".

The functional F_ε has been studied by Modica and Mortola (1977) and it is one of the first example of Γ-convergence in this context.

<u>Theorem</u> (Modica-Mortola): let $W \in C^1(\mathbb{R};[0,+\infty))$ s.t. $W(t) = 0$ if and only if $t = 0,1$ and $\exists\, c_1, c_2 > 0$ s.t.

$$c_1(|t|^2 - 1) \leq W(t) \leq c_2(|t|^2+1) \qquad \forall t \in \mathbb{R}.$$

Let $F_\varepsilon : L^1(\Omega) \to [0,+\infty]$ be

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \, dx & u \in W^{1,2}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\Gamma(L^1)\text{-}\lim\limits_{\varepsilon \to 0} F_\varepsilon = F_0$ where

$$F_0(u) := \begin{cases} c_W P(\{u=1\};\Omega) & \text{if } u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$.

<u>Rmk</u> (sets of finite perimeter): the (relative) perimeter functional is defined as

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}(\varphi(x))\,dx : \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ |\varphi| \le 1 \right\}.$$

The class of sets s.t. $P(\cdot;\Omega)$ is well-defined is called sets of finite (relative) perimeter in Ω.
To have an intuition of this, if $P(E;\Omega) < +\infty$ then the linear functional $\varphi \mapsto \int_E \operatorname{div}(\varphi(x))\,dx$ is bounded (and densely defined) on $C_0(\Omega;\mathbb{R}^n)$. By Riesz's Theorem it is in duality with a (vector valued) Radon measure $\mu_E \in \mathcal{M}(\Omega;\mathbb{R}^n)$.
If we consider χ_E as a distribution and denote $\nabla\chi_E$ as its distributional gradient we can write, $\forall\, \varphi \in C_0^1(\Omega;\mathbb{R}^n)$

$$\int_\Omega \varphi(x)\cdot d\mu_E(x) = \int_\Omega \chi_E(x)\operatorname{div}(\varphi(x))\,dx = -\int_\Omega \nabla\chi_E(x)\cdot\varphi(x)\,dx.$$

Loosely speaking, sets of finite perimeter are those sets whose distributional derivative is a Radon measure.

For smooth sets E (e.g. C^1), $P(E;\Omega)$ coincides with the "standard" notion of perimeter of a set, i.e. the surface area of $\partial E \cap \Omega$

$$P(E;\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega),$$

and $d\mu_E(x) = \nu_{\partial E}(x)\, d(\mathcal{H}^{n-1} \llcorner \partial E)(x)$.
The same holds in general for sets of finite perimeter utilizing a suitable notion of boundary.

Sets of finite perimeter is a subclass of the space of functions of bounded variations, $BV(\Omega) \subset L^1(\Omega)$. These are L^1 functions s.t. their distributional derivative is a Radon measure. $\chi_E \in BV(\Omega)$ if and only if E is a set of finite (relative) perimeter in Ω.

<u>Rmk</u> (one-dimension): sets of finite (relative) perimeter have a particularly easy form on the real line, i.e. if $\Omega = (a,b)$ for some $a, b \in \mathbb{R}$.
In this case $E \subseteq (a,b)$ is of finite perimeter if and only if it is a finite union of (disjoint) intervals, $E = \bigcup_{i=1}^{N} (a_i, b_i)$, and

$$P(E, \Omega) = \# \{ a_i \neq a, b_i \neq b : i = 1 \dots N \},$$

that is, the number of extreme different from a and b.

<u>proof</u> (ideas): We consider first the one dimensional case, i.e. $\Omega = (a,b)$. Our energies read

$$F_\varepsilon(u) = \int_a^b \tfrac{1}{\varepsilon} W(u(x)) + \varepsilon |u'(x)|^2 \, dx$$

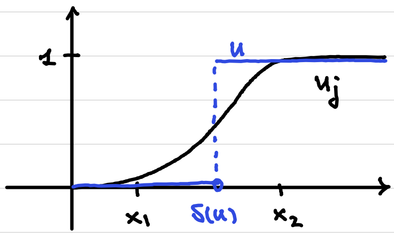
and the limit energy is $F(u) = c_W \# S(u)$, where $S(u) := \{ x \in (a,b) : u \text{ is discontinuous in } x \}$.

We start with the liminf inequality. Let u_j s.t. $F_{\varepsilon_j}(u_j) \leq c_1$ for some constant (otherwise there is nothing to prove). This implies that

$$\int_a^b W(u_j(x)) \, dx \leq \varepsilon_j c_1$$

thus $u_j \to u$ in $L^1(a,b)$ (up to s) where $u \in \{0,1\}$. In principle, this u may oscillate a lot (i.e. $\{u=1\}$ may not be of finite perimeter, which in 1D means that $\# S(u) = +\infty$ where $S(u)$ is the set of jumps of u).

Assume that $u \neq$ const (otherwise nothing to prove). Then since u_j converges a.e. (up to s) $\exists\, x_1, x_2 \in (a,b)$ s.t. $u_j(x_1) \to 0$ and $u_j(x_2) \to 1$.



Loosely speaking, u_j approximate a jump of u.
We control the energy of this jump by the so called "Modica-Mortola trick". Exploiting the inequality $a^2+b^2 \geq 2ab$,

$$\int_{x_1}^{x_2} \frac{1}{\varepsilon_j} W(u_j(x)) + \varepsilon_j |u_j'(x)|^2 \, dx \geq 2\int_{x_1}^{x_2} \sqrt{W(u_j(x))}\, |u_j'(x)|\, dx$$
$$= 2\int_{u_j(x_1)}^{u_j(x_2)} \sqrt{W(s)}\, ds$$

where we used the change of variable $s = u_j(x)$.
Taking the limit as $j \to +\infty$ we get

$$C_1 \geq \varliminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \geq c_W.$$

Repeating this calculation around every jump point of u (and working a little bit more) we can prove that $\{u=1\}$ is a set of finite perimeter, and

$$\# S(u)\, c_W \leq \varliminf_{j\to+\infty} F_{\varepsilon_j}(u_j)$$

which prove the liminf inequality.

We prove now the limsup inequality. Let v be a solution of the ODE

$$v'(t) = \sqrt{W(v(t))}, \qquad v(0) = 1/2.$$

By Picard's theorem $\exists$ a solution $v \in C^1(\mathbb{R}; (0,1))$ with $\lim_{t\to-\infty} v(t) = 0$, $\lim_{t\to+\infty} v(t) = 1$.
Here we also used that, by (C_2), $\sqrt{W}$ is sublinear so we have a global solution, and that v is monotone and bounded thus $\lim_{t\to\pm\infty} v'(t) = 0 \Rightarrow \lim_{t\to\pm\infty} \sqrt{W(v(t))} = 0$.

Now, let $\{x_i\}_{i=1}^N = S(u)$, let $[u](x_i) = u(x_i^+) - u(x_i^-)$ the amplitude of the jump. Since $u \in \{0,1\}$, $[u](x_i) \in \{\pm 1\}$.
We construct a recovery sequence for u as follows: we define $v_\varepsilon^{(i)} \in C^1(\mathbb{R};(0,1))$ as

$$v_\varepsilon^{(i)}(x) := \begin{cases} v\left(\frac{x-x_i}{\varepsilon}\right) & \text{if } [u](x_i) = 1 \\ v\left(\frac{x_i - x}{\varepsilon}\right) & \text{if } [u](x_i) = -1. \end{cases}$$

Each $v_\varepsilon^{(i)}$ approximate u around the jump x_i.
By a cut-off procedure we can build $u_\varepsilon \in C^1((a,b);(0,1))$ s.t. $u_\varepsilon(x) = v_\varepsilon^{(i)}(x)$ $\forall\, |x - x_i| < \delta$ for some δ sufficiently small.
This is a recovery sequence by the optimality of the profile v: indeed

$$F_\varepsilon(u_\varepsilon) \sim \sum_{i=1}^N \int_{x_i-\delta}^{x_i+\delta} \tfrac{1}{\varepsilon} W(u_\varepsilon(x)) + \varepsilon |u_\varepsilon'(x)|^2 \, dx$$

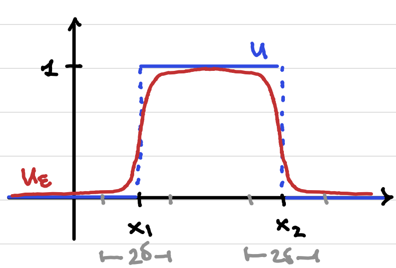
$$\overset{t = \pm\left(\frac{x-x_i}{\varepsilon}\right)}{=} \sum_{i=1}^N \int_{-\delta/\varepsilon}^{\delta/\varepsilon} W(v(t)) + |v'(t)|^2 \, dt$$

$$= \sum_{i=1}^N \int_{-\delta/\varepsilon}^{\delta/\varepsilon} 2\sqrt{W(v(t))}\, v'(t)\, dt$$

$$\overset{s = v(t)}{=} \sum_{i=1}^N \int_{v(-\delta/\varepsilon)}^{v(\delta/\varepsilon)} 2\sqrt{W(s)}\, ds.$$

By taking the limit as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \# S(u)\, c_W.$$



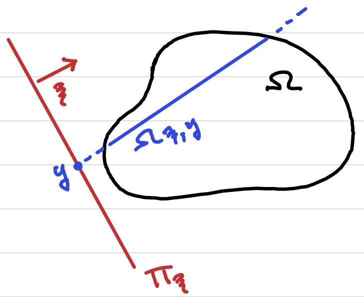
Now we briefly see some ideas on how to work in higher dimensions.
We work using a slicing method for which we can reduce to the one-dimensional case.

- $\forall \xi \in S^{n-1}$, $\forall\ y \in \Pi_\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}$, $\forall u \in W^{1,2}(\Omega)$ we define,

$$F_\varepsilon^\xi(u) := \int_\Omega \tfrac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x) \cdot \xi|^2 \, dx$$

$$= \int_{\Pi_\xi} \Big(\underbrace{\int_{\Omega_{\xi,y}} \tfrac{1}{\varepsilon} W(u_{\xi,y}(t)) + \varepsilon |u'_{\xi,y}(t)|^2 \, dt}_{=: F_\varepsilon^{\xi,y}(u_{\xi,y};\, \Omega_{\xi,y})} \Big) dy$$

where $u_{\xi,y}(t) := u(y + t\xi)$ which is a.e. (in y) well-defined.
Notice that $F_\varepsilon^\xi \leq F_\varepsilon \quad \forall \xi \in S^{n-1}$.



We controlled from below our energy by reducing to a one-dimensional one.
Here we denoted $\forall v \in W^{1,2}(I)$, $I \subseteq \mathbb{R}$

$$F_\varepsilon^{\xi,y}(v, I) := \int_I \tfrac{1}{\varepsilon} W(v(t)) + \varepsilon |v'(t)|^2 \, dt.$$

- by the one-dimensional case, $\forall v \in L^1(I; \{0,1\})$ s.t. $\{v = 1\}$ is of finite perimeter, we have

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon^{\xi,y}(v; I) = c_W \,\#S(v).$$

By Fatou's lemma, $\forall u_\varepsilon \xrightarrow{L^1} u$ (one can prove that the slices $(u_\varepsilon)_{\xi,y} \xrightarrow{L^1} u_{\xi,y}$ for a.e. $y \in \Pi_\xi$, $\forall \xi \in S^{n-1}$) then

$$\varliminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \varliminf_{\varepsilon \to 0} F_\varepsilon^\xi(u_\varepsilon)$$

$$\geq \int_{\Pi_\xi} \varliminf_{\varepsilon \to 0} \int_{\Omega_{\xi,y}} \tfrac{1}{\varepsilon} W((u_\varepsilon)_{\xi,y}(t)) + \varepsilon |(u_\varepsilon)'_{\xi,y}(t)|^2 dt \, dy$$

$$\geq \int_{\Pi_\xi} c_W \,\# S(u_{\xi,y}) \, d\mathcal{H}^{n-1}(y)$$

$\forall \xi \in S^{n-1}$.

By the previous inequality we get that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \geq c_W \int_{\pi_\xi} \# S(u_{\xi,y})\, d\mathcal{H}^{n-1}(y), \quad \forall \xi \in S^{n-1}.$$

Essentially, we are counting the jumps "trasversal" to ξ and integrating this in all possible slices.
This is not yet the result since the part of the perimeter of $\{u=1\}$ that are parallel to ξ are not counted.
Working a bit and using the characterization of sets of finite perimeter by slicing we obtain

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \geq c_W \operatorname{Per}(\{u=1\};\Omega).$$

Finally, for the limsup inequality we used that, C^∞ sets are dense in the class of sets of finite perimeter (wrt L^1 convergence of the characteristic functions), hence we can reduce working for $\{u=1\}$ being C^∞.
In this case, we set

$$v_\varepsilon(x) := v\left(\frac{d(x)}{\varepsilon}\right), \quad d(x) := \operatorname{dist}(x, \Omega\setminus\{u=1\}) - \operatorname{dist}(x,\{u=1\}).$$

By an application of the coarea formula, we can prove (similarly to the one-dimensional case) that this is optimal.