Lecture 1: 15.04.2024

Γ- convergence of integral functionals

Main references for the course:

- Γ- convergence for Beginners, Braides (2000)
- Homogenization of Multiple Integrals, Braides and Defranceschi (1998)

The framework of this course is the Calculus of Variations. Let's give a quick overview of the standard problem in CalcVar and what is the usual strategy to attack it.

Calculus of Variations: study of minimum problems.

(for simplicity a sequential space, e.g. a metric space)

Let X be a topological space, consider a function

$$F : X \longrightarrow (-\infty, +\infty], \text{ so called energy (functional).}$$

We may be interested in minimizing the energy F in X:

$$\begin{cases} \inf \{ F(u) : u \in X \} =: M > -\infty, \\ \text{(if possible) find } \bar{u} \in X : F(\bar{u}) = M. \end{cases}$$

An immediate comment is that we are not solving minimum problems "per se" but solving problems that can be rewritten as minimization of a certain functional.

Rmk (solving PDEs): since minimizers satisfies "$DF(\bar{u})=0$" we can relate minimum problems to PDEs.

solving (variational) PDEs $\Leftrightarrow$ solving a min. pb.

A classical example is Poisson's equation

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \Leftrightarrow \min_{u|_{\partial\Omega}=0} \int_\Omega \frac{1}{2}|\nabla u|^2 - fu \, dx.$$

In general, solving (distributionally) elliptic PDEs in divergence form corresponds to minimize a suitable "Lagrangean".

Direct Method (of CalcVar): this is the main tool to solve minimum problems. It consists in an application of (a variant of) Weierstrass' Theorem: if

- F is coercive (i.e. sublevels are precompact)
  $\{u \in X : F(u) \leq t\}, t \in \mathbb{R}$ — the closure is compact
- F is lower-semicontinuous (i.e. sublevels are closed)

then $X_T := \{u \in X : F(u) \leq T\}$ is compact, for $T \in \mathbb{R}$ (e.g. T chosen so that $X_T \neq \emptyset$).
By Weierstrass Thm $\exists \, \bar{u} \in X_T : F(\bar{u}) = \min_{X_T} F = \min_X F$.

Note: compactness and lower-semicontinuity are crucial notions throughout the whole course.
They depend on the choice of the topology.

Γ- convergence ...

Many mathematical problems are characterized by the presence of a parameter that may be artificial (coming from an approximation or discretization) or natural (e.g. a geometric quantity).

Let $\varepsilon > 0$ and consider $F_\varepsilon : X \longrightarrow (-\infty, +\infty]$ and study

$$\begin{cases} \inf \{ F_\varepsilon(u) : u \in X \} =: M_\varepsilon > -\infty \\ u_\varepsilon \in X \ : \ F_\varepsilon(u_\varepsilon) = M_\varepsilon \end{cases}$$

Due to the presence of an additional parameter, these problems are usually difficult.
We look for an "effective" energy $F_0 : X \longrightarrow (-\infty, +\infty]$ whose min. pb. is easier to solve but is "close" to that of F_ε:

$$\begin{cases} \inf \{ F_0(u) : u \in X \} =: M_0 > -\infty \\ u_0 \in X : F_0(u_0) = M_0 \end{cases} \qquad \text{[easier]}$$

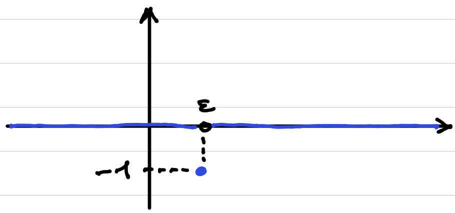
and such that $\lim_{\varepsilon \to 0} M_\varepsilon = M_0$ and $\lim_{\varepsilon \to 0} u_\varepsilon = u_0$.

The research of such an effective energy can be formalized with a notion of convergence which implies the convergence of minimum problems and of (global) minimizers.

For these reasons De Giorgi and Franzoni (1975) introduce the notion of Γ- convergence.

Do we need a "new" notion of convergence for this porpouse? Is any elementary convergence (e.g. pointwise convergence) doing this job?

Ex (what about pointwise convergence?): consider



$$F_\varepsilon(u) = \begin{cases} 0 & u \neq \varepsilon \\ -1 & u = \varepsilon, \end{cases}$$

clearly $\inf_{\mathbb{R}} F_\varepsilon = -1$. Let's compute the pointwise limit of F_ε:

- $\forall u \neq 0$, $F_\varepsilon(u) = 0$ when $\varepsilon < |u| \Rightarrow \lim_{\varepsilon \to 0} F_\varepsilon(u) = 0$

- $u = 0$, $F_\varepsilon(u) \equiv 0$ $\forall \varepsilon > 0 \Rightarrow \lim_{\varepsilon \to 0} F_\varepsilon(u) = 0$

so $F_\varepsilon \to G$, $G \equiv 0$, pointwise. But $\inf_{\mathbb{R}} G = 0$, so the pointwise limit of F_ε is <u>not</u> a good approximation for the minimum problem of F_ε.

Note: pointwise convergence is not good enough. We have to check how F_ε behaves along converging sequences, since the characterize the topology.

Def (Γ-convergence, but Naïve): let $F_\varepsilon, F_0 : X \to (-\infty, +\infty]$ we say that F_ε Γ-converges to F_0 ($F_\varepsilon \xrightarrow{\Gamma} F_0$) as $\varepsilon \to 0$ if the following to properties hold:

(i) F_0 is a lower bound, i.e. $\forall u \in X$ and $\forall u_\varepsilon \to u$

$$F_0(u) \leq F_\varepsilon(u_\varepsilon) + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0 \, ;$$

(ii) F_0 is the "optimal" lower bound, i.e. $\forall u \in X$
$\exists\ \bar{u}_\varepsilon \to u$ (so called recovery sequence) such that

$$F_\varepsilon(\bar{u}_\varepsilon) \leq F_0(u) + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0.$$

Naively (for this we would need additional assumptions)

(i) $\Rightarrow$ $\inf F_0 \leq \inf F_\varepsilon + o_\varepsilon(1)$ as $\varepsilon \to 0$

(ii) $\Rightarrow$ $\inf F_\varepsilon \leq \inf F_0 + o_\varepsilon(1)$ as $\varepsilon \to 0$

Rmk: from the previous comment, a consequence (up to requesting extra assumptions on F_ε) of the fact that $F_\varepsilon \xrightarrow{\Gamma} F_0$ is that

$$\inf F_0 \overset{(i)}{\leq} \liminf_{\varepsilon \to 0} \inf F_\varepsilon \leq \limsup_{\varepsilon \to 0} \inf F_\varepsilon \overset{(ii)}{\leq} \inf F_0$$

which yields

the Γ-limit is always lsc

$$\lim_{\varepsilon \to 0} \inf F_\varepsilon = \inf F_0 \left(= \min F_0\right).$$

This is the property we wanted to be ensured by the notion of convergence we were looking for.

Note: again, all of this depend heavily on the choice of the topology.

... of integral functionals

In this course we will mainly focus on the case

$$X \subseteq L^1(\Omega;\mathbb{R}^m), \text{ e.g. } \begin{array}{l} X = L^p(\Omega;\mathbb{R}^m) \\ X = W^{1,p}(\Omega;\mathbb{R}^m) \\ X = W^{1,p}_g(\Omega;\mathbb{R}^m) \end{array}$$

endowed with different topology (strong, weak etc.). The type of functionals we will deal with is integral functionals, i.e.

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, u, \nabla u)\, dx.$$

<u>Note</u>: techniques to find the Γ-limit changes from problem to problem. Intuition is important but sometimes might be misleading

This is an idea of the program of the course:

1. definiton and properties of Γ-convergence
2. Relaxation in Lebesgue/Sobolev spaces
3. (variational) Homogenization
4. discrete-to-continuum limit
5. gradient theory of phase transitions

We see one of the "applications" of Γ-convergence and how we cannot trust too much our intuition.

Ex (Homogenization): asymptotic description of variational problems with highly oscillating solutions.

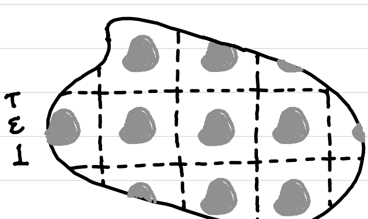
In its main formulation, we have an integral functional whose density depends explicity on the space variable x;

$$F_\varepsilon(u) = \int_\Omega g_\varepsilon(x, u, \nabla u)\, dx.$$

We say that it is nonhomogeneous, since in different points of Ω, the way in which g_ε acts on u and ∇u is different.
If the dependence in x is e.g. periodic, with period ε, when taking $\varepsilon \to 0$ we expect an "homogeneous" behavior (i.e. not depending on x);

$$F_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} F_0(u) = \int_\Omega g_{hom}(u, \nabla u)\, dx.$$

(i) nonhomogeneous Poisson's equation: Suppose we want to study the conductivity of a nonhogeneous material $\Omega \subseteq \mathbb{R}^n$.



$f : \Omega \to \mathbb{R}$ source term

$a : \mathbb{R}^n \to \{\alpha, \beta\}$ $\alpha, \beta > 0$ conductivity $(0,1)^n$- periodic.

The conductivity of the material is α in the gray regions and β elsewhere.
As $\varepsilon \to 0$ the periodicity of the conductivity increases.

The way electricity is conducted through Ω can be described by

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \\ u_\varepsilon|_{\partial\Omega} = 0 \end{cases} \iff \min_{u|_{\partial\Omega}=0} \underbrace{\int_\Omega \tfrac{1}{2} a\left(\tfrac{x}{\varepsilon}\right)|\nabla u|^2 - fu \, dx}_{=: F_\varepsilon(u)}$$

The dependence on x inside the divergence, and the fact that $a(x/\varepsilon)$ is oscillating faster and faster, makes the problem hard.

Via homog. techniques, we can prove that

$$F_\varepsilon(u) \xrightarrow{\Gamma} F_0(u) := \int_\Omega (A_{hom}\nabla u)\cdot\nabla u - fu \, dx$$

for some explicit, elliptic $A_{hom} \in \mathbb{R}^{n\times n}$. So that, as $\varepsilon \ll 1$ the electricity is conducted (up to small errors) as the solution to the following effective problem

$$\begin{cases} -\operatorname{div}(A_{hom}\nabla u_0) = f \\ u_0|_{\partial\Omega} = 0 \end{cases}.$$

(ii) a 1-dimensional case: let $n=1$ and $\Omega = (0,1)$,

$$F_\varepsilon(u) = \int_0^1 \tfrac{1}{2} a\left(\tfrac{x}{\varepsilon}\right)|u'|^2 - fu \, dx, \quad u \in H_0^1(0,1), \; f \in L^2(0,1).$$

Assume that $F_\varepsilon \xrightarrow{\Gamma} F_0$ (using weak H^1 topology) and try to find how F_0 looks like.

- (wrong) intuition: by Riemann-Lebesgue lemma, we know that $a(x/\varepsilon) \xrightarrow[L^\infty(0,1)]{*} \fint a =: \bar{a}$.

Since $u \in H_0^1(0,1)$, $|u'|^2 \in L^1(0,1)$ so, for every fixed u we have

$$\int_0^1 \frac{1}{2} a\left(\frac{x}{\varepsilon}\right)|u'|^2 - fu\,dx \xrightarrow[\varepsilon \to 0]{} \int_0^1 \frac{1}{2}\bar{a}|u'|^2 - fu\,dx =: G(u).$$

So we may bet on the fact that G is the Γ-limit of F_ε.
That's not true, G is only the pointwise limit.

- (correct) intuition: since Γ-convergence implies conv. of (global) minimizers, we study the convergence of minimizers of F_ε ($\exists!$ because of convexity).

$$\begin{cases} -\left(a\left(\frac{x}{\varepsilon}\right) u_\varepsilon'(x)\right)' = f(x) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases}$$ is the unique min. of F_ε in H_0^1.

By integration

$$u_\varepsilon'(x) = \frac{1}{a\left(\frac{x}{\varepsilon}\right)} F(x), \quad \text{where} \quad F(x) = -\int_0^x f(t)\,dt + c.$$

By Riemann-Lebesgue (and continuity of F)

$$u_\varepsilon'(x) \xrightarrow[L^2]{} \int \frac{1}{a}\, F(x).$$

Since $\|u_\varepsilon\|_{L^2} \leq C$ (easy to check) $\Rightarrow u_\varepsilon \xrightarrow[H^1]{} u_0$ for some u_0 such that $u_0' = \int \frac{1}{a} F$ and $u_0(0) = u_0(1) = 0$.
Thus u_0 solves

$$\begin{cases} -\left(\left(\int \frac{1}{a}\right)^{-1} u_0'(x)\right)' = f(x) \\ u_0(0) = u_0(1) = 0 \end{cases} \iff u_0 \text{ is the (unique) min. of } F_0.$$

But F_0 s.t. this equation is its Euler-Lagrange eq. is

$$F_0(u) = \int_0^1 \frac{1}{2} a^* |u'|^2 - fu\,dx, \quad a^* := \left(\int \frac{1}{a}\right)^{-1} \text{ harmonic mean of } a.$$

So, by studying the minimizers of F_ε we found its Γ-limit, which is a bit counterintuitive since it is different from the pointwise limit, $F_0 \neq G$ e.g. when $\{\alpha, \beta\} = \{1,2\}$.

Γ-convergence

Main references for this lecture:

- Ch.1 of [B,2000] (see also Ch.1,7 of [BD,1998])

In this lecture we see the definition (and some properties) of lower semicontinuity and Γ-convergence. To fix the notation we start by recalling the notion of upper and lower limits.

Let (X,d) be a metric space.
Unless otherwise specified, when we write $x_j \to x$ we mean $d(x_j,x) \to 0$.

<u>Def</u> (upper and lower limits). Let $f: X \to [-\infty,+\infty]$ and $x \in X$

$$\varliminf_{y\to x} f(y) := \inf\left\{ \varliminf_{j\to+\infty} f(x_j) : x_j \to x \right\}$$
$$= \inf\left\{ \lim_{j\to+\infty} f(x_j) : \begin{array}{l} x_j \to x \text{ and} \\ \exists \text{ the limit of } f(x_j) \end{array} \right\},$$

$$\varlimsup_{y\to x} f(y) := \sup\left\{ \varlimsup_{j\to+\infty} f(x_j) : x_j \to x \right\}$$
$$= \sup\left\{ \lim_{j\to+\infty} f(x_j) : \begin{array}{l} x_j \to x \text{ and} \\ \exists \text{ the limit of } f(x_j) \end{array} \right\}.$$

<u>Note</u>: the fact that we can restrict the inf (or sup) to sequences s.t. the limit of $f(x_j)$ exists is consequence of the fact that we can always extract a subsequence converging to the $\varliminf$ (or $\varlimsup$).

Rmk (immediate properties):

(1.1) by choosing $x_j = x$ we get $\liminf_{y \to x} f(y) \leq f(x)$

(1.2) $\liminf_{y \to x} f(y) + g(y) \geq \liminf_{y \to x} f(y) + \liminf_{y \to x} g(y)$

(1.3) $\liminf_{y \to x} f(y) + g(y) \leq \liminf_{y \to x} f(y) + \limsup_{y \to x} g(y)$

(1.4) $\limsup_{y \to x} f(y) = - \liminf_{y \to x} -f(y)$.

By (1.4) one can get the analogous properties (with opposite sign) for the $\limsup$.

It may be useful to give an alternative definition of upper and lower limits just in terms of neighbors. This topological definitions coincide with the previous ones in metric spaces but may differ in general topological spaces.

Rmk (topological definitions): it holds that

$$\liminf_{y \to x} f(y) = \sup_{U \in N(x)} \inf_{y \in U} f(y)$$

$$\limsup_{y \to x} f(y) = \inf_{U \in N(x)} \sup_{y \in U} f(y).$$

Notice that $U \mapsto \inf_{y \in U} f(y)$ is decreasing (w.r.t. inclusion) so the $\sup_{U \in N(x)}$ resembles a limit process as U shrinks to x. Analogous for the $\limsup$.

Lower semicontinuty

Def (lsc): $f : X \to [-\infty, +\infty]$ is lower semicontinuous at $x \in X$ iff

$$f(x) \leq \varliminf_{j \to +\infty} f(x_j) \quad \forall x_j \to x. \qquad \text{(LSC)}$$

If f is lsc $\forall x \in X$ we say that f is lsc.

Note: lsc depends on the metric (more in general topology) we choose. If we want to highlight the role of metric we write d-lsc.

Note: the definition above is that of sequential lsc. In topological spaces f is lsc iff $f(x) \leq \inf_{y \in U} f(y)$ $\forall U \in N(x)$. This coincide with sequential lsc in metric spaces, thus we simply say lsc.

Rmk (characterization): the following are equivalent

(2.1) f is lsc

(2.2) $f(x) = \varliminf_{y \to x} f(y)$

(2.3) $\forall t \in \mathbb{R}$, $\{x \in X : f(x) \leq t\} = \{f \leq t\}$ is closed

Indeed: (2.1) $\Rightarrow$ (2.2) by taking the inf over $x_j \to x$ in (LSC) we get $f(x) \leq \varliminf_{y \to x} f(y) \overset{(1.1)}{\leq} f(x)$.

(2.2) $\Rightarrow$ (2.1) immediate from definition of lower limit.

(2.1) $\Rightarrow$ (2.3) $\forall\, t \in \mathbb{R}$, take any $x_j \in \{f \le t\}$ s.t. $x_j \to x \overset{(LSC)}{\Longrightarrow} f(x) \le \varliminf_{j\to+\infty} f(x_j) \le t \Rightarrow x \in \{f \le t\}$.
Every converging sequence in $\{f \le t\}$ has limit still in $\{f \le t\} \Rightarrow \{f \le t\}$ is closed.

(2.3) $\Rightarrow$ (2.1) assume by contradiction that $\exists$ $x_j \to x$ s.t. $f(x) > \varliminf_{j\to+\infty} f(x_j)$.
Then $\exists\, t_0 \in \mathbb{R}$ s.t. $f(x) > t_0 > \varliminf_{j\to+\infty} f(x_j)$.
By properties of $\varliminf$ $\exists\, \{x_{j_k}\} \subseteq \{x_j\}$ s.t.
$\lim_{k\to+\infty} f(x_{j_k}) = \varliminf_{j\to+\infty} f(x_j)$.
For k large enough $x_{j_k} \in \{f \le t_0\}$ but converges to $x \notin \{f \le t_0\} \Rightarrow \{f \le t_0\}$ is not closed, which is a contradiction.

<u>Rmk</u> (immediate properties):

(3.1) f, g are lsc $\Rightarrow f + g$ is lsc

(3.2) let $\mathcal{J}$ be a family of indicies (also uncountable).
Let f_α be lsc $\forall\, \alpha \in \mathcal{J} \Rightarrow \sup_{\alpha\in\mathcal{J}} f_\alpha$ is lsc

<u>Indeed</u>: for ever $x_j \to x$, $\forall\, \alpha \in \mathcal{J}$

$$f_\alpha(x) \overset{(LSC)}{\le} \varliminf_{j\to+\infty} f_\alpha(x_j) \le \varliminf_{j\to+\infty} \sup_{\alpha\in\mathcal{J}} f_\alpha(x_j).$$

Taking the sup over α we get the result

(3.3) let $E \subseteq X$ and $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$
χ_E is lsc $\Leftrightarrow$ E is open

<u>Indeed</u>: $\{\chi_E \le t\} = \begin{cases} X & t \ge 1 \\ E^c & 0 \le t < 1 \\ \emptyset & t < 0 \end{cases}$ is closed $\forall\, t \in \mathbb{R} \Leftrightarrow$ E is open and the result comes from (2.3).

Def (lsc envelope): let $f: X \to [-\infty, +\infty]$, its lower semicontinuous envelope is the function $sc(f): X \to [-\infty, +\infty]$ defined as

$$sc(f)(x) := \sup \left\{ g: X \to [-\infty, +\infty] : \begin{matrix} g \leq f \\ g \text{ is lsc} \end{matrix} \right\}.$$

$sc(f)$ is the largest lsc function smaller than f.
$sc(f)$ is also called relaxed functional.

Rmk: by (3.2) $sc(f)$ is lsc

Ex: it is important to consider functions which can attain $-\infty$.
Let $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $sc(f)(x) = \begin{cases} 1/x & x \neq 0 \\ -\infty & x = 0 \end{cases}$.

Γ-convergence

Def (Γ-convergence): let $f_j, f_\infty : X \to [-\infty, +\infty]$, we say that f_j Γ-converges to f_∞ at $x \in X$ iff

(i) liminf inequality: $\forall\, x_j \to x$

$$f_\infty(x) \leq \liminf_{j \to +\infty} f_j(x_j)$$

(ii) limsup inequality: $\exists\, \bar{x}_j \to x$ (recovery sequence) s.t.

$$f_\infty(x) \geq \limsup_{j \to +\infty} f_j(\bar{x}_j).$$

If f_j Γ-converges to f_∞ at any $x \in X$ then we say that f_j Γ-converges to f_∞ in X.

We will write $f_j \xrightarrow{\Gamma} f_\infty$ or $\Gamma\text{-}\lim_{j\to+\infty} f_j = f_\infty$. If we want to highlight the metric we write $f_j \xrightarrow{\Gamma(d)} f_\infty$ or $\Gamma(d)\text{-}\lim_{j\to+\infty} f_j = f_\infty$.

If we have a family of functions parametrized by a continuous parameter $f_\varepsilon : X \to [-\infty,+\infty]$, $\varepsilon>0$ $f_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} f_\infty$ iff $\forall\, \varepsilon_j \to 0$ $f_{\varepsilon_j} \xrightarrow{\Gamma} f_\infty$.

<u>Rmk</u>: condition (ii) can be replaced by one of the following conditions.
By this we mean that (i)&(ii) $\Leftrightarrow$ (i) & (ii)' $\Leftrightarrow$ (i) & (ii)"

(ii)' $\exists\, \bar{x}_j \to x$ s.t. $f_\infty(x) = \lim_{j\to+\infty} f_j(\bar{x}_j)$

<u>Indeed</u>: clearly (ii)' $\Rightarrow$ (ii). If (i)&(ii) hold then

$$f_\infty(x) \leq \liminf_{j\to+\infty} f_j(\bar{x}_j) \leq \limsup_{j\to+\infty} f_j(\bar{x}_j) \leq f_\infty(x)$$

so $\exists \lim_{j\to+\infty} f_j(\bar{x}_j) = f_\infty(x)$.

(ii)" approximate limsup inequality: $\forall \varepsilon>0$ $\exists\, x_j^\varepsilon \to x$ s.t.

$$f_\infty(x) \geq \limsup_{j\to+\infty} f_j(x_j^\varepsilon) - \varepsilon$$

<u>Indeed</u>: clearly (ii) $\Rightarrow$ (ii)". If (ii)" holds then $\forall k \in \mathbb{N}$ (take $\varepsilon = 1/k$) $\exists\, x_j^{(k)} \to x$ s.t.

$$f_\infty(x) \geq \limsup_{j\to+\infty} f_j(x_j^{(k)}) - 1/k \quad \forall k \in \mathbb{N},$$

We find a sequence "close" to $x_j^{(k)}$ when k is large along which f_j is "close" to the $\overline{\lim}$. We do it with a sort of diagonal argument.

Let $\sigma_0 := 0$ and define iteratively

$$\sigma_k := \min\left\{ h \geq \sigma_{k-1} : \begin{array}{l} d(x_j^{(k)}, x) \leq 1/k \quad \forall j \geq h \\ f_h(x_h^{(k)}) \leq \overline{\lim}_{j\to+\infty} f_j(x_j^{(k)}) + 1/k \end{array} \right\}.$$

By definition of $\overline{\lim}$ and since $x_j^{(k)} \to x$, σ_k is well defined $\forall k \in \mathbb{N}$.
We then define

$$\overline{x}_j := x^{(k)}_{\sigma_k} \qquad \text{if} \quad \sigma_k \leq j \leq \sigma_{k+1}.$$

Then $\overline{x}_j \to x$ and $f_j(\overline{x}_j) \leq \overline{\lim}_{j\to+\infty} f_j(x_j^{(k)}) + \frac{1}{k}$ $\forall k \in \mathbb{N}$. So

$$f_\infty(x) \geq \overline{\lim}_{j\to+\infty} f_j(x_j^{(k)}) - 1/k$$

$$\geq \overline{\lim}_{j\to+\infty} f_j(\overline{x}_j) - 1/2k.$$

Taking the limit as $k \to +\infty$ we get the result.

We can motivate the definition of Γ-limit as the research of a functions which is a lower bound for f_j along every sequence (condition).
This lower bound is somehow optimal (condition in the sense that f_j converges to f_∞ along at least one sequence.

<u>Note</u>: as for lsc, the definition above is that of sequential Γ-convergence, which coincide with the "topological" definition in metric spaces, but may differ in general topological spaces

<u>Rmk</u> (dependence on the metric): let d and d' be two metrics on X, with d' stronger than d (i.e. if $d'(x_j,x)\to 0 \Rightarrow d(x_j,x)\to 0$) then

$$\Gamma(d)-\lim_{j\to+\infty} f_j \leq \Gamma(d')-\lim_{j\to+\infty} f_j$$

if the Γ-limits exist.

In particular, let $d'(x,y):=\begin{cases}0 & x=y\\ 1 & x\neq y\end{cases}$ (i.e. $x_j\xrightarrow{d'}x \Leftrightarrow x_j\equiv x$).
It's immediate to prove that $\Gamma(d')-\lim_{j\to+\infty} f_j$ is the pointwise limit $\lim_{j\to+\infty} f_j$.
Since every metric d is weaker than d' we infer that

$$\Gamma(d)-\lim_{j\to+\infty} f_j \leq \lim_{j\to+\infty} f_j.$$

The Γ-limit is always smaller than the pointwise limit (if they exist).

Some easy examples/exercise on the real line (with the metric induced by the modulus).

<u>Exrc 1</u>: let $f_j(x)=\begin{cases}0 & x\neq 1/j\\ -1 & x=1/j\end{cases}$, "prove" that

$$\Gamma-\lim_{j\to+\infty} f_j=\begin{cases}0 & x\neq 0\\ -1 & x=0.\end{cases}$$

<u>Ex</u>: let $f_j(x)=\begin{cases}0 & x\notin 1/j\,\mathbb{Z}\\ -1 & x\in 1/j\,\mathbb{Z}\end{cases}$, $\Gamma-\lim_{j\to+\infty} f_j\equiv -1$.

- the liminf inequality is obvious

- $x \in \mathbb{R}$, $(x - \frac{1}{2j}, x + \frac{1}{2j}] \cap \frac{1}{j}\mathbb{Z} =: \bar{x}_j$ is a recovery sequence.
  Indeed, $|\bar{x}_j - x| < \frac{1}{j}$ so $\bar{x}_j \to x$ and $f_j(\bar{x}_j) \equiv -1$, so (ii)' holds.

Ex (Γ-limit may not exist): let $f_j(x) = \begin{cases} 0 & x \notin \frac{1}{j}\mathbb{Z} \\ (-1)^{j+1} & x \in \frac{1}{j}\mathbb{Z} \end{cases}$.
Let f be a function s.t. (i) holds.
Since, for every $x \in X$, $\tilde{x}_j := (x - \frac{1}{2j}, x + \frac{1}{2j}] \cap \frac{1}{j}\mathbb{Z}$ then $f_j(\tilde{x}_j) = (-1)^{j+1} \Rightarrow f(x) \leq -1$.
But $\forall x_j \to x$, $f_j(x_j) \geq 0$ as j is odd so

$$\overline{\lim_{j \to +\infty}} f_j(x_j) \geq 0 > f(x) \quad \forall x_j \to x, \ \forall f \leq -1$$

so $\nexists f$ which satisfies both (i) and (ii), so f_j does not Γ-converge.

Ex (Γ-converging seq. without pointwise limit): let

$$f_j(x) = \begin{cases} 1 & x \in \frac{1}{j}(2\mathbb{Z}+1) \\ -1 & x \in \frac{1}{j}(2\mathbb{Z}) \\ 0 & \text{otherwise.} \end{cases}$$

This sequence does not converge pointwise. Indeed $f_j(1) = \begin{cases} 1 & j \text{ is odd} \\ -1 & j \text{ is even} \end{cases}$ so $\nexists \lim_{j \to +\infty} f_j(1)$. Similar $\forall x \in \mathbb{Q}$.

But $\Gamma\text{-}\lim_{j \to +\infty} f_j \equiv -1$ as in the previous exercise by taking $\bar{x}_j := (x - j, x + j] \cap \frac{1}{j}(2\mathbb{Z})$ as a recovery sequence.

Alternative definitions

As we have seen from previous examples, Γ-limit not always exist.
It is convenient to introduce quantities that always exists that may help in the computation of the Γ-limit. On the other hand they also provide an alternative (equivalent) definition of Γ-limit.

<u>Def</u> (upper and lower Γ-limits): let $f_j : X \to [-\infty, +\infty]$ we define

$$\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j(x) := \inf \left\{ \underline{\lim}_{j\to+\infty} f_j(x_j) : x_j \to x \right\}$$

$$\Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j(x) := \inf \left\{ \overline{\lim}_{j\to+\infty} f_j(x_j) : x_j \to x \right\}$$

<u>Proposition</u>: $f_j \xrightarrow{\Gamma} f_\infty \iff \Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j = \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j = f_\infty$

<u>proof</u>: ($\Rightarrow$) if $f_j \xrightarrow{\Gamma} f_\infty$ then

$$f_\infty(x) \overset{(i)}{\leq} \Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j(x) \leq \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j(x) \leq$$
$$\leq \overline{\lim}_{j\to+\infty} f_j(\bar{x}_j) \overset{(ii)}{\leq} f_\infty(x).$$

($\Leftarrow$) point (i) trivially holds. We prove (ii)'', by characterization of inf $\forall \varepsilon > 0 \; \exists \; x_j^\varepsilon \to x$ s.t.

$$f_\infty(x) = \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j(x) > \overline{\lim}_{j\to+\infty} f_j(x_j^\varepsilon) - \varepsilon$$

which is exactly (ii)'' so $f_j \xrightarrow{\Gamma} f_\infty$. $\square$

We give also a topological version of upper and lower Γ-limits that might be useful in the sequel.

<u>Rmk</u> (topological definitions): in general topological spaces

$$\Gamma\text{-}\liminf_{j\to+\infty} f_j(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{j\to+\infty} \inf_{y\in U} f_j(y)$$

$$\Gamma\text{-}\limsup_{j\to+\infty} f_j(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{j\to+\infty} \inf_{y\in U} f_j(y).$$

As for the lsc, these quantities coincide with the previous definitions but may differ in general topological spaces.

Lecture 3: 29.04.2024

We concluded last lecture with an alternative definition of Γ-convergence by means of Γ-upper and lower limit. It is immediate to deduce some propertie of Γ-conver- -gence by exploiting them.

<u>Rmk</u>: (1.1) the Γ-limit (if $\exists$) is unique, e.g. since it coincides with the Γ-liminf, which is a well- -defined function

(1.2) if $f_j \leq g_j$ $\forall j$ $\Rightarrow$ $\Gamma\text{-}\liminf_{j\to+\infty} f_j \leq \Gamma\text{-}\liminf_{j\to+\infty} g_j$. Same for the Γ-$\limsup$ (and hence for the Γ-lim if exists)

(1.3) let $\{f_{j_k}\}\subseteq\{f_j\}$ then

$$\Gamma\text{-}\liminf_{j\to+\infty} f_j \leq \Gamma\text{-}\liminf_{k\to+\infty} f_{j_k}\,,\quad \Gamma\text{-}\limsup_{k\to+\infty} f_{j_k} \leq \Gamma\text{-}\limsup_{j\to+\infty} f_j .$$

In particular, if $\exists$ $\Gamma\text{-}\lim_{j\to+\infty} f_j = f_\infty$ then $\forall$ subsequence $\{f_{j_k}\}\subseteq\{f_j\}$ $\exists$ $\Gamma\text{-}\lim_{k\to+\infty} f_{j_k} = f_\infty$.

<u>Exercise</u>: deduce (1.1) directly from the definition of Γ-limit.

<u>Some properties of Γ-convergence</u>

Now we see some of the main propertres of Γ convergences that will be useful in the sequel.

Proposition 1: $\Gamma\text{-}\varliminf\limits_{j\to+\infty} f_j$, $\Gamma\text{-}\varlimsup\limits_{j\to+\infty} f_j$ are lsc.
In particular if $\exists\ \Gamma\text{-}\lim\limits_{j\to+\infty} f_j$ it is lsc.

Proof: we use the notation $f'(x) := \Gamma\text{-}\varliminf\limits_{j\to+\infty} f_j(x)$ and
and $f''(x) := \Gamma\text{-}\varlimsup\limits_{j\to+\infty} f_j(x)$.

Step 1 (lsc of f'): let $x_k \to x$. $\forall x_k\ \exists x_j^{(k)} \to x_k$ s.t.

$$f'(x_k) \leq \varliminf_{j\to+\infty} f_j(x_j^{(k)}) < f'(x_k) + 1/k.$$

As in the last lecture, we define the seq. of indices

$$\sigma_0' = 0, \quad \sigma_k' := \min\left\{ h \geq \sigma_{k-1}' : \begin{array}{l} d(x_j^{(k)}, x_k) \leq 1/k \quad \forall j \geq h \\ |f'(x_k) - f_h(x_h^{(k)})| < 2/k \end{array} \right\}$$

and define $\bar{x}_j = x^{(k)}_{\sigma_k'}$ when $\sigma_k' \leq j < \sigma_{k+1}'$. By definition
$\bar{x}_j \to x$ and

$$f'(x) \overset{\substack{\text{def of}\\ \Gamma\text{-}\underline{\lim}}}{\leq} \varliminf_{j\to+\infty} f_j(\bar{x}_j) \leq \varliminf_{k\to+\infty} f_{\sigma_k'}(x^{(k)}_{\sigma_k'}) \leq \varliminf_{k\to+\infty} f'(x_k).$$

Step 2 (lsc of f''): for x_k and $x_j^{(k)}$ as above we have

$$f''(x_k) \leq \varlimsup_{j\to+\infty} f_j(x_j^{(k)}) < f''(x_k) + 1/k.$$

Now we define a slightly different seq. of indeces

$$\sigma_0'' = 0, \quad \sigma_k'' := \min\left\{ h \geq \sigma_{k-1}'' : \begin{array}{l} d(x_j^{(k)}, x_k) \leq 1/k \\ f_j(x_j^{(k)}) - f''(x_k) < 2/k \end{array} \ \forall j \geq h \right\}$$

and $\tilde{x}_j = x^{(k)}_{\sigma_k''}$ when $\sigma_k'' \leq j < \sigma_{k+1}''$. As above

$$f''(x) \overset{\substack{\text{def of}\\ \Gamma\text{-}\overline{\lim}}}{\leq} \varlimsup_{j\to+\infty} f_j(\tilde{x}_j) \leq f''(x_k) + 2/k \quad \forall k$$

and taking the $\varliminf$ as $k \to +\infty$ we get the result. $\square$

Note: in the previous proof, when dealing with f' it was sufficient that $f'(x_k) \sim f_j(\bar{x}_j)$ along a subsequence (indeed we used monotonicity of $\underline{\lim}$).
Since $\overline{\lim}$ has the inverse monotonicity, we need that the whole sequence $f_j(\bar{x}_j) \lesssim f''(x_k)$.
This is why we used two sligthly different de-finitions for σ_k' and σ_k''.

Corollary 2: $\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j \leq sc(\underline{\lim}_{j\to+\infty} f_j)$ and $\Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j \leq sc(\overline{\lim}_{j\to+\infty} f_j)$.
In particular, if there exist the Γ-limit and the pointwise limit, then $\Gamma\text{-}\lim_{j\to+\infty} f_j \leq sc(\lim_{j\to+\infty} f_j)$.

proof: by definition of Γ-$\underline{\lim}$, taking $x_j \equiv x$ as a test sequence we get $\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j(x) \leq \underline{\lim}_{j\to+\infty} f_j(x)$. Same for the Γ-$\overline{\lim}$. $\square$

The lsc of the Γ-$\overline{\lim}$ allows to simplify the computation of the Γ-limit, by restring its computation only on a dense set.

Rmk (density argument for limsup inequality): let d' stronger than d (i.e. $d'(x_j,x)\to 0 \Rightarrow d(x_j,x)\to 0$), and let

(2.1) $D \subseteq X$ be a d'-dense set

(2.2) $\Gamma(d')\text{-}\overline{\lim}_{j\to+\infty} f_j(x) \leq f(x) \quad \forall x\in D$, f d'-continuous

then $\Gamma(d)\text{-}\overline{\lim}_{j\to+\infty} f_j(x) \leq f(x) \quad \forall x \in X$.

Indeed: $\forall x\in X \ \exists x_k \in D : x_k \xrightarrow{d'} x \Rightarrow x_k \xrightarrow{d} x$, so

$$f''(x) \overset{lsc}{\leq} \varliminf_{k\to+\infty} f''(x_k) \overset{(2.2)}{\leq} \varliminf_{k\to+\infty} f(x_k) \overset{cont}{=} f(x).$$

We will use this a lot with $D = C^\infty(\bar{\Omega})$, dense in $H^1(\Omega)$ in $\|\cdot\|_{H^1}$, to get $\Gamma(\|\cdot\|_{L^2})$ - limit.

In general, the sum of two Γ-limits differs from the Γ-limit of the sum. A trivial example is the following: we saw in the last lecture that

$$f_j(x) = \begin{cases} -1 & x = 1/j \\ 0 & x \neq 1/j \end{cases} \xrightarrow{\Gamma} f_\infty(x) = \begin{cases} -1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

It can be easily checked that $-f_j(x) \xrightarrow{\Gamma} g_\infty \equiv 0$.
But $f_\infty = f_\infty + g_\infty \neq \Gamma\text{-}\lim_{j\to+\infty} f_j + (-f_j) \equiv 0$.

Adding a uniformly converging sequence though, do not affect Γ-convergence.

<u>Proposition</u> 3: let $f_j \xrightarrow{\Gamma} f_\infty$ and $g_j \rightrightarrows g$, and $g : X \to [-\infty, +\infty]$ continuous. Then

$$\Gamma\text{-}\lim_{j\to+\infty} (f_j + g_j) = f_\infty + g.$$

<u>Proof</u>: Since $g_j \rightrightarrows g$, $\forall \varepsilon > 0$ $\sup_{x\in X} |g_j(x) - g(x)| < \varepsilon$ for j large enough.
In particular, $\forall x_j \to x$, $|g_j(x_j) - g(x)| < \varepsilon$.

$$f_\infty(x) + g(x) \overset{(i)}{\leq} \varliminf_{j\to+\infty} f_j(x_j) + \lim_{j\to+\infty} g(x_j)$$
$$\leq \varliminf_{j\to+\infty} \left(f_j(x_j) + g_j(x_j) \right) + \varepsilon,$$

since it holds $\forall \varepsilon > 0$ it yields (i) for $f_j + g_j$.

from (ii)'' for f_j we have $\forall \varepsilon > 0\ \exists\, x_j^\varepsilon \to x$ s.t

$$f_\infty(x) + g(x) \overset{(ii)''+\text{cont}}{\geq} \overline{\lim_{j\to+\infty}} f_j(x_j^\varepsilon) - \varepsilon + \lim_{j\to+\infty} g_j(x_j^\varepsilon) - \varepsilon$$
$$\geq \overline{\lim_{j\to+\infty}} \left(f_j(x_j^\varepsilon) + g_j(x_j^\varepsilon)\right) - 2\varepsilon$$

which implies (ii)'' for $f_j + g_j$. $\square$

<u>Corollary</u> 4 (stability under continuous perturbations): let $g: X \to [-\infty, +\infty]$ be a continuous function and let $f_j \xrightarrow{\Gamma} f_\infty$, then

$$\Gamma\text{-}\lim_{j\to+\infty} (f_j + g) = f_\infty + g.$$

As we have seen, the Γ-limit not always exists. There are some special sequences for which it is easy to prove the existence of the Γ-limit and to compute it.

<u>Proposition</u> 5 (constant sequence): $\Gamma\text{-}\lim_{j\to+\infty} f(x) = sc(f)(x) = \underline{\lim}_{y\to x} f(y)$

<u>proof</u>: we work in two steps

Step 1 ($\exists\, \Gamma\text{-}\lim_{j\to+\infty} f(x) = \underline{\lim}_{y\to x} f(y)$): (i) is obvious by definition of $\underline{\lim}_{y\to x} f(y)$.
From characterization of inf, $\forall \varepsilon > 0\ \exists\, x_j^\varepsilon \to x$ s.t

$$\underline{\lim}_{y\to x} f(y) > \lim_{j\to+\infty} f(x_j^\varepsilon) - \varepsilon$$

which proves (ii)''.

Step 2 ($\underline{\lim}_{y\to x} f(y) = sc(f)(x)$): by Step 1 $\underline{\lim}_{y\to x} f(y)$ is lsc

(since it is a Γ-limit) and $\varliminf_{y\to x} f(y) \leq f(x)$ so $\varliminf_{y\to x} f(y) \leq sc(f)(x)$.
But $\forall\, g \leq f$, g lsc there holds that $g(x) \leq \varliminf_{y\to x} g(y) \leq \varliminf_{y\to x} f(y)$. Taking the sup over g we get the opposite inequality. $\square$

Another useful result is the following.

<u>Proposition</u> 6: $\Gamma\text{-}\varliminf_{j\to+\infty} f_j = \Gamma\text{-}\varliminf_{j\to+\infty} sc(f_j)$ (same for $\Gamma\text{-}\varlimsup$).

<u>proof</u>: by using the topologic definition of $\Gamma\text{-}\varliminf$ and $\varliminf$ and the fact that $sc(f)(x) = \varliminf_{y\to x} f(y)$ we get

$$\Gamma\text{-}\varliminf_{j\to+\infty} sc(f_j)(x) = \sup_{U\in N(x)} \varliminf_{j\to+\infty} \inf_{y\in U} sc(f_j)(y)$$

$$= \sup_{U\in N(x)} \varliminf_{j\to+\infty} \inf_{y\in U} \sup_{V\in N(y)} \inf_{z\in V} f_j(z),$$

since $V \mapsto \inf_V f_j$ is decreasing we can reduce to neighbours that are contained in U so

$$= \sup_{U\in N(x)} \varliminf_{j\to+\infty} \inf_{y\in U} \sup_{V\in N(y)} \inf_{z\in U\cap V} f_j(z)$$

$$\geq \sup_{U\in N(x)} \varliminf_{j\to+\infty} \inf_{y\in U} f_j(y) = \Gamma\text{-}\varliminf_{j\to+\infty} f_j(x).$$

Analogously for the $\Gamma\text{-}\varlimsup$. $\square$

<u>Note</u>: working similarly as in the proof of the lsc of $\Gamma\text{-}\varliminf$ we can prove the result above also reasoning with sequences (same for $\Gamma\text{-}\varlimsup$).

Proposition 7 (limits of monotone sequences): let $\{f_j\}$ be a a monotone sequence, then the Γ-limit $\exists$ and

(1) if $f_{j+1} \leq f_j$ (decreasing) $\Rightarrow$ $\Gamma\text{-}\lim\limits_{j\to+\infty} f_j = sc(\lim\limits_{j\to+\infty} f_j)$

(2) if $f_{j+1} \geq f_j$ (increasing) $\Rightarrow$ $\Gamma\text{-}\lim\limits_{j\to+\infty} f_j = \lim\limits_{j\to+\infty} sc(f_j)$

proof: my monotonicity, there exists the pointwise limit in both cases.

(1) by Corollary 2, $\Gamma\text{-}\overline{\lim\limits_{j\to+\infty}} f_j \leq sc(\lim\limits_{j\to+\infty} f_j)$.
Moreover by Proposition 5

$$sc(\lim_{j'\to+\infty} f_{j'}) = \Gamma\text{-}\lim_{j\to+\infty} (\lim_{j'\to+\infty} f_{j'}) \leq \Gamma\text{-}\underline{\lim_{j\to+\infty}} f_j$$

since $\{f_j\}$ is decreasing hence $\lim\limits_{j'\to+\infty} f_j \leq f_j \ \forall j$.

(2) since $\{f_j\}$ is increasing, so is $\{sc(f_j)\}$ hence $\forall j$ $sc(f_j) \leq \lim\limits_{j'\to+\infty} sc(f_{j'}) = \sup\limits_{j'} sc(f_{j'})$ which is lsc since it is sup of lsc functions.
By Corollary 2, $\Gamma\text{-}\overline{\lim\limits_{j\to+\infty}} sc(f_j) \leq \lim\limits_{j\to+\infty} sc(f_j)$.
Moreover, $\forall j' < j$ $sc(f_j) \geq sc(f_{j'})$ so

$$\Gamma\text{-}\underline{\lim_{j\to+\infty}} sc(f_j) \geq sc(f_{j'}) \quad \forall j'$$

and the result follows by taking the sup and by Proposition 6. $\square$

Convergence of minimum problems

It is possible to apply the Direct Method to Γ-converging, since the Γ-limit is lsc. It is necessary though to be sure that we can reduce to study infima on a compact set.

Def (coerciveness): $f : X \to [-\infty, +\infty]$ is coercive if $\forall\, t \in \mathbb{R}$, $\{f \leq t\}$ are precompact.
f is mildly-coercive if $\exists\, K \subseteq X$ compact s.t. $\inf_X f = \inf_K f$.
$f_j : X \to [-\infty, +\infty]$ is equi mildly-coercive if $\exists\, K \subseteq X$ cmpt s.t. $\inf_X f_j = \inf_K f_j$.

Rmk: f coercive $\Rightarrow$ f mildly-coercive
indeed, this is obvious if $f \equiv +\infty$, if not take $t \in \mathbb{R}$ s.t. $\{f \leq t\} \neq \emptyset$ and $K := \overline{\{f \leq t\}}$. Then

$$\inf_X f = \inf_{\{f \leq t\}} f \geq \inf_K f \geq \inf_X f.$$

Actually, it is sufficient that $\exists\, t_0$ s.t. $\emptyset \neq \{f \leq t_0\}$ is precompact to imply mild-coerciveness.

Theorem (fundamental Thm of Γ-convergence): let (X, d) be a metric space and $f_j, f_\infty : X \to [-\infty, +\infty]$ be s.t. $\{f_j\}$ is equi mildly coercive and $f_j \xrightarrow{\Gamma} f_\infty$. Then

$$\exists \lim_{j \to +\infty} \inf_X f_j = \min_X f_\infty.$$

Moreover, $\forall\, \{x_j\}$ s.t. $\lim_{j \to +\infty} f_j(x_j) = \min_X f_\infty$, every cluster point of $\{x_j\}$ is a global minimizer of f_∞.

proof: let $K \subset\subset X$ s.t. $\inf_K f_j = \inf_X f_j$ $\forall j$. By characterization of inf $\exists\, \tilde{x}_j \in K$ s.t.

$$\inf_X f_j \leq f_j(\tilde{x}_j) < \inf_X f_j + 1/j \quad \forall j.$$

Taking the $\underline{\lim}$ we get

$$(*) \quad \varliminf_{j\to+\infty} \inf_X f_j = \varliminf_{j\to+\infty} f_j(\tilde{x}_j).$$

Since K is compact $\exists\, \{\tilde{x}_{j_k}\} \subseteq \{\tilde{x}_j\}$ and $x_\infty \in K$ s.t. $\tilde{x}_{j_k} \to x_\infty$ and $\varliminf_{j\to+\infty} f_j(\tilde{x}_j) = \lim_k f_{j_k}(\tilde{x}_{j_k})$.
Now define $x_j := \begin{cases} \tilde{x}_j & j = j_k \text{ for some } k \\ x_\infty & \text{otherwise} \end{cases}$, then

$$(*_1) \quad f_\infty(x_\infty) \overset{(i)}{\leq} \varliminf_{j\to+\infty} f_j(x_j) \leq \lim_{k\to+\infty} f_{j_k}(\tilde{x}_{j_k}) = \varliminf_{j\to+\infty} \inf_X f_j.$$

By def. of Γ-convergence, $\forall x \in X$ $\exists\, \bar{x}_j \to x$ a rec. seq., then

$$(*_2) \quad f_\infty(x) \overset{(ii)}{\geq} \varlimsup_{j\to+\infty} f_j(\bar{x}_j) \geq \varlimsup_{j\to+\infty} \inf_X f_j, \quad \forall x \in X.$$

By $(*_1)$ and taking the inf in $(*_2)$ we get

$$\inf_X f_\infty \leq f_\infty(x_\infty) \leq \varliminf_{j\to+\infty} \inf_X f_j$$
$$\leq \varlimsup_{j\to+\infty} \inf_X f_j \leq \inf_X f_\infty$$

which yields $\exists \lim_{j\to+\infty} \inf_X f_j = \inf_X f_\infty \left(= f_\infty(x_\infty)\right) = \min_X f_\infty$.

We can repeat the argument above $\forall \{x_j\}$ satisfying $(*)$ since then $\forall x_0$ cluster point of $\{x_j\}$ we repeat all the passages with x_0 in place of x_∞. This proves the full result. $\square$

Rmk: (3.1) notice that, in the case in which $\inf_x f_j = \min_x f_j$ $\forall j$ and let x_j be a (global) minimizer of f_j, then

$$\lim_{j\to+\infty} f_j(x_j) = \min_x f_\infty .$$

In particular, for j large enough $x_j \in K$ (as in the proof, so $\exists \{x_{j_k}\} \subseteq \{x_j\}$ converging to some x_0 a global minimizers of f_∞.

Loosely speaking, if $f_j \xrightarrow{\Gamma} f_\infty$ then global minimizers converg to global minimizers.

(3.2) Γ-convergence does not give any information about local minimizers.

Local minimizers (in general) do not converge to local minimizers.

Ex: $f_j(x) = \begin{cases} -1 & x \in \frac{1}{j}\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$, we saw that $\Gamma\text{-}\lim_{j\to+\infty} f_j \equiv -1$.

Let $g_j(x) = x^2 + f_j$, by stability under continuous perturbation $g_j \xrightarrow{\Gamma} x^2 - 1$.

In this case every $x \in \frac{1}{j}\mathbb{Z}$ is a local minimizer of g_j but they converge to any point in $\mathbb{R}$, while the unique local minimizer of the Γ-limit is $x = 0$.

Equi (mildly-) coerciveness is essential to prove the fund. thm, since it ensures that min. seq. stay bounded. If minimizers "escape" to infinity, Γ-convergence alone is (in general) not sufficient.

Ex (lack of coerciveness): take $f_j(x) = \begin{cases} 0 & x \neq j \\ -1 & x = j \end{cases}$. It is immediate to prove that

$$f_\infty := \Gamma\text{-}\lim_{j \to +\infty} f_j \equiv 0.$$

Even though each f_j is mildly-coercive, since $\{f \leq -1/2\} = \{j\}$, the sequence is not equi mildly-coercive, indeed $\forall K \Subset \mathbb{R}$, for j large enough $\inf_K f_j = 0 \neq -1 = \inf_x f_j$.
For such a Γ-converging sequence, the fund. thm does not hold

$$\lim_{j \to +\infty} \inf_x f_j = -1 \neq 0 = \min_x f_\infty.$$

Lecture 4: 06.05.2024

An immediate corollary of the Fundamental Theorem of Γ-convergence is when considering a constant sequence $f_j \equiv f$. In this case, it provides a relaxation result, with which we can rigorously solve infimum problems for non lsc functionals.

Corollary 1 (relaxation): let $f : X \longrightarrow [-\infty, +\infty]$ be s.t. $\exists\, t_0 \in \mathbb{R}$: $\{f \leq t_0\}$ is precompact. Then

$$\inf_X f = \min_X sc(f).$$

Moreover, $\forall \{x_j\}$ inf. seq. $\exists x_\infty, \{x_{j_k}\} \subseteq \{x_j\}$ s.t. $x_{j_k} \longrightarrow x_\infty$ and x_∞ is a global minimizer of $sc(f)$.

proof: $f_j \equiv f$, $f_j \xrightarrow{\Gamma} sc(f)$ and f_j is equi mildly-coercive. So by the Fund. Thm of Γ-conv:

$$\inf_X f = \lim_{j \to +\infty} \inf_X f_j = \min_X sc(f).$$

By def. of inf $\exists\, x_j$ s.t. $\lim_{j \to +\infty} f(x_j) = \inf_X f$. For j large enough, $x_j \in \{f \leq t_0\} \subseteq \overline{\{f \leq t_0\}}$ which is compact by coercivity.

So $\{x_j\}$ is precompact $\Rightarrow \exists \{x_{j_k}\} \subseteq \{x_j\}$ and x_∞ s.t. $x_{j_k} \longrightarrow x_\infty$ and, again by Fund. Thm, x_∞ is a global minimizer of $sc(f)$. $\square$

"Compactness" of Γ-convergence

It is often useful (and at this point we did it already quite some times in this course) to assume the existence of the Γ-limit of a sequence of functionals.
With the result below, if the space X is separable, this is a reasonable assumption, which is actually true up to sub-sequence.

Theorem 2: let (X,d) be a separable metric space, let $f_j : X \longrightarrow [-\infty, +\infty]$.
Then $\exists \{f_{j_k}\} \subseteq \{f_j\}$ s.t. $\exists\, \Gamma\text{-}\lim_{k\to+\infty} f_{j_k}$.

proof: let $\mathcal{B} := \{U_h\}_{h\in\mathbb{N}}$ countable basis of topology
consider $\inf_{U_0} f_j \in [-\infty,+\infty]$, $\exists\, \sigma_j^{(0)} \to +\infty$ s.t.

$$\exists \lim_{j\to+\infty} \inf_{U_0} f_{\sigma_j^{(0)}}.$$

We can do the same for $\inf_{U_1} f_{\sigma_j^{(0)}}$, $\exists \{\sigma_j^{(1)}\} \subseteq \{\sigma_j^{(0)}\}$ s.t.

$$\exists \lim_{j\to+\infty} \inf_{U_1} f_{\sigma_j^{(1)}},\ \lim_{j\to+\infty} \inf_{U_0} f_{\sigma_j^{(0)}}.$$

Iterating this, we can define $\{\sigma_j^{(k)}\} \subseteq \{\sigma_j^{(h)}\}$ s.t.

$$\exists \lim_{j\to+\infty} \inf_{U_h} f_{\sigma_j^{(k)}} \quad \forall\, h \leq k.$$

By a diagonal argument, we define $j_k := \sigma_k^{(k)}$ s.t.

$$\exists \lim_{k\to+\infty} \inf_{U} f_{j_k} \quad \forall\, U \in \mathcal{B}.$$

Taking the sup over $U \in \mathcal{N}(x) \cap \mathcal{B}$ we get, $\forall x \in X$

$$\sup_{U\in N(x)} \underline{\lim}_{k\to+\infty} \inf_U f_{j_k} = \sup_{U\in N(x)} \overline{\lim}_{k\to+\infty} \inf_U f_{j_k}$$

This implies the claim, indeed since $N(x) \supseteq N(x)\cap \mathcal{B}$ then

$$\sup_{U\in N(x)} \underline{\lim}_{k\to+\infty} \inf_U f_{j_k} \geq \sup_{U\in N(x)\cap\mathcal{B}} \underline{\lim}_{k\to+\infty} \inf_U f_{j_k}.$$

Moreover, since $\mathcal{B}$ is a base, $\forall\, U\in N(x)\ \exists\, U'\in N(x)\cap\mathcal{B}$ s.t. $U'\subseteq U$, thus

$$\underline{\lim}_{k\to+\infty} \inf_U f_{j_k} \leq \underline{\lim}_{k\to+\infty} \inf_{U'} f_{j_k}$$

which implies the proof by taking first the sup in $U'\in N(x)\cap\mathcal{B}$ and then in $U\in N(x)$. $\square$

<u>Proposition</u> 3 (Uryshon property): let $f_j : X \to [-\infty,+\infty]$, then

$$\Gamma\text{-}\lim_{j\to+\infty} f_j = f_\infty \iff \forall \{j_k\}\ \exists \{j_k'\}\subseteq\{j_k\} \text{ s.t. } \Gamma\text{-}\lim_{k\to+\infty} f_{j_k'} = f_\infty.$$

proof: ($\Rightarrow$) by (1.3) of last lecture.

($\Leftarrow$) by assumption, and again by (1.3), $\exists\, j_k$ s.t.

$$\Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j \leq \Gamma\text{-}\underline{\lim}_{k\to+\infty} f_{j_k} =$$

$$= \Gamma\text{-}\overline{\lim}_{k\to+\infty} f_{j_k} = \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j.$$

Suppose by contradiction that $\nexists\, \Gamma\text{-}\lim_{j\to+\infty} f_j$, then one of the following holds:

(1) $f_\infty(x) < \Gamma\text{-}\overline{\lim}_{j\to+\infty} f_j$ (2) $f_\infty(x) > \Gamma\text{-}\underline{\lim}_{j\to+\infty} f_j$.

If (1) holds then $f_\infty(x) < \sup_{U \in N(x)} \overline{\lim_{j\to+\infty}} \inf_U f_j$. Since the inequality is strict, $\exists\, U \in N(x)$ s.t. $f_\infty(x) < \overline{\lim_{j\to+\infty}} \inf_U f_j$. So $\exists \{j_k\}$ s.t. $f_\infty(x) < \lim_{k\to+\infty} \inf_U f_{j_k}$ and by hypothesis $\exists \{j_k'\} \subseteq \{j_k\}$ s.t. $f_{j_k'}$ is Γ-converging. So

$$f_\infty(x) < \lim_{k\to+\infty} \inf_U f_{j_k'} < \Gamma\text{-}\lim_{k\to+\infty} f_{j_k}(x)$$

which leads to a contradiction.
If (2) holds, $\exists\, x_j \to x$ s.t. $\underline{\lim_{j\to+\infty}} f_j(x_j) < f_\infty(x)$. By the properties of the liminf $\exists \{j_k\}$ s.t.

$$\lim_{k\to+\infty} f_{j_k}(x_{j_k}) < f_\infty(x).$$

Again by hypothesis, $\exists \{j_k'\} \subseteq \{j_k\}$ s.t. the Γ-limit exists $\Rightarrow$ $\Gamma\text{-}\underline{\lim_{k\to+\infty}} f_{j_k'}(x) < f_\infty(x)$ which is again a contradiction. $\square$

Recollecting all the "topological" property of the Γ-limit we can associate it to a topology.

<u>Rmk</u> (the topology of Γ-convergence): in (X,d) separable

- Γ-limit is unique
- Γ-limit of constant seq. (lsc) is the function itself
- compactness: $\forall \{f_j\}$ $\exists$ subseq. $\{f_{j_k}\}$ Γ-converging
- Uryshon property.

Let $S(X) := \{ f : X \to [-\infty, +\infty] : f \text{ is lsc} \}$, then Γ-convergence defines a topology τ, and $(S(X), \tau)$ is compact and T_2.

[Ch. 10 of "An Introduction to Γ-convergence", Dal Maso (1993)]

Exercise: given $a : \mathbb{R} \to \{\alpha, \beta\}$ being a 1-periodic, measurable function, where $0 < \alpha \leq \beta < +\infty$.
Consider $F_j : L^2(0,1) \to [0, +\infty]$ the energy defined as

$$F_j(u) = \begin{cases} \frac{1}{2} \int_0^1 a(jx) |u'(x)|^2 \, dx & u \in H_0^1(0,1) \\ +\infty & \text{otherwise.} \end{cases}$$

Prove that

$$\Gamma(L^2) - \lim_{j \to +\infty} F_j(u) = \begin{cases} \frac{1}{2} \int_0^1 a^* |u'(x)|^2 \, dx & u \in H_0^1(0,1) \\ +\infty & \text{otherwise} \end{cases}$$

where $a^* := \left(\int_0^1 a^{-1} \right)^{-1}$ is the harmonic mean of a.

Hint: use compactness of Γ-convergence, Urysohn property, Fund. Thm of Γ-conv. and look at the Euler-Lagrange equations of F_j.

Integral functionals in Lebesgue spaces

Main reference for this lecture:

- Sec. 2.1–2.3, [B, 2000]

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, connected domain and $n, m \in \mathbb{N}$ $n, m \geq 1$. We will consider $F: L^p(\Omega; \mathbb{R}^m) \longrightarrow (-\infty, +\infty]$ of the form

$$F(u) = \int_\Omega f(x, u(x))\, dx$$

with $f: \Omega \times \mathbb{R}^m \longrightarrow (-\infty, +\infty]$ measurable in the first and Borel in the second variable.

Studying functional as F with the Direct Method is more suited for weak convergences rather than strong convergences since, for the former, mild-coerciveness, i.e. $\{F \leq t\}$ being pre-compact, it's easier to obtain.
The same consideration holds also when working in Sobolev spaces (that we will treat from lecture 7 on).

We then recall some definitions and properties regarding weak convergences in Lebesgue spaces.

Weak convergence

Def: let $p \in [1, \infty]$ and $u_j, u \in L^p(\Omega; \mathbb{R}^m)$ u_j converges weakly (or weakly* if $p = \infty$), $u_j \rightharpoonup u$ (or $u_j \overset{*}{\rightharpoonup} u$ if $p = \infty$) iff

$$\lim_{j \to +\infty} \int_\Omega (u_j - u) \cdot v \, dx = 0, \quad \forall\, v \in L^{p'}(\Omega; \mathbb{R}^m)$$

where $p' = \begin{cases} \infty & p = 1 \\ p/p-1 & 1 < p < \infty \end{cases}$

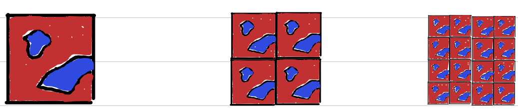
Rmk: some facts to recall are the following

(1.1) by Banach-Alaoglu, if $p>1$, if $\sup_{j\in\mathbb{N}} \|u_j\|_{L^p} < +\infty$ then $u_j \rightharpoonup u$ (or $u_j \overset{*}{\rightharpoonup} u$ if $p=\infty$) uts (i.e. up to subsequences).
Conversely, by Banach Steinhaus, for $1\le p\le\infty$, if $u_j \rightharpoonup u$ (or $u_j \overset{*}{\rightharpoonup} u$ if $p=\infty$) $\Rightarrow$ $\sup_{j\in\mathbb{N}} \|u_j\|_{L^p} < +\infty$ and $\|u\|_{L^p} \le \liminf_{j\to+\infty} \|u_j\|_{L^p}$.

(1.2) given $u\in L^p(\Omega;\mathbb{R}^m)$ which is $[0,1]^n$-periodic, then $u_j(x) := u(jx)$ it holds $\forall\, E\subseteq\Omega$ open

$$u_j \xrightarrow[L^p(E;\mathbb{R}^m)]{} \int_{[0,1]^n} u(x)\,dx =: \bar{u} \in \mathbb{R}.$$

Let's give some intuition for this. For this consider $u\in\{z_1,z_2\}$ taking just two values. If we periodize it and reduce the period, this is similar to a "ZOOM OUT". From distance we only see the average.



period $=1$ period $=1/2$ period $=1/4$

(Seq.) Weak lowersemicontinuity

We saw lsc and Γ-convergence in metric spaces. Since weak topologies are not metrizable in Lebebesgue spaces, we are not in the setting of the previous lectures.
We then consider sequential lower semicontinuity. We will see that this coincides with the topological lsc when f complies with some growth conditions.

For simplicity we deal with the autonomous, i.e. case $f(x,z) = f(z)$ but the same results hold for the nonautonomous case.

Proposition 3 (necessity): Let $F : L^\infty(\Omega;\mathbb{R}^m) \to (-\infty,+\infty]$ as

$$F(u) = \int_\Omega f(u(x))\,dx.$$

If F is (seq.) weakly* lsc $\Rightarrow$ f is lsc and convex.

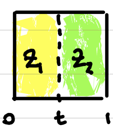
proof: (lsc) Let $z_j \to z \in \mathbb{R}^m$ and $u_j \equiv z_j$ and $u \equiv z$
$u_j \xrightarrow{*} u$ (actually $u_j \rightrightarrows u$) then

$$|\Omega| f(z) = \int_\Omega f(u)\,dx \leq$$

$$\leq \varliminf_j \int_\Omega f(u_j)\,dx \leq \varliminf_j |\Omega| f(z_j).$$

(convexity) consider $z_1, z_2 \in \mathbb{R}^m$ s.t. $f(z_1), f(z_2) < +\infty$ otherwise the convexity is obvious. Let $t \in [0,1]$, $z := tz_1 + (1-t)z_2$ and $v \in L^\infty(\mathbb{R}^n;\mathbb{R}^m)$ a $[0,1]^n$-periodic function defined as

$$v(x) = \begin{cases} z_1 & 0 \leq x_1 < t \\ z_2 & t \leq x_1 < 1 \end{cases},$$



and $u_j(x) := v(jx)$. By Remark (1.2) we have

$$u_j \xrightarrow{*} \int_{[0,1]^n} v(x)\,dx = tz_1 + (1-t)z_2 = z,$$

so denoting $u(x) \equiv z$, $u_j \xrightarrow{*} u$. Analogously

$$f(v(x)) = \begin{cases} f(z_1) & 0 \leq x_1 < t \\ f(z_2) & t \leq x_1 < 1 \end{cases}$$

and $f(u_j(x)) = f(v(jx))$ so $f(u_j) \xrightarrow{*} tf(z_1) + (1-t)f(z_2)$ again by Remark (1.2). Then

$$|\Omega| f(z) = F(u) \leq \varliminf_j F(u_j)$$

$$= \lim_j \int_\Omega f(u_j)\,dx = |\Omega|\big(tf(z_1) + (1-t)f(z_2)\big). \quad \square$$

Rmk (wellposedness): we do not consider f that can attain value $-\infty$ since in this case the Lebesgue integral may not be well-defined.
Moreover, even in the case it is (e.g. if $f \leq C < +\infty$) if $\exists\, z \in \mathbb{R}^m$ s.t. $f(z) = -\infty$, if F is (seq.) weakly* lsc then $F \equiv -\infty$.
Indeed, $\forall\, u \in L^\infty(\Omega; \mathbb{R}^m)$, let $x_0 \in \Omega$ then consider

$$u_j(x) := \begin{cases} z & x \in B_{1/j}(x_0) \\ u(x) & \text{otherwise}, \end{cases} \quad \text{for } j \text{ large enough},$$

$$\int_\Omega (u_j - u) v \, dx = \int_{B_{1/j}(x_0)} z v \, dx \leq |z| \, \|v\|_{L^1(B_{1/j}(x_0))} \to 0. \text{ So } u_j \overset{*}{\rightharpoonup} u$$

but $F(u_j) \equiv -\infty$.

It may sound strange to specify (as we did in the statement of Proposition 3) that f is convex and lsc since, in the case in which $f : \mathbb{R}^m \to \mathbb{R}$, convexity implies at least continuity (actually even more regularity).
In the case in which f takes also value $+\infty$ this is not true and there exist convex functions that are not lsc.

Ex (convex nonlsc function): take $f(z) := \begin{cases} 0 & |z| < 1 \\ +\infty & |z| \geq 1. \end{cases}$
It is trivial to check convexity and that f is not lsc.

We continue our study of weak lsc in Lebesgue spaces. For this we need the following approximation result.

Lemma 1: let $f:\mathbb{R}^m \to [0,+\infty]$ be convex and lsc $\Rightarrow$ $\exists$ an increasing sequence $f_k:\mathbb{R}^m \to [-1,+\infty)$ of convex, Lipschitz, C^1 functions s.t. $f(z)=\sup_k f_k(z)$ $\forall z\in\mathbb{R}^m$.

We will not prove the result above. It can be seen as a corollary of a well known property of the Yosida tranform which says that lsc functions are sup of their Yosida transform [Proposition 1.7, BD, 1998]. Using other properties [Remark 1.6 (a), (c), BD, 1998] and a mollification argu- -ment, one gets the result.

Proposition 2 (sufficiency): let $f:\mathbb{R}^m\to(-\infty,+\infty]$ be convex and lsc. Then $F:L^1(\Omega;\mathbb{R}^m)\to(-\infty,+\infty]$ defined as $F(u)=\int_\Omega f(u(x))\,dx$ is (seq.) weakly lsc.

proof: we work in two steps

Step 1 (f assume f Lipschitz and C^1): from convexity and regularity of f, it holds $f(z_1)\leq f(z_2)+\nabla f(z_2)\cdot(z_1-z_2)$ for every $z_1,z_2\in\mathbb{R}^m$. Let $u_j \underset{L^1(\Omega)}{\rightharpoonup} u$ then

$$\int_\Omega f(u(x))\,dx \leq \int_\Omega f(u_j(x))\,dx + \int_\Omega \nabla f(u(x))\cdot(u(x)-u_j(x))\,dx.$$

Since f is Lipschitz, $\nabla f\in L^\infty(\mathbb{R}^m;\mathbb{R}^m)$ so $\nabla f(u)\in L^\infty(\Omega;\mathbb{R}^m)$, thus it is a good test function for weak convergene, i.e.

$$\int_\Omega \nabla f(u(x))\cdot(u(x)-u_j(x))\,dx \xrightarrow[j\to+\infty]{} 0.$$

which yields

$$F(u)=\int_\Omega f(u(x))\,dx \leq \varliminf_{j\to+\infty}\int_\Omega f(u_j(x))\,dx = \varliminf_{j\to+\infty} F(u_j).$$

Step 2 (general f): by convexity $\exists\, a \in \mathbb{R}^m$, $b \in \mathbb{R}$ s.t. $f(z) \geq a \cdot z + b$. Define $g(z) := f(z) - (a \cdot z + b) + 1$ and note that $g(z) \geq 1$ and convex.
By Lemma 1, $\exists\, g_k \geq 0$, convex, Lipschitz, e' $g_k(z) \uparrow g(z)$ and denote

$$F_k(u) := \int_\Omega g_k(u(x))\, dx.$$

By Step 1, F_k are (seq.) weakly lsc in $L^1(\Omega;\mathbb{R}^m)$. Since $\forall\, u \in L^1(\Omega;\mathbb{R}^m)$, $g_k(u(x)) \uparrow f(u(x)) - a \cdot u(x) - b + 1$ for a.e. $x \in \Omega$, by Monotone Convergence

$$\int_\Omega f(u(x)) - a \cdot u(x) - b + 1\, dx = \sup_k F_k(u).$$

Hence, $F(u) = \sup_k F_k(u) + a \cdot \int_\Omega u(x)\, dx + (b-1)|\Omega|$. Since $u \mapsto \int_\Omega u(x)\, dx$ is continuous wrt L^1 weak (even strong) convergence, and the sup of (seq.) lsc functionals $\Rightarrow$ F is (seq.) lsc. $\square$

<u>Rmk</u> (well-posedness): if f is convex, we authomatically have a growth condition from below, this yields that our functional F is well-defined.
Indeed, f convex $\Rightarrow$ $f(z) \geq -a|z| - b$ for some $a, b > 0$, $\forall\, z \in \mathbb{R}^m$, then

$$F(u) \geq -a \int_\Omega |u(x)|\, dx - b|\Omega| > -\infty, \quad \forall\, u \in L^1(\Omega;\mathbb{R}^m).$$

Collecting the results of Proposition 3 of last lecture and Proposition 2 above we obtain the following characterization.

Corollary 3: Let $F: L^p(\Omega;\mathbb{R}^m) \to (-\infty,+\infty]$ be defined as $F(u) = \int_\Omega f(u(x))\,dx$, for $f:\mathbb{R}^m \to (-\infty,+\infty]$ Borel. Then F is (seq.) weakly lsc (wrt L^p convergence) $\Leftrightarrow$ f is lsc and convex.

Proof: ($\Rightarrow$) consider the restriction $G := F|_{L^\infty(\Omega;\mathbb{R}^m)}$. For any $u_j \xrightarrow[L^\infty]{*} u \Rightarrow u_j \xrightarrow[L^p]{} u$, so by hypothesis

$$G(u) = F(u) \leq \liminf_{j\to+\infty} F(u_j) = \liminf_{j\to+\infty} G(u_j)$$

so G is (seq.) weakly* lsc. By Proposition 3 of last lecture, f is lsc and convex.

($\Leftarrow$) similarly, consider the extension $H(u) := \int_\Omega f(u(x))\,dx$ for $u \in L^1(\Omega;\mathbb{R}^m)$. By Proposition 2 above, H is (seq.) weak lsc. In particular, $\forall\, u_j \xrightarrow[L^p]{} u$ it holds

$$F(u) = H(u) \leq \liminf_{j\to+\infty} H(u_j) = \liminf_{j\to+\infty} F(u_j). \qquad \square$$

The characterization above holds in general for functionals $F(u) = \int_\Omega f(x,u(x))\,dx$ with $f(x,\cdot)$ convex and lsc for a.e. $x \in \Omega$.

Rmk ((seq.) weak continuity): a functional is upper semicontinuous if its opposite is lsc.
So by Corollary 3, F is upper semicontinuous if $-f$ is convex and lsc.
In particular, F is (seq.) weakly continuous $\Leftrightarrow$ f and $-f$ are both convex $\Leftrightarrow$ f is affine.

It is useful to check which conditions imply strong lsc. We will make use of this often when dealing with functionals is Sobolev spaces.
Strong lsc is a consequence of Fatou's lemma, growth conditions and lsc of the density f.
Notice that we do not need to specify (seq.) strong lsc since strong convergence is induced by a metric and in metric spaces lsc $\Leftrightarrow$ (seq.) lsc.

<u>Rmk</u> (strong lsc): if $f(x,z) \geq a(x) + b|z|$ for some $b \in \mathbb{R}$ and $a \in L^1(\Omega)$ and $f(x,\cdot)$ is lsc for a.e. $x \in \Omega$ then F is strong lsc in $L^1(\Omega;\mathbb{R}^m)$.
Indeed, let $u_j \xrightarrow[L^1]{} u \Rightarrow$ uts $u_j \xrightarrow[a.e.]{} u$. Denote by $A = \int_\Omega a(x)\,dx$, $g(x,z) := f(x,z) - a(x) - b|z| \geq 0$. Then

$$
\begin{aligned}
F(u) &= \int_\Omega g(x,u(x))\,dx + A + b\|u\|_{L^1(\Omega;\mathbb{R}^m)} \\
&\overset{\text{lsc of } f(x,\cdot)}{\leq} \int_\Omega \varliminf_{j\to+\infty} g(x,u_j(x))\,dx + A + b\|u\|_{L^1(\Omega;\mathbb{R}^m)} \\
&\overset{\text{Fatou's Lemma}}{\leq} \varliminf_{j\to+\infty} \int_\Omega g(x,u_j(x))\,dx + A + b\|u\|_{L^1(\Omega;\mathbb{R}^m)} \\
&= \varliminf_{j\to+\infty} F(u_j)
\end{aligned}
$$

where in the last step we used that $u \mapsto \|u\|_{L^1(\Omega;\mathbb{R}^m)}$ is strongly continuous.

Working analogously we get that if $f(x,z) \geq a(x) + b|z|^p$ and $f(x,\cdot)$ is lsc $\Rightarrow$ F is strongly continuous in L^p.

An immediate consequence is that if $|f(x,z)| \leq a(x) + b|z|^p$ and f is a Caratheodory function $\Rightarrow$ F is strongly continuous in L^p.

Convex (lsc) envelope

We proved that convexity and lsc (of the density) characterizes (seq.) weak lsc (of the functional).
It is reasonable to expect that (seq.) weak lsc envelopes are linked to convex, lsc envelopes of the densities.

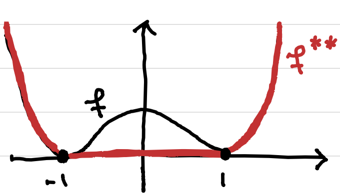
Def: let $f: \mathbb{R}^m \to [-\infty, +\infty]$, we define $f^{**}: \mathbb{R}^m \to [-\infty, +\infty]$ is the convex, lsc envelope of f as

$$f^{**} := \sup \{ g: \mathbb{R}^m \to [-\infty, +\infty] : \text{convex, lsc, } g \leq f \}.$$

Note that f^{**} is convex and lsc since sup of convex (and lsc) functions is still convex (and lsc).
The notation comes from the conjugation (i.e. Legendre transform) operation; f^{**} is obtained by doing the Legendre transform twice.

If $f: \mathbb{R}^m \to \mathbb{R}$ then $f^{**} = f^c$, where f^c denotes the convex envelope, since for functions not attaining $+\infty$ convexity $\Rightarrow$ lsc.

Ex: the easiest example of convex envelope, $f(t) = (t^2-1)^2$ and its convex envelope is



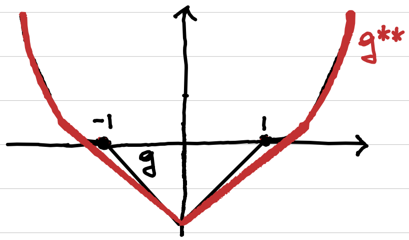
$$f^{**}(t) = \begin{cases} (t^2-1)^2 & |t| > 1 \\ 0 & |t| \leq 1. \end{cases}$$

The convex envelope is a nonlocal operation, if f changes in a compact, f^{**} may change also outside the compact.
For this reason, especially in higher dimensions, its computation is not easy.

For instance, consider

$$g(t) = \begin{cases} (t^2-1)^2 & |t|>1 \\ |t|-1 & |t|\leq 1. \end{cases}$$

This is a modification of f inside B_1, i.e. $f(t)=g(t)$ $\forall\ |t|\geq 1$. But $f^{**} \neq g^{**}$ in $|t|\geq 1$. Indeed



$$g^{**}(t) = \begin{cases} (t^2-1)^2 & |t|> t_0 \\ \sigma_0 |t|-1 & |t|\leq t_0 \end{cases}$$

where $t_0 = \left(\frac{1+7^{1/2}}{3}\right)^{1/2}$ and $\sigma_0 = 4t_0(t_0^2-1)$.

Some useful (standard) properties of convex functions are listed below.

__Rmk__: (1.1) f convex $\Rightarrow$ $\forall\ t\in(-\infty,+\infty]$, $\{f<t\}$ is convex

(1.2) f convex, lsc $\Rightarrow$ inside $\{f<+\infty\}$ f is locally Lipschitz, i.e. $\forall\ K \Subset \{f<+\infty\}$ $\exists\ C_K>0$ s.t.

$$|f(x)-f(y)| \leq C_K\,|x-y|, \quad \forall\ x,y\in K.$$

__Lemma__ 4: let $f:\mathbb{R}^m \to \mathbb{R}$ then

$$f^{**}(z) = \inf\left\{\sum_{i=1}^{k} \lambda_i f(z_i) : \lambda_i>0,\ z_i\in\mathbb{R}^m,\ \sum_{i=1}^{k}\lambda_i=1,\ \sum_{i=1}^{k}\lambda_i z_i = z\right\}.$$

We do not prove the Lemma above since it is standard in convex analysis.

Relaxation in Lebesgue spaces

Main reference for this part: since in [B, 2000] only the 1D case it's treated, the proof of next Theorem (in m dimension) can be found in:

- Sec. 4.3, ["Gamma convergence - Lecture notes", R. Cristoferi]

If our functional is not (seq.) weak lsc, then its relaxation (i.e. its (seq.) weak lsc envelope, $\bar{F}$ in the statement below) is obtained by the functional whose density is the convex (lsc) envelope of the density.

We treat the case of f not attaining $+\infty$ for simplicity. The same results holds true, also for f attaining $+\infty$.

<u>Theorem</u> 5: let $p \in [1, \infty]$, $\Omega \subseteq \mathbb{R}^n$ open and bounded and let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a Borel function. Consider the functional $F : L^p(\Omega; \mathbb{R}^m) \longrightarrow [-\infty, +\infty]$ defined as

$$F(u) = \int_\Omega f(u(x))\, dx.$$

Then

$$(R) \qquad \bar{F}(u) = \int_\Omega f^{**}(u(x))\, dx$$

where $\bar{F}(u) := \sup\{G : L^p(\Omega; \mathbb{R}^m) \to [-\infty, +\infty] : G \le F$ and G is (seq.) weak (weak* if $p = \infty$) lsc$\}$.

<u>proof</u>: we give a name to the right-hand side of (R so let $G(u) := \int_\Omega f^{**}(u(x))\, dx$.

Since $f^{**} \leq f \Rightarrow G \leq F$ and since f^{**} is convex, lsc by Corollary 3, G is (seq.) weakly lsc so $G \leq \bar{F}$.
To prove (R) it is sufficient to show that $\bar{F} \leq G$.

Step 1 (u=const): let $u \equiv z \in \mathbb{R}^m$. By Lemma 4 $\forall \varepsilon > 0$ $\exists\, \lambda_i^\varepsilon > 0$, $z_i^\varepsilon \in \mathbb{R}^m$ s.t. $\sum_{i=1}^k \lambda_i^\varepsilon z_i^\varepsilon = z$ and

$$(*) \quad f^{**}(z) > \sum_{i=1}^k \lambda_i^\varepsilon f(z_i^\varepsilon) - \varepsilon.$$

We consider

$$v^\varepsilon(x) = z_j^\varepsilon \quad \text{as} \quad \sum_{i=1}^{j-1} \lambda_i^\varepsilon < x_1 < \sum_{i=1}^{j} \lambda_i^\varepsilon$$

and extend it 1-periodically in x_1. Let $u_j^\varepsilon(x) := v^\varepsilon(jx)$, by Riemann-Lebesgue Lemma we have

$$u_j^\varepsilon \xrightarrow{*} u \quad \text{and} \quad f(u_j^\varepsilon) \xrightarrow{*} \sum_{i=1}^k \lambda_i^\varepsilon f(z_i^\varepsilon).$$

So, using also $(*)$, we get $\forall \varepsilon > 0$

$$\bar{F}(u) \leq \lim_{j \to +\infty} F(u_j^\varepsilon) = \int_\Omega \sum_{i=1}^k \lambda_i^\varepsilon f(z_i^\varepsilon)\, dx$$
$$< \int_\Omega f^{**}(z)\, dx + \varepsilon|\Omega| = G(u) + \varepsilon|\Omega|$$

and we have the desired estimate by arbitrariness of ε.

Step 2 (u simple function): let $u = \sum_{i=1}^N z_i \chi_{E_i}$ for $z_i \in \mathbb{R}^m$ and $E_i \subseteq \Omega$ open s.t. $|\Omega \setminus \bigcup_{i=1}^N E_i| = 0$.
$\forall \varepsilon > 0$ let $u_j^{\varepsilon,i}$ the sequence defined in Step 1 for $u \equiv z_i$ in E_i. Then $u_j^\varepsilon := \sum_{i=1}^N u_j^{\varepsilon,i} \chi_{E_i} \xrightarrow{*} u$, so

$$\bar{F}(u) \leq \lim_{j \to +\infty} F(u_j^\varepsilon) = \lim_{j \to +\infty} \int_\Omega f\Big(\sum_{i=1}^N u_j^{\varepsilon,i}(x) \chi_{E_i}(x)\Big)\, dx$$
$$= \lim_{j \to +\infty} \sum_{i=1}^N \int_{E_i} f(u_j^{\varepsilon,i}(x))\, dx$$
$$\overset{\text{Step 1}}{<} \sum_{i=1}^N \Big(\int_{E_i} f^{**}(z_i)\, dx + \varepsilon |E_i| \Big) = G(u) + \varepsilon|\Omega|,$$

and again we conclude by arbitrariness of ε. Notice that, in the second line above, we used that the limits exist, thanks to Riemann-Lebesgue, so that we could switch lim and summation.

Step 3 (reduction to simple functions): let $u \in L^p(\Omega;\mathbb{R}^m)$, if we prove that $\exists\, \bar{u}_j$ simple functions $\bar{u}_j \xrightarrow[L^p]{} u$ s.t. $\overline{\lim_{j\to+\infty}}\, G(\bar{u}_j) \leq G(u)$ then we obtain (R). Indeed,

$$G(u) \leq \bar{F}(u) \leq \underline{\lim}_{j\to+\infty} \bar{F}(\bar{u}_j) \overset{\text{Step 2}}{\leq} \underline{\lim}_{j\to+\infty} G(\bar{u}_j)$$

$$\leq \overline{\lim_{j\to+\infty}}\, G(\bar{u}_j) \leq G(u).$$

So, we are left to prove that $\forall u \in L^p(\Omega;\mathbb{R}^m)$ $\exists\, \bar{u}_j \to u$ of simple functions s.t. $\overline{\lim_{j\to+\infty}}\, G(\bar{u}_j) \leq G(u)$.

If $G(u) = +\infty$ there is nothing to prove, so let $G(u) < +\infty$ that is $f^{**}(u) \in L^1(\Omega)$.

$\forall \varepsilon > 0$ $\exists\, \Omega_\varepsilon \subset\subset \Omega$ s.t. $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$

$$\int_{\Omega\setminus\Omega_\varepsilon} f^{**}(u)\,dx < \varepsilon \quad \text{and} \quad |u(x)| \leq 1/\varepsilon \quad \forall x \in \Omega_\varepsilon.$$

By density of simple functions in $L^p(\Omega;\mathbb{R}^m)$, we can find u_j simple functions $u_j \to u$ a.e. and by Egorov Theorem (up to reduce Ω_ε) we can assume $u_j \rightrightarrows u$ on Ω_ε. In particular, $|u_j(x)| \leq 2/\varepsilon$ $\forall x \in \Omega_\varepsilon$, for j large enough.

Defining

$$\bar{u}_j^\varepsilon(x) = \begin{cases} u_j(x) & x \in \Omega_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

we get

$$|G(\bar{u}_j^\varepsilon) - G(u)| \leq \int_\Omega |f^{**}(\bar{u}_j^\varepsilon(x)) - f^{**}(u(x))|\,dx$$

(*$_1$)

$$\leq \int_{\Omega_\varepsilon} |f^{**}(u_j(x)) - f^{**}(u(x))|\,dx + \int_{\Omega\setminus\Omega_\varepsilon} |f^{**}(0)| + |f^{**}(u(x))|\,dx.$$

Since f^{**} is convex (and real valued) $\exists\, C_\varepsilon > 0$ s.t.

$$|f^{**}(z_1) - f^{**}(z_2)| \leq C_\varepsilon |z_1 - z_2|$$

$\forall z_1, z_2 \in \mathbb{R}^m : |z_1|, |z_2| \leq 2/\varepsilon.$

Then, by $(*_1)$ and the properties of Ω_ε we get

$$|G(\bar{u}_j^\varepsilon) - G(u)| \leq C_\varepsilon \int_{\Omega_\varepsilon} |u_j(x) - u(x)|\,dx + (1 + f^{**}(0))\,\varepsilon.$$

Taking the limit in j we have

$$\lim_{j \to +\infty} |G(\bar{u}_j^\varepsilon) - G(u)| \leq (1 + f^{**}(0))\,\varepsilon.$$

We can then extract a sequence $\bar{u}_j \rightharpoonup u$ s.t. $G(\bar{u}_j) \to G(u)$ which yields the claim. $\square$

Rmk: if f is not bounded below by an affine function $f^{**} \equiv -\infty$ and this result becomes trivial.
If instead $f(z) \geq a \cdot z + b$ then both F and $\bar{F}$ are well-defined (as we saw previously).

Rmk (f attaining $+\infty$): the identical result holds also in the case in which $f : \mathbb{R}^m \to (-\infty, +\infty]$.

Lecture 6: 13.05.2024

In past lectures (as we often commented on) we worked with (seq.) weak lsc.
This is because weak topologies are not metrizable on the whole $L^p(\Omega;\mathbb{R}^m)$. They are metrizable on bounded (wrt $\|\cdot\|_{L^p}$) subsets of $L^p(\Omega;\mathbb{R}^m)$ though.

We see now that, if the density f comply with some growth condition from below this ambiguity is removed. Additionally, growth conditions provide coerciveness (in the weak-L^p topology) of the functionals F.

Before proceeding we recall the notation we have been using. We consider $F: L^p(\Omega;\mathbb{R}^m) \to (-\infty,+\infty]$ defined as

$$(*) \quad F(u) := \int_\Omega f(u(x))\,dx$$

with $f:\mathbb{R}^m \to (-\infty,+\infty]$ a Borel functions.

Weak coerciveness

Proposition 1: let F be as in $(*)$ and f such that:
for $1<p<\infty$

(GC$_p$) $\exists\, c_1, c_2>0$ s.t. $c_1|z|^p - c_2 \leq f(z)$, $\forall z\in\mathbb{R}^m$

then F is (L^p-)weakly coercive;
for $p=1$

(GC$_1$ $\exists\, c>0$, $\varphi:[0,+\infty)\to[0,+\infty)$ increasing, convex with $\lim_{t\to+\infty}\frac{\varphi(t)}{t}=+\infty$ s.t. $c(\varphi(|z|)-1)\leq f(z)$, $\forall z\in\mathbb{R}^m$

then F is (L^1-)weakly coercive;

if $p=\infty$

(GC_∞) $\exists\, c, R>0$ s.t. $f(z) \geq \begin{cases} -c & |z|\leq R \\ +\infty & |z|>R \end{cases}$

then F is $(L^\infty-)$ weakly* coercive.

<u>proof</u>: $(1<p<\infty)$ let $u\in\{F\leq t\}$ then, by integrating (GC_p)

$$c_1\|u\|^p_{L^p} - c_2|\Omega| \leq F(u)\leq t,$$

so $\{F\leq t\}\subseteq \left\{\|u\|_{L^p}\leq \left(\frac{t+c_2|\Omega|}{c_1}\right)^{1/p}\right\}$ which is weakly precompact by Banach-Alaoglu.

$(p=\infty)$ any $u\in\{F\leq t\}$ for $t>-c \Rightarrow |u(x)|\leq R$ a.e. so $\{F\leq t\}\subseteq\{\|u\|_{L^\infty}\leq R\}$ and again Banach-Alaoglu gives the result.

$(p=1)$ for $u\in\{F\leq t\}$, integrating (GC_1) we get

$$\int_\Omega \varphi(|u(x)|)\,dx \leq \frac{t}{c}+1,$$

so by de la Vallee Poussin Theorem, $\{F\leq t\}$ is uniformly integrable $\Rightarrow$ by Dunford-Pettis Theorem it is weakly precompact in L^1. $\square$

<u>Rmk</u>: Dunford-Pettis Theorem states that a necessary and sufficient condition for weak precompactness in $L^1(\Omega)$ is uniform integrability.
Of course boundedness alone is not enough, take e.g. $u_j(x)=j\chi_{(0,1/j)}(x)$ in $L^1(-1,1)$.

Under the growth conditions above, (seq.) weak lsc (or weak* if $p=\infty$) coincides to the standard weak lsc.
Intuitively, we can restrict ourselfs to the sublevels (that are bounded) and in there the two notions coincide.

We show just a sketch of the proof for the case $p>1$.

<u>Proposition</u> 2: Let F be as in $(*)$ with f complying with either (GC_1), (GC_p) or (GC_∞).
Then $sc(F)$ (wrt weak topology, or weak* if $p=\infty$) coincides with $\bar{F}$.
In particular:

(i) F is weakly lsc (weakly* if $p=\infty$) $\Leftrightarrow$ f is lsc, convex;

(ii) $sc(F)(u) = \int_\Omega f^{**}(u(x))\,dx$, $\forall\, u \in L^p(\Omega;\mathbb{R}^m)$.

<u>proof</u> (sketch, $p>1$): $sc(F) \leq \bar{F}$ since weakly closed sets are (seq.) weakly closed.
Given u s.t. $sc(F)(u) < +\infty$ then $\exists\, t \in \mathbb{R}$ s.t. $sc(F)(u) < t$.
By $(GC_{p,\infty})$ and Proposition 1 the set $K := \overline{\{F \leq t\}}^{weak}$ is weakly compact and since $sc(F) \leq F$, $u \in K$. Then $\exists\, u_j \in K$, $u_j \rightharpoonup u$ s.t.

$$sc(F)(u) = sc_K(F_{|K})(u) = \overline{F_{|K}}(u) = \lim_{j\to+\infty} F_{|K}(u_j)$$

$$= \lim_{j\to+\infty} F(u_j) \geq \bar{F}(u)$$

where sc_K denotes the lsc envelope in K. $\square$

<u>Rmk</u>: part (ii) of Proposition 2 is a consequence of Theorem 5 (and the Remark on f attaining $+\infty$) of last lecture.

Γ- convergence in Lebesgue spaces

We now state the main Γ-convergence result for integral functionals defined in Lebesgue spaces.
To deal with a unified set of hypotheses, we consider only the case $1<p<\infty$ but analogous results work for the case $p=1,\infty$.

Theorem 3: let $1<p<\infty$ and $F_j : L^p(\Omega;\mathbb{R}^m) \longrightarrow (-\infty,+\infty]$ defined as

$$F_j(u) = \int_\Omega f_j(u(x))\,dx$$

with $f_j : \mathbb{R}^m \longrightarrow (-\infty,+\infty]$ complying with (GC_p) uniformly, i.e.

$$\exists\, c_1,c_2>0 \text{ s.t. } f_j(z) \geq c_1|z|^p - c_2,\ \forall z\in\mathbb{R}^m,\ \forall j$$

and s.t. $\sup_j f_j(0) < +\infty$.
Then $F_j \xrightarrow[j\to+\infty]{\Gamma} F$ wrt weak L^p-topology if and only if $f_j^{**} \xrightarrow[j\to+\infty]{\Gamma} f_\infty$ where

$$F(u) = \int_\Omega f_\infty(u(x))\,dx.$$

Rmk: (1.1) condition $\sup_j f_j(0)<+\infty$ can be replaced by: $\exists\, z_0\in\mathbb{R}^m$ s.t. $\sup_j f_j(z_0) < +\infty$.

(1.2) if f_j are locally equibounded, i.e. $\forall R>0\ \exists\, C_R>0$ s.t. $|f_j(z)| \leq C_R\ \ \forall |z|\leq R,\ \forall j$, then $f_j^{**} \xrightarrow{\Gamma} f_\infty$ if and only if $f_j^{**} \to f_\infty$ pointwise.

Before proving this result, we introduce the notion of the Yosida transform, that will be used in the proof.

<u>Def</u> (Yosida transform): $f: \mathbb{R}^m \to (-\infty, +\infty]$, $\lambda > 0$
the Yosida transform of f is

$$T_\lambda f(z) := \inf_{w \in \mathbb{R}^m} \{ f(w) + \lambda |z - w| \}.$$

<u>Rmk</u> (some properties): given f lsc and bounded below

(2.1) $T_{\lambda'} f(z) \leq T_\lambda f(z) \leq f(z)$, $\forall\, 0 < \lambda' < \lambda$

(2.2) $T_\lambda f$ is λ-Lipschitz

(2.3) $T_\lambda f \longrightarrow f$ pointwise as $\lambda \to +\infty$

(2.4) if f is convex $\Rightarrow$ $T_\lambda f$ is convex.

<u>proof</u>: ($\Leftarrow$) we first notice that, by Proposition 6 of Lecture 3,

$$\Gamma\text{-}\lim_{j \to +\infty} F_j(u) = \Gamma\text{-}\lim_{j \to +\infty} sc(F_j)(u)$$

and by Proposition 2 (ii) $sc(F_j)(u) = \int_\Omega f_j^{**}(u(x))\, dx$.
So we reduce to the functionals $sc(F_j)$.
We work in several steps.

Step 1 (using Yosida transform): let $k \in \mathbb{N}$. $\forall z \in \mathbb{R}^m$ the function $w \mapsto k|z-w|$ is continuous, so by stability under continuous perturbations we have

$$f_j^{**} + k|z - \cdot| \xrightarrow{\Gamma} f_\infty + k|z - \cdot|$$

Moreover, as a consequence of (GC_p), the functions $f_j^{**} + k|z-\cdot|$ are equicoercive.
Indeed, since $g(z) := c_1|z|^p - c_2$ is convex (and lsc) and $g(z) \le f_j(z) \Rightarrow f_j^{**}(z) \ge g(z)$.
So let $w \in \{f_j^{**} + k|z-\cdot| \le t\}$ then

$$t \ge f_j^{**}(w) + k|z-w| \ge c_1|w|^p - c_2 + k|z-w|$$

which implies $|w| \le \left(\frac{t+c_2}{c_1}\right)^{1/p} =: C_p$.
So by Fund. Thm of Γ-convergence

$$T_k f_j^{**}(z) \xrightarrow[j\to+\infty]{} T_k f_\infty(z), \quad \forall z \in \mathbb{R}^n$$

and $T_k f_\infty$ is convex (since pointwise limit of convex functions).
Also, by a standard result in convex analysis is that pointwise convergence of convex functions $\Rightarrow$ locally uniform convergence, so $\forall R>0$

$$(*_1) \quad \sup_{|z|\le R} \left|T_k f_j^{**}(z) - T_k f_\infty(z)\right| \xrightarrow[j\to+\infty]{} 0.$$

To simplify the notation we write $g_j^{(k)} := T_k f_j^{**}$ and $g^{(k)} := T_k f_\infty$.
Notice that, by (GC_p) it is easy to prove that $g_j^{(k)}(z), g^{(k)}(z) \ge \tilde{c}_1|z| - \tilde{c}_2$ for some $\tilde{c}_1, \tilde{c}_2 > 0$.

Let $\ell \in \mathbb{N}$, we do again Yosida transform on $g_j^{(k)}$ and $g^{(k)}$.
Working as before (since local uniform convergence implies Γ-convergence) $T_\ell g_j^{(k)}(z) \xrightarrow[j\to+\infty]{} T_\ell g^{(k)}(z)$.
Moreover, by convexity and coerciveness $\forall j, \forall z \in \mathbb{R}^n$
$\exists!\, w_j^{(z)} \in \mathbb{R}^n : |w_j^{(z)}| \le C_\ell$ s.t. $T_\ell g_j^{(k)}(z) = g_j^{(k)}(w_j^{(z)}) + k|z-w_j^{(z)}|$
and analogously $\exists!\, w^{(z)} \in \mathbb{R}^n : |w^{(z)}| \le C_\ell$ s.t. $T_\ell g^{(k)}(z) =$
$= g^{(k)}(w^{(z)}) + k|z-w^{(z)}|$.

By definition of Yosida transform

$$T_\ell g_j^{(k)}(z) - T_\ell g^{(k)}(z) \leq g_j^{(k)}(w^{(z)}) + k|z - w^{(z)}| - g^{(k)}(w^{(z)}) - k|z - w^{(z)}|$$
$$= g_j^{(k)}(w^{(z)}) - g^{(k)}(w^{(z)})$$

Analogously $T_\ell g^{(k)}(z) - T_\ell g_j^{(k)}(z) \leq g^{(k)}(w_j^{(z)}) - g_j^{(k)}(w_j^{(z)})$.
Then we get, $\forall\, z \in \mathbb{R}^m$

$$|T_\ell g_j^{(k)}(z) - T_\ell g^{(k)}(z)| \leq \sup_{|w| \leq C_\ell} |g_j^{(k)}(w) - g^{(k)}(w)|$$

which implies uniform convergence of $T_\ell g_j^{(k)}$ to $T_\ell g^{(k)}$ by $(*_1)$.

Step 2 (liminf inequality): let $u \in L^p(\Omega;\mathbb{R}^m)$ and consider any $u_j \xrightarrow[L^p]{} u$. Then $\forall\, k, \ell \in \mathbb{N}$, by Step 1 we get

$$\lim_{j\to+\infty} sc(F_j)(u_j) = \lim_{j\to+\infty} \int_\Omega f_j^{**}(u_j(x))\,dx$$
$$\overset{(2.1)}{\geq} \lim_{j\to+\infty} \int_\Omega T_k f_j^{**}(u_j(x))\,dx$$
$$\overset{(2.1)}{\geq} \lim_{j\to+\infty} \int_\Omega T_\ell g_j^{(k)}(u_j(x))\,dx$$

converges to 0 by uniform conv.

$$= \lim_{j\to+\infty} \left(\int_\Omega T_\ell g^{(k)}(u_j(x))\,dx + \int_\Omega T_\ell g_j^{(k)}(u_j(x)) - T_\ell g^{(k)}(u_j(x))\,dx\right)$$
$$= \lim_{j\to+\infty} \int_\Omega T_\ell g^{(k)}(u_j(x))\,dx \geq \int_\Omega T_\ell g^{(k)}(u(x))\,dx,$$

where the last inequality is a consequence of weak lsc since $T_\ell g^{(k)}$ is convex (and lsc).
By sending first $\ell \to +\infty$ and then $k \to +\infty$, the desired inequality follows by Monotone Convergence (as a consequence of (2.1)).

Step 3 (limsup inequality for constants): let $z \in \mathbb{R}^m$ s.t. $f_\infty(z) < +\infty$ (otherwise there is nothing to prove) and let $u \equiv z$.
Since $f_j^{**} \xrightarrow{\Gamma} f_\infty$, $\exists\, z_j \to z$ a recovery sequence s.t.

$$\lim_{j\to+\infty} f_j^{**}(z_j) = f_\infty(z).$$

Then, define $u_j \equiv z_j$, applying reverse Fatou's lemma

$$\overline{\lim_{j\to+\infty}} \int_\Omega f_j^{**}(u_j(x))\,dx \overset{\text{Fatou}}{\leq} \int_\Omega \lim_{j\to+\infty} f_j(z_j)\,dx$$

$$= \int_\Omega f_\infty(u(x))\,dx.$$

Notice that we can apply reverse Fatou since $f_j^{**}(z_j)$ converges (so its bounded).

Working as in Step 2 of the proof of Theorem 5 of last lecture, the limsup inequality extends to simple functions.

Step 4 (general case): as in Step 3 of the proof of Theorem 5 of last lecture, it is sufficient to prove that $\forall\, u \in L^p(\Omega;\mathbb{R}^m)$ we can find $\bar{u}_j$ simple functions s.t. $\bar{u}_j \xrightarrow[L^p]{} u$ and $\lim_{j\to+\infty} F(u_j) \leq F(u)$. Let $u \in L^p(\Omega;\mathbb{R}^m)$: $f_\infty(u) \in L^1(\Omega)$ (otherwise there is nothing to prove).
$\forall\, \varepsilon > 0$ we can find $\Omega_\varepsilon \subset\subset \Omega$ s.t. $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and

$$\int_{\Omega\setminus\Omega_\varepsilon} |f_\infty(u(x))|\,dx < \varepsilon, \quad |u(x)|, |f_\infty(u(x))| \leq 1/\varepsilon \quad \text{for } x \in \Omega_\varepsilon.$$

We can find simple functions $u_j^\varepsilon \rightrightarrows u$ in Ω_ε and such that $|f_\infty(u_j^\varepsilon(x))| \leq 1/\varepsilon \;\; \forall\, x \in \Omega_\varepsilon$.

An example of such a sequence is the following: denote $J_\varepsilon := \{ z \in \mathbb{R}^m : f_\infty(z) \leq 1/\varepsilon \}$ which is a compact convex set.

Let $Y := \frac{1}{j}\mathbb{Z}^m \cap J_\varepsilon = \{z_i\}_i$, $Q := \{z_i + (0, \frac{1}{j})^m\}_i$ and $\Omega_i \subseteq \Omega$ as $\Omega_i := \{x \in \Omega : u(x) \in z_i + (0, \frac{1}{j})^m\}$.
Then $u_j^\varepsilon(x) := \sum_i z_i \chi_{\Omega_i}(x)$ is a desired sequence, for ε small enough.

Defining $\bar{u}_j^\varepsilon(x) := \begin{cases} u_j^\varepsilon(x) & x \in \Omega_\varepsilon \\ 0 & \text{otherwise} \end{cases}$ we have

$$|F(\bar{u}_j^\varepsilon) - F(u)| = \left|\int_\Omega f_\infty(\bar{u}_j^\varepsilon(x))\,dx - \int_\Omega f_\infty(u(x))\,dx\right|$$

$$\leq \int_{\Omega_\varepsilon} |f_\infty(u_j^\varepsilon(x)) - f_\infty(u(x))|\,dx + \int_{\Omega \setminus \Omega_\varepsilon} |f_\infty(0)| + |f_\infty(u(x))|\,dx$$

$$\leq C_\varepsilon \int_{\Omega_\varepsilon} |u_j^\varepsilon(x) - u(x)|\,dx + (\sup_j |f_j(0)| + 1)\,\varepsilon$$

where $C_\varepsilon > 0$ is s.t. $|f_\infty(z) - f_\infty(z')| \leq C_\varepsilon |z - z'|$ $\forall$ $|z|, |z'| \leq 2/\varepsilon$ (which $\exists$ by local Lipschitzianity).
Notice that since $f_j^{**} \xrightarrow{\Gamma} f_\infty$ then

$$f_\infty(0) \leq \sup_j f_j^{**}(0) \leq \sup_j f_j(0).$$

Again since $\lim_{j \to +\infty} |F(\bar{u}_j^\varepsilon) - F(u)| \leq C_1 \varepsilon$ (for some $C_1 > 0$) we can extract a sequence as desired.

($\Rightarrow$) assume now that F_j Γ-converges to some functional $G : L^p(\Omega; \mathbb{R}^m) \to (-\infty, +\infty]$.
By compactness of Γ convergence $\exists$ $\{f_{j_k}^{**}\} \subseteq \{f_j^{**}\}$ s.t. $f_{j_k}^{**} \xrightarrow{\Gamma} g$ for some $g : \mathbb{R}^m \to (-\infty, +\infty]$ convex and lsc.
By ($\Leftarrow$) just proved

$$F_{j_k}(u) \xrightarrow{\Gamma} G(u) = \int_\Omega g(u(x))\,dx$$

$\forall$ $\{f_{j_k}^{**}\}$ again by compactness of Γ conv. $\exists$ $\{f_{j_k'}^{**}\}$ a subsequence s.t. $f_{j_k'}^{**} \xrightarrow{\Gamma} \tilde{g}$, again by ($\Leftarrow$)

$$F_{j_k'}(u) \xrightarrow{\Gamma} \int_\Omega \tilde{g}(u(x))\,dx.$$

Since F_j Γ-converges, $\tilde{g} = g \Rightarrow$ by Urysohn property of Γ-convergence $f_j^{**} \xrightarrow{\Gamma} g$ and the result is proved. $\square$

Rmk (the nonautonomous case): the same result holds also in the case in which f_j have a space dependence, i.e.

$$F_j(u) := \int_\Omega f_j(x, u(x))\, dx.$$

In this case $F_j \xrightarrow{\Gamma} F \iff f_j^{**}(x,\cdot) \xrightarrow{\Gamma} f_\infty(x,\cdot)$ for a.e. $x \in \Omega$ and $F(u) := \int_\Omega f_\infty(x, u(x))\, dx$.

The same generalization holds true for the relaxation result, i.e. (provided e.g. some growth conditions)

$$sc(F)(u) = \int_\Omega f^{**}(x, u(x))\, dx$$

where $f^{**}(x,\cdot)$ is the convex, lsc envelope of $f(x,\cdot)$.

Lecture 7: 27.05.2024

Main reference for this lecture:

- Sec. 4.1, 4.3, 5.3 of [BD, 1998]

We finally consider functionals in Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^m)$. Now $\Omega\subseteq\mathbb{R}^n$ denotes a bounded, open set with Lipschitz boundary (this will be needed to apply some well known compactness results).

Also, we restrict to the case $1<p<\infty$ since for the cases $p=1,\infty$ we may have a change of nature of the Γ-limit. For simplicity, we also take into account positive energies, but it is easy to reduce to functionals bounded below (not necessarily positive).

So we will consider $F_j: W^{1,p}(\Omega;\mathbb{R}^m)\longrightarrow[0,+\infty]$ defined as

$$F_j(u):=\int_\Omega f_j(x,\nabla u(x))\,dx$$

with $f_j:\Omega\times\mathbb{R}^{m\times n}\longrightarrow[0,+\infty]$ a Borel function complying with

$$c_1|M|^p\leq f_j(x,M)\leq c_2(|M|^p+1),\qquad \text{for a.e. } x\in\Omega,\ \forall\, M\in\mathbb{R}^{m\times n}$$

where $c_1,c_2>0$.

Ex (running example): we want to understand the limit of

$$F_j(u)=\tfrac{1}{p}\int_\Omega a_j(x)|\nabla u(x)|^p-g(x)u(x)\,dx \underset{\min}{\longmapsto} (EL_j)\ -\mathrm{div}(a_j(x)|\nabla u(x)|^{p-2}\nabla u(x))=g(x)$$

under some boundary conditions.

Question: ▶ are solutions to (EL_j) converging to a solution of some variational problem?

or equivalently

▶ is F_j Γ-converging to an integral functional of the same type?

(seq.) Weak lower-semicontinuity

As in Lebesgue spaces, the most natural topology to consider is the weak topology (in this case weak-Sobolev).
We recall some notions and results known is Sobolev spaces.

<u>Def</u>: given $u_j, u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we say that u_j weak converges to u, $u_j \underset{W^{1,p}}{\rightharpoonup} u$ as $j \to +\infty$ iff $u_j \underset{L^p}{\rightharpoonup} u$ and $\nabla u_j \underset{L^p}{\rightharpoonup} \nabla u$.

<u>Rmk</u>: by Rellich compactness theorem, $W^{1,p}(\Omega;\mathbb{R}^m) \subset\subset L^p(\Omega;\mathbb{R}^m)$,
$u_j \underset{W^{1,p}}{\rightharpoonup} u \iff u_j \underset{L^p}{\to} u$ and $\nabla u_j \underset{L^p}{\rightharpoonup} \nabla u$.

We already saw that oscillating periodic functions weak converge to their average when sending the period to 0 (Riemann-Lebesgue's Lemma).
There is an equivalent result in Sobolev spaces: piecewis affine functions with oscillating periodi gradient weak(-Sobolev) converge to the affine function whose gradient is the average their gradient.

We will be using this result quite a lot, so we spend few lines to prove it and understand it.

<u>Rmk</u> (oscillating sequence): let $Q \subseteq \mathbb{R}^n$ a cube, let $v \in W^{1,p}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ s.t. ∇v is Q-periodic. Let $v_j(x) := \frac{1}{j} v(jx)$ then

$$v_j \xrightarrow[W^{1,p}_{loc}]{} \left(\fint_Q \nabla v \right) x, \qquad \text{as } j \to +\infty.$$

Indeed, for simplicity we consider $Q=[0,1]^n$ but the general case follows identically. $\forall\, i=1,\dots,n$ let $g_i(x) := v(x+e_i) - v(x)$, by periodicity of ∇v, $\nabla g_i = 0 \Rightarrow v(x+e_i)-v(x)$ is constant. Let $A_0 \in \mathbb{R}^{m\times n}$ be the matrix defined as $A_0 e_i = v(x+e_i) - v(x)$ and define $w(x) := v(x) - A_0 x$.

Notice that $A_0 = \int_{[0,1]^n} \nabla v(x)\,dx$, indeed since $g_i = \text{const}$

$$v(x+e_i)-v(x) = \int_{[0,1]^n} v(x+e_i) - v(x)\,dx$$

$$= \int_{[0,1]^n}\int_0^1 \nabla v(x+te_i)\cdot e_i\,dt\,dx$$

$$\overset{\text{Fubini}}{=} \int_0^1 \int_{[0,1]^n} \nabla v(x+te_i)\,dx \cdot e_i\,dt$$

$$\overset{\nabla v \text{ periodic}}{=} \int_{[0,1]^n} \nabla v(y)\,dy \cdot e_i.$$

w is $[0,1]^n$-periodic since $w(x+e_i) - w(x) = v(x+e_i) - v(x) - A_0(x+e_i) - A_0 e_i = v(x+e_i) - v(x) - A_0 e_i = 0$, $\forall\, i=1,\dots,n$.
We also have

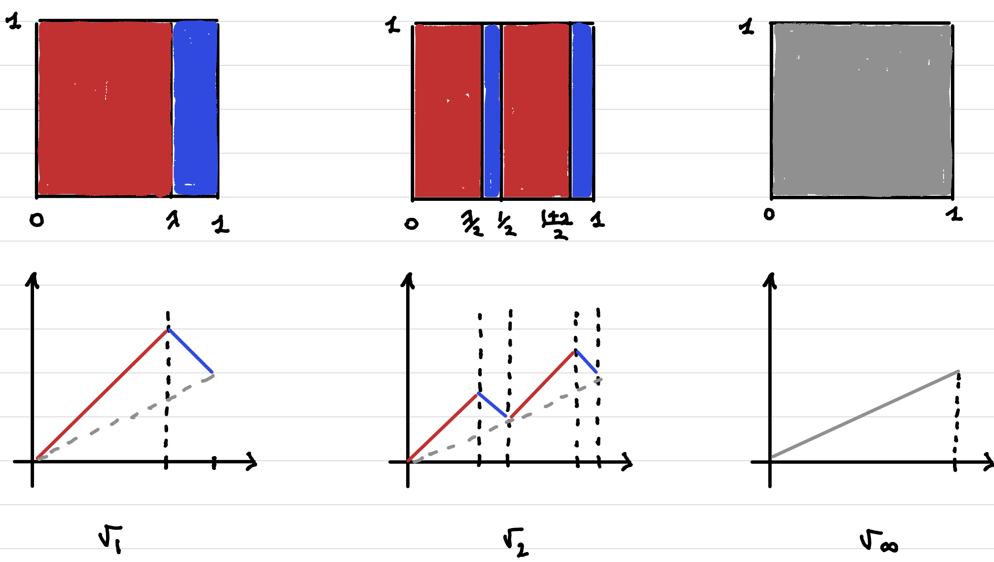
$$v_j(x) = \tfrac{1}{j} v(jx) = \tfrac{1}{j} w(jx) + A_0 x, \qquad \nabla v_j(x) = \nabla w(jx) + A_0.$$

By Riemann-Lebesgue, $\nabla v_j \xrightarrow{L^\infty_{loc}} A_0$ and $w(j\cdot) \xrightarrow{L^\infty_{loc}} \int_{[0,1]^n} w$ so $\tfrac{1}{j} w(j\cdot) \xrightarrow{L^\infty_{loc}} 0$ and we obtain the result.

It is good to visualize it, as an example take, for $\lambda \in (0,1)$

$$v(x_1,x_2) = \begin{cases} (x_1 + (2\lambda-2)j,\ 0) & j \le x_1 < j+\lambda,\ j\in\mathbb{Z} \\ (-x_1 + 2\lambda(j+1),\ 0) & j+\lambda \le x_1 < j+1,\ j \in \mathbb{Z}. \end{cases}$$

It is easy to check that $v \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$ and that ∇v is $[0,1]^2$-periodic with $\int_{[0,1]^2} \nabla v(x)\,dx = \begin{pmatrix} 2\lambda-1 & 0 \\ 0 & 0 \end{pmatrix}$.
Then $v_j \xrightharpoonup[W^{1,\infty}_{loc}]{*} ((2\lambda-1)x_1, 0) =: v_\infty$.



Before studying the Γ-convergence, it is useful to check conditions on densities for (seq.) weak lsc of the integral energies. For this we restric to homogeneous case (i.e. $f(x,A)=f(A)$).

This result actually holds for every $p\in[1,\infty]$.

<u>Theorem</u> 1 (necessity): let $f:\mathbb{R}^{m\times n}\to[0,+\infty]$ be Borel and let $F(u):=\int_\Omega f(\nabla u(x))\,dx$.
If F is (seq.) weak-lsc (weak* if $p=\infty$), then

(i) f is lsc,

(ii) $\forall Q\subseteq\mathbb{R}^n$ cube $f\left(\fint_Q \nabla v\right)\leq\fint_Q f(\nabla v)$ $\forall v\in W^{1,p}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ with ∇v is Q-periodic.

<u>Proof</u>: (i) it follows exactly as in the Lebesgue case, that is as in the first part of the proof of Proposition 3 of Lecture 4.

(ii) let v be as in the statement, let $u_j(x):=\frac{1}{j}v(jx)$ and $u(x):=\left(\fint_Q\nabla v\right)x$.
By the previous Remark, $u_j\longrightarrow u$ in $W^{1,p}(Q;\mathbb{R}^m)$ so by lsc

$$|\Omega|\,f\left(\fint_Q\nabla v\right)=\int_\Omega f(\nabla u(x))\,dx$$

$$=F(u)\leq\varliminf_{j\to+\infty}F(u_j)=\varliminf_{j\to+\infty}\int_\Omega f(\nabla u_j).$$

If $f(\nabla v)\in L^1(\Omega)\Rightarrow f(\nabla u_j(x))=f(\nabla v(jx))\xrightarrow[L^1_{loc}]{}\fint_Q f(\nabla v)$ by Riemann-Lebesgue's Lemma, which yields (ii).
If $f(\nabla v)\notin L^1(\Omega)$ there is nothing to prove since the right-hand side in (ii) is $+\infty$. $\square$

Rmk (difference between Lebesgue and Sobolev case): we saw that the proof of necessary conditions for weak lsc of functionals $\int_\Omega f(u(x))\,dx$ and $\int_\Omega f(\nabla u(x))\,dx$ is similar but brings to different conclusion:

- in the Lebesgue case f is convex;
- in the Sobolev case f complies with (ii).

Let us first comment on the fact that a function is convex if and only if satisfies Jensen's inequality. Notice also that, condition (ii) is a Jensen's inequality for gradients, so we can interpret condition (ii) as a convexity condition for f "on gradients".

We can say that weak lsc $\Rightarrow$ convexity of "along oscillating sequence".
While in Lebesgue space every oscillation is admissible, this is not true in Sobolev space since not all piecewise constant function are a gradient in dimensions higher than one.
So, since for functions acting on ∇u, f need to be convex "only on gradients", weak lsc does not imply convexity but the weaker condition (ii).
We give a name to condition (ii) in the sequel.

Exercise (Hadamard's jump condition): as we commented on, not every $v: \mathbb{R}^n \to \mathbb{R}^{m\times n}$ piecewise constant is a gradient, i.e. $\exists\, u: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $v(x) = \nabla u(x)$.

A characterization of which piecewise constant functions are gradient is given by Hadamard's jump conditions:

Let $A, B \in \mathbb{R}^{m\times n}$, $\nu \in S^{n-1}$ and let $v: B_1 \to \{A, B\}$ defined as

$$v(x) = \begin{cases} Ax & x\cdot\nu \leq 0 \\ Bx & x\cdot\nu > 0 \end{cases}.$$

Then $\exists\, u \in W^{1,\infty}(B_1;\mathbb{R}^m)$ s.t. $\nabla u = v \iff \exists\, a \in \mathbb{R}^m$ s.t. $A-B = a\otimes\nu$. In particular $\operatorname{rank}(A-B)=1$.

This says that a piecewise constant function is a gradient if the two matrices are rank-1 connected, that is their difference is rank-1.

Try to prove this result.

Notions of convexity

As we saw in Theorem 1, and as discussed in the Remark below, we need different notions of convexity to deal with functionals acting on the gradient.

Def (quasiconvexity): $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is quasiconvex iff is continuous and $\forall\, E\subseteq\mathbb{R}^n$ open, bounded, $\forall A\in\mathbb{R}^{m\times n}$ it holds that

$$|E|\,f(A)\leq\int_E f(A+\nabla\varphi(x))\,dx,\qquad\forall\,\varphi\in C_c^\infty(E;\mathbb{R}^m).$$

Notice that condition (ii) of Theorem 1 is very close to quasi-convexity, but not quite it since φ complies with boundary conditions (and is regular) while v has periodic, L^p gradient. We will see in which cases the two coincide.

Def (rank-1 convexity): $f:\mathbb{R}^{m\times n}\to(-\infty,+\infty]$ is rank-1 convex iff $\forall\, A,B\in\mathbb{R}^{m\times n}$ s.t. $\operatorname{rank}(A-B)=1$ and $t\in(0,1)$ it holds

$$f(tA+(1-t)B)\leq t\,f(A)+(1-t)\,f(B).$$

Def (polyconvexity): $f : \mathbb{R}^{m\times n} \to (-\infty, +\infty]$ is polyconvex iff there exists a convex function $g : \mathbb{R}^{\tau(n,m)} \to (-\infty, +\infty]$ s.t.

$$f(A) = g(M(A)), \quad \forall A \in \mathbb{R}^{m\times n}$$

where $M(A) \in \mathbb{R}^{\tau(n,m)}$ is the vector of all minors of A of order $1, \ldots, \min\{n,m\} = n \wedge m$ and $\tau(n,m) := \sum_{k=1}^{n\wedge m} \binom{n}{k}\binom{m}{k}$.

Notice that $\tau(n,m)$ is the number of minors of a matrix of order $m\times n$.

Ex: if $m=n=2$, then $\tau(n,m)=5$ and $M(A) = (A_{11}, A_{12}, A_{21}, A_{22}, \det(A))$. So f is polyconvex iff $\exists\, g : \mathbb{R}^5 \to [-\infty, +\infty]$ s.t. $\forall A \in \mathbb{R}^{2\times 2}$ $f(A) = g(A, \det(A))$, where on the right-hand side we see A as a vector of $\mathbb{R}^4$.

Rmk: there are relations between these notions of convexity. We list here the most important:

(1.1) if $n,m \geq 2$ and f is finite (i.e. not taking value $+\infty$) then convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-1 convexity;

(1.2) these "new" notions of convexity are different from the standard one only in matrix spaces, i.e. if $n=1$ or $m=1$ convexity $\Leftrightarrow$ polyconvexity $\Leftrightarrow$ quasiconvexity $\Leftrightarrow$ rank-1 convexity.

We said we study functionals whose densities comply with some growth conditions.
In presence of growth condition from above, quasiconvexity can be slightly improved.

Rmk (quasiconvexity and growth conditions): if f is quasiconvex and $f(A) \leq C(|A|^p+1)$ for some $C>0$, $p\geq 1$ then $\forall E \subseteq \mathbb{R}^n$ bounded, open set with $|\partial E|=0$, $\forall A \in \mathbb{R}^{m\times n}$ it holds that

$$|E|f(A) \leq \int_E f(A+\nabla v(x))\,dx, \qquad \forall v \in W_0^{1,p}(E;\mathbb{R}^m).$$

Indeed, by definition of $W_0^{1,p}(E,\mathbb{R}^m)$ $\exists$ $C_c^\infty(E;\mathbb{R}^m) \ni \varphi_j \xrightarrow{W^{1,p}} v$. By quasiconvexity, $\forall j$

$$|E|f(A) \leq \int_E f(A+\nabla\varphi_j(x))\,dx.$$

Since we can assume $\varphi_j \to v$ a.e. in E, taking the limit as j goes to $+\infty$, by reverse Fatou and the continuity of f we obtain

$$|E|f(A) \leq \overline{\lim_{j\to+\infty}} \int_E f(A+\nabla\varphi_j(x))\,dx \leq \int_E f(A+\nabla v(x))\,dx.$$

We will not deal with polyconvexity in this course but it is an important notion, especially in elasticity.
We therefore mention the main result about polyconvexity, that is that polyconvexity is a sufficient condition for (seq.) weak lsc in absence of growth conditions from above.

Rmk: let $f: \mathbb{R}^{m\times n} \to [0,+\infty]$ be polyconvex, lsc and complying with $f(A) \geq c(|A|^p-1)$ $\forall A \in \mathbb{R}^{m\times n}$, for some $c>0$ and $p \geq n\wedge m$. Then the functional $F(u)=\int_\Omega f(\nabla u(x))\,dx$ is (seq.) weak lsc on $W^{1,p}(\Omega;\mathbb{R}^m)$.

The lack of a growth condition from above is useful to incorporate a constraint. The main examples are energies in continuum elasticity:
$\Omega \subseteq \mathbb{R}^3$ represents the undeform configuration of an elastic solid and $u: \Omega \to \mathbb{R}^3$ its deformation.

The elastic energy of the deformed body is

$$E_{el}(u) := \int_\Omega W(\nabla u(x))\,dx$$

where $W : \mathbb{R}^{3\times3} \to [0,+\infty]$ with $c_1 \operatorname{dist}^2(A, SO(3)) \leq W(A) \leq c_2(|A|^2+1)$ $\forall A \in \mathbb{R}^{3\times3}$ with $\det(A) > 0$ and $W(A) = +\infty$ $\forall A \in \mathbb{R}^{3\times3}$ with $\det(A) \leq 0$.
This last condition is called "noninterpenetration condi--tion" and it is crucial when dealing with a realistic model.

In this cases it has not been proved (nor disproved, hence it is not known) whether quasiconvexity implies esc but the result stated above says that esc implies poly--convexity.