Lecture 8: 05.05.2026

Last lecture we proved existence of minimizers for the (constraint) isoperimetric problem, and that, we can reduce to consider only smooth sets.

We now claim that balls are isoperimetric sets (among regular uniformly bounded sets of volume 1).

Lemma (optimality of the ball): for every $R > \omega_n^{-1/n}$, $\gamma_R^{reg}(1) = n\omega_n^{1/n}$ and the equality holds $\iff |E \Delta B_r(x)| = 0$ for some $x \in B_R$ and $r > 0$.

note: the proof of this lemma is non-trivial. We will see three different proofs during the course:
(i) with PDE methods (on Lecture 9);
(ii) via optimal transport (at the beginning of block 3);
(iii) via symmetrization (at the beginning of block 4).

With this lemma (whose proof is postponed to the next lectures) we can prove the validity of the isoperimetric inequality.

Theorem (isoperimetric inequality): if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $|E| < +\infty$, then

$$P(E) \geq n\omega_n^{1/n} |E|^{\frac{n-1}{n}}.$$

Moreover if E is bounded, equality holds true $\iff |E \Delta B_r(x)| = 0$ for some $x \in \mathbb{R}^n$, $r > 0$.

note: to prove the uniqueness of the ball also among unbounded sets we will need Steiner symmetrization, that we will see further in the course (in block 4).

proof: if E is bounded, let $F = E/|E|^{\frac{1}{n}}$, then for some R, $F \subseteq B_R$.
Then $|F|=1$ and, by the optimality of the ball and Lemma 2 of last lecture,

$$P(E) = |E|^{\frac{n-1}{n}} P(F) \geq |E|^{\frac{n-1}{n}} \mathcal{J}_R(1) = |E|^{\frac{n-1}{n}} \mathcal{J}_R^{reg}(1) = n(\omega_n)^{1/n} |E|^{\frac{n-1}{n}}.$$

Moreover, the equality holds $\Leftrightarrow \frac{E}{|E|^{1/n}} \simeq B_{r_n}(x) \Leftrightarrow E \simeq B_{r'}(x')$ with $r' = |E|^{1/n} r_n$, $x' = |E|^{1/n} x$.

If E is unbounded and $P(E) = +\infty$ the statement is obvious
If $P(E) < +\infty$, by approximation $\exists\, E_h$ bounded and smooth s.t. $P(E_h) \to P(E)$, $E_h \to E$, so that

$$P(E) = \lim_{h\to\infty} P(E_h) \geq \lim_{h\to\infty} n(\omega_n)^{1/n} |E_h|^{\frac{n-1}{n}} = n(\omega_n)^{1/n} |E|. \qquad \boxtimes$$

note (two-dimensional case): in Lecture 1 we saw Hurwitz's theorem, which gives the optimality of the ball for $n=2$, among connected sets.

Since one can easily prove, using the sublinearity of $V^{\frac{1}{2}}$, that minimizers are connected (left as an excercise), Hurwit's theorem proves the optimality of the ball (i.e. the Lemma above) for $n=2$.

This prove completely the isoperimetric inequality (as stated in the theorem above) for $n=2$.

1. The reduced boundary

We conclude this first block by showing a fundamental result from De Giorgi which characterizes completely the structure of (some notion of) boundary of sets of finite perimeter, and of their Gauss-Green measure.

We show that many properties of the regular case are present also in this context, even if in a relaxed version.

We have seen that, if E has C^1 boundary, as a consequence of the divergence theorem, that

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E.$$

For general sets of finite perimeter we can give notions of outer normal vector and boundary so that $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$. This will be the goal of this lecture.

<u>Def</u>: Let $E \subseteq \mathbb{R}^n$ be of locally finite perimeter. We define the reduced boundary of E as the set

$$\partial^* E := \left\{ x \in \operatorname{spt}(\mu_E) : \exists \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} \in \mathbb{S}^{n-1} \right\}.$$

The function $\nu_E : \partial^* E \longrightarrow \mathbb{S}^{n-1}$ defined as

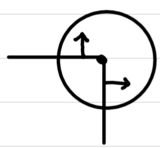
$$\partial^* E \ni x \longmapsto \nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} \in \mathbb{S}^{1}$$

is called the measure-theoretic outer unit normal vector to E at x.

<u>note</u>: by Lebesgue-Besicovitch differentiation theorem and Riesz theorem $D_{|\mu_E|} \mu_E = \nu_E = g$ a.e. on $\operatorname{spt}(\mu_E)$, so ν_E is Borel integral and $\mu_E = \nu_E |\mu_E| \llcorner \partial^* E$.

note: clearly $\partial^* E \subseteq \overline{\partial^* E} \subseteq \mathrm{spt}(\mu_E)$, since the support is a closed set.
As, by the previous observation, μ_E concentrates on $\partial^* E$ and, by definition $\mathrm{spt}(\mu_E)$ is the intersection of closed sets on which μ_E is concentrating, then $\mathrm{spt}(\mu_E) \subseteq \overline{\partial^* E}$.
So $\overline{\partial^* E} = \mathrm{spt}(\mu_E)$, the reduced boundary is dense in the support of the Gauss-Green measure (with coincide with the topological boundary for a good Lebesgue representant).

Ex: let $E = (0,1)^2$, and let $V := \{(0,0), (1,0), (0,1), (1,1)\}$ the set of vertices.
For $x \in \partial E \setminus V$, $\nu_E(x)$ is well defined, so it is immediate to prove that $\partial^* E \supseteq \partial E \setminus V$.
Take now any point in V, e.g. $(1,1)$ and compute



$$\frac{\mu_E(B_r(1,1))}{|\mu_E|(B_r(1,1))} = \frac{e_1 \mathcal{H}^1(\{1\}\times(1-r,1)) + e_2 \mathcal{H}^1((1-r,1)\times\{1\})}{\mathcal{H}^1((\{1\}\times(1-r,1))\cup((1-r,1)\times\{1\}))}$$
$$= \frac{e_1+e_2}{2} \notin S!$$

So V are not point of the reduced boundary, and actually $\partial^* E = \partial E \setminus V$.

Exc: find the reduced boundary of $E^{(\pm)} := \{x \in \mathbb{R}^2 : \pm x_2 < \sqrt{|x_1|}\}$.

1.2. Tangential properties of blow-ups

The reduced boundary is the set of points in which an outer normal vector to E is well-defined. In these points, by this fact we can prove that there exists an approximate tangent space in the sense of rectifiable sets.

We first recall some notation. Given $x \in \mathbb{R}^n$, $r > 0$, $\Phi_{x,r}(y) := \frac{y-x}{r}$.
For any μ Radon measure, its ($(n-1)$-dimensional) blow-up is

$$\mu_{x,r} := \frac{1}{r^{n-1}} (\Phi_{x,r})_\# \mu.$$

We also define blow-ups for sets. Given $E \subseteq \mathbb{R}^n$ we write

$$E_{x,r} := \Phi_{x,r}(E).$$

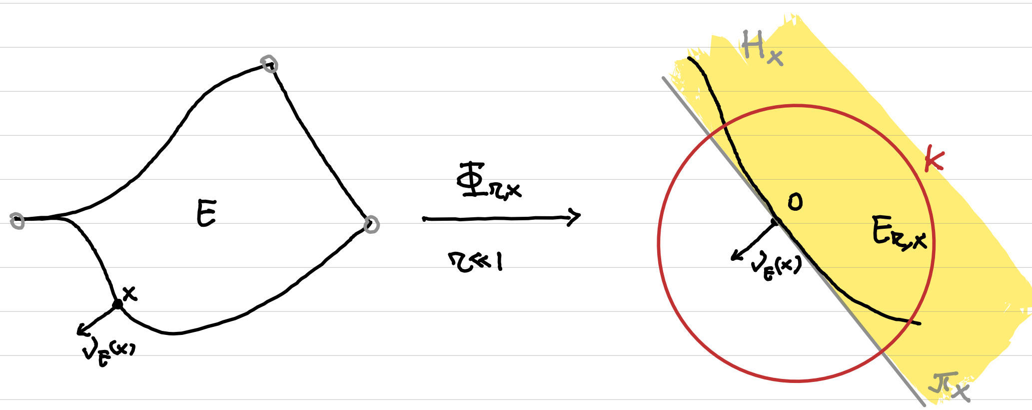
We can now state the following fundamental result.

<u>Theorem 1</u>: if E is of locally finite perimeter, $x \in \partial^* E$ then

$$E_{x,r} \xrightarrow{loc} H_x := \{ y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0 \}.$$

Moreover, let $\pi_x = \partial H_x = \nu_E(x)^\perp$, then

$$\mu_{E_{x,r}} \xrightarrow{*} \nu_E(x) \mathcal{H}^{n-1} \llcorner \pi_x, \quad |\mu_{E_{x,r}}| \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \pi_x.$$



<u>note</u>: in particular $P(E_{x,r}; B_R) \longrightarrow P(H_x; B_R) = n\omega_n R^{n-1} \ \forall R > 0$.

We can interpret Theorem 1 as follows, when "zooming-in" around points of the reduced boundary, our set of finite perimeter "resembles" an halfspace oriented orthogonal to the outer normal vector.
This process is somehow regular enough, so that the perimeter of the set around that point converge to that of the halfspace.

We have seen, at the beginning of Lecture 5 (see also the addendum to Lecture 5), the rectifiability criterion, that is

- if a Radon measure μ, concentrated on M, is s.t. $\mu_{x,r} \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \pi_x$ $\forall x \in M$, $\pi_x \subseteq \mathbb{R}^n$ an hyperplane $\Rightarrow$ M is (locally) $\mathcal{H}^{n-1}$-rectifiable and $\mu = \mathcal{H}^{n-1} \llcorner M$.

We will use this to recover, from Theorem 1, the following structure result.

<u>Theorem</u> (De Giorgi's structure theorem I): E of locally finite perimeter, then $\partial^* E$ is (locally) $\mathcal{H}^{n-1}$-rectifiable and $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$. Moreover,

$$T_x(\partial^* E) = \nu_E(x)^\perp, \quad \forall x \in \partial^* E.$$

<u>proof</u>: we work into two steps.

step 1: we first prove that the Gauss-Green measures of the blow-up (sets) are the blow-ups of the Gauss-Green measures. Namely, $\mu_{E_{x,r}} = (\mu_E)_{x,r} = \frac{1}{r^{n-1}} (\Phi_{x,r})_\# \mu_E$.

We preliminarily note that $\nabla \Phi_{x,r}(y) \equiv \frac{1}{r} I_n$, and $J\Phi_{x,r}(y) = \frac{1}{r^n}$. Also, $\forall \varphi \in C^1_c(\mathbb{R}^n)$, by the chain rule $\nabla \varphi \circ \Phi_{x,r}(y) = \frac{1}{r} \nabla \varphi(\Phi_{x,r}(y))$. So, by the area formula

$$\int_{\mathbb{R}^n} \varphi \, d\mu_{E_{x,r}} = \int_{E_{x,r}} \nabla \varphi(y) \, dy = \int_{\Phi_{x,r}(E)} \nabla \varphi(y) \, dy$$

$$\overset{AF}{=} \frac{1}{r^n} \int_E \nabla \varphi(\Phi_{x,r}(y)) \, dy$$

$$\overset{CR}{=} \frac{1}{r^{n-1}} \int_E \nabla \varphi \circ \Phi_{x,r}(y) \, dy$$

$$= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \varphi \circ \Phi_{x,r} \, d\mu_E = \int_{\mathbb{R}^n} \varphi \, d(\mu_E)_{x,r}.$$

By taking the sup over φ we also get $|\mu_{E_{x,r}}| = |\mu_E|_{x,r}$.

step2: by Theorem 1, $|\mu_E|_{x,r} = |\mu_{E_{x,r}}| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_x \quad \forall x \in \partial^* E$.
Then by the rectifiability criterion, $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$.
By the observation before, $\mu_E = \nu_E |\mu_E| = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$. ☒

We conclude by just mentioning another (more classical) reformulation of this result, whose proof we omit and uses Whitney's extension theorem, see [Chapter 15, Maggi (2012)] for details.

<u>Theorem</u> (De Giorgi's structure theorem II): E of locally finite perimeter, then

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Moreover, $\exists$ countably many C^1-hypersurfaces $M_h \subseteq \mathbb{R}^n$, compact sets $K_h \subseteq M_h$, and a Borel set F, $\mathcal{H}^{n-1}(F) = 0$ s.t.

$$\partial^* E = F \cup \bigcup_h K_h$$

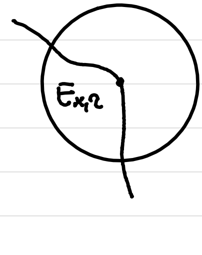
and, for every $x \in K_h$, $\nu_E(x)^\perp = T_x M_h$.

1.3. <u>A crucial estimate</u>

We see one main ingredient to the proof of Theorem 1. See the detailed proof in [Chapter 15, Maggi (2012)]

<u>note</u>: we will need the following estimate, for r sufficiently small

$$P(E_{x,r}; B_R) \leq C_1 R^{n-1}$$



this is proved by $x \in \partial^* E \subseteq \operatorname{spt}(\mu_E)$.
It is a sort of measure theoretic way to say that $\partial^* E$ is not oscillating too much.

By the previous estimate and the coercivity of the perimeter, up to subsequences, $\exists$ F of locally finite perimeter s.t.

$$E_{x,R} \xrightarrow{loc} F, \qquad \mu_{E_{x,R}} \xrightarrow{*} \mu_F$$

By compactness of Radon measures, we also have $|\mu_{E_{x,R}}| \xrightarrow{*} \lambda$. We need to prove that $F = H_x$ and $\lambda = |\mu_F|$.
Notice also that, for a.e. $R>0$ (namely those s.t. $\lambda(\partial B_R)=0$, which one can prove to be a.e.)

$$\lim_{R\to 0} \mu_{E_{x,R}}(B_R) = \mu_F(B_R), \quad \lim_{R\to 0} |\mu_{E_{x,R}}|(B_R) = \lambda(B_R).$$

As $x \in \partial^* E$, and $\mu_{E_{x,R}} = (\mu_E)_{x,R}$, we have

$$\lim_{R\to 0^+} \frac{\mu_{E_{x,R}}(B_R)}{|\mu_{E_{x,R}}|(B_R)} = \lim_{R\to 0^+} \frac{\mu_E(B_{RR}(x))}{|\mu_E|(B_{RR}(x))} = \nu_E(x) \in \mathbb{S}^{n-1}.$$

Multiplying scalarly the limit above by $\nu_E(x)$, and taking the reciprocals we get

$$\lim_{R\to 0^+} \frac{\nu_E(x)\cdot \mu_{E_{x,R}}(B_R)}{|\mu_{E_{x,R}}|(B_R)} = 1 \iff \lim_{R\to 0^+} \frac{|\mu_{E_{x,R}}|(B_R)}{\nu_E(x)\cdot \mu_{E_{x,R}}(B_R)} = 1.$$

By the lower-semicontinuity of the perimeter we obtain

$$P(F;B_R) \leq \lim_{R\to 0^+} P(E_{x,R};B_R) = \lambda(B_R) = \lim_{R\to 0^+} \nu_E(x)\cdot \mu_F(B_R)$$

$$= \nu_E(x)\cdot \mu_F(B_R) \leq |\mu_F|(B_R) = P(F;B_R),$$

where we used Cauchy-Schwartz in the last inequality. This chain of inequalities proves that $\lambda(B_R) = |\mu_F|(B_R)$ for a.e. R so (e.g. by Lebesgue-Besicovitch differentiation) $\lambda = |\mu_F|$, and that, $|\mu_F|(B_R) = \nu_E(x)\cdot \mu_F(B_R)$. Since $\mu_F = \nu_F |\mu_F|$ we have

$$\int_{B_R} d|\mu_F|(y) = \int_{B_R} \nu_E(x)\cdot \nu_F(y)\, d|\mu_F|(y) \iff \int_{B_R} (1-\nu_E(x)\cdot \nu_F(y))\, d|\mu_F|(y) = 0.$$

As, by Cauchy-Schwartz, the integrand above is non-negative it has to be zero $|\mu_F|$- a.e..
We can conclude (see [Proposition 15.15, Maggi (2012)]) by the fact that if $\nu_F \equiv \nu_E(x) \Rightarrow F$ is an halfspace orthogonal to $\nu_E(x)$.

<u>note</u>: by the following two estimates

$$\frac{|E\cap B_r(x)|}{r^n} \geq \alpha > 0, \qquad \frac{|(\mathbb{R}^n\setminus E)\cap B_r(x)|}{r^n} \geq \alpha > 0,$$

we can conclude that $F = H_x$. Indeed, loosely speaking, if F were an halfspace not centered in the origin, $E_{x,r}$ would have either 0 or 1 density for r sufficiently contradicting the previous estimates.

2. Federer's theorem

We conclude by just mentioning another notion of boundary that might be useful to perform some computation (this was not discussed during lecture).

<u>note</u> (recall of Lecture 3): given $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, we denote $\forall t \in [0,1]$

$$E^{(t)} := \left\{ x \in \mathbb{R}^n : \exists\ \theta_n(E)(x) := \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{r^n \omega_n} = t \right\}$$

the set of points of density t.
We also saw that $|E \triangle E^{(1)}| = 0$ and $|(\mathbb{R}^n \setminus E) \triangle E^{(0)}| = 0$.

<u>Exc.</u>: let $E := \{x \in \mathbb{R}^2 : x_2 < \alpha |x_1|\}$, for any $\alpha \in \mathbb{R}$ compute the density of E at the origin.

<u>Exc.</u>: prove that

$$x \in E^{(1)} \iff E_{x,r} \xrightarrow[r \to 0^+]{loc} \mathbb{R}^n, \quad x \in E^{(0)} \iff E_{x,r} \xrightarrow[r \to 0^+]{loc} \emptyset.$$

The set of points of density 1 is the measure-theoretic interior whereas those of density 0 are the measure-theoretic exterior. By this, Federer introduced the notion of essential boundary as

$$\partial^e E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

We have the three new notions of boundary $\partial^* E \subseteq \partial^e E \subseteq \mathrm{spt}(\mu_E)$ which (luckily) coincide up to $\mathcal{H}^{n-1}$-negligible sets.

<u>Theorem</u>: if E is of locally finite perimeter, then $\partial^* E \subseteq E^{(1/2)} \subseteq \partial^e E$ and

$$\mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0.$$