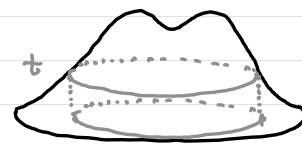
Lecture 7: 04.05.2025

1. The coarea formula and approximation

A fundamental technical tool in GMT, that we will use in our course too, is the coarea formula.

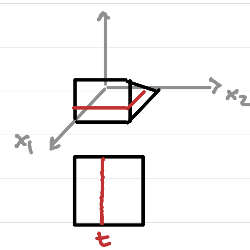
Theorem (coarea formula): if $u: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $A \subseteq \mathbb{R}^n$ is open, then $\mathbb{R} \ni t \mapsto P(\{u>t\}; A)$ is a Borel function and

$$\int_A |\nabla u(x)|\, dx = \int_{\mathbb{R}} P(\{u>t\}; A)\, dt.$$



Note: the same holds true for $u \in W^{1,1}_{loc}$!

Note: one possible interpretation of the coarea formula is that it is a generalization of Fubini's theorem, or integration via slicing, e.g. $A = (0,1)^2$ and $u(x) = x_1$, we have



$$\int_A dx = \int_0^1 P(\{x_1>t\}; A)\, dt = \int_0^1 \mathcal{H}^1(\{t\} \times (0,1))\, dt.$$

To prove the coarea formula we will be using the layer-cake formula. From the proof, we suggest another interpretation of the coarea formula, that is a sort of "differentiation" of the Lebesgue integration.

Note (layer-cake formula): if $u \in L^1(\mathbb{R}^n)$, $u \geq 0$, $v \in L^\infty(\mathbb{R}^n)$, as a consequence of Fubini's theorem, it holds that

$$\int_{\mathbb{R}^n} u(x) v(x)\, dx = \int_0^\infty \int_{\{u>t\}} v(x)\, dx\, dt.$$

Proof (coarea formula): for simplicity, we give the proof for $u \geq 0$, and leave the general case as an excercise (hint: use $u = u^+ - u^-$).

Step 1: for any $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, the map $t \mapsto \int_{\{u>t\}} \operatorname{div} T(x)\, dx$ is Borel as some of monotone functions, i.e.

$$\int_{\{u>t\}} \operatorname{div} T(x)\, dx = \int_{\{u>t\}} (\operatorname{div} T(x))^+ dx - \int_{\{u>t\}} (\operatorname{div} T(x))^- dx.$$

As $P(\{u>t\};A)$ is sup (restricting on a maximazing sequence) of countable Borel functions, is Borel.

Step 2: given $T \in C^\infty_c(A;\mathbb{R}^n)$ with $\|T\|_\infty \leq 1$, by definition of distributional derivative and by the layer-cake formula

$$-\int_A \nabla u(x) \cdot T(x)\, dx = \int_{\mathbb{R}^n} u(x) \operatorname{div} T(x)\, dx \overset{LCF}{=} \int_{\mathbb{R}} \int_{\{u>t\}} \operatorname{div} T(x)\, dx\, dt$$

$$\leq \int_{\mathbb{R}} P(\{u>t\};A)\, dt,$$

where in the last step we used the definition of (distributional) perimeter. By the duality of the L^1-norm, taking the sup over T we get

$$\int_A |\nabla u(x)|\, dx \leq \int_{\mathbb{R}} P(\{u>t\};A)\, dt.$$

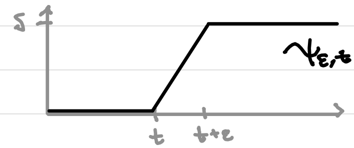
Step 3: let $m(t) := \int_{A \cap \{u \leq t\}} |\nabla u(x)|\, dx$. This is an increasing function (in t) so it is differentiable a.e. and by the fundamental theorem of calculus for one-dimensional BV functions (increasing functions are BV$_{loc}$) it holds

$$\int_0^\infty m'(t)\, dt \leq \lim_{t \to +\infty} m(t) - m(0) = \int_A |\nabla u(x)|\, dx.$$

If we control m' from below with $P(\{u>t\};A)$ we conclude the proof.

For this, we consider the function

$$\psi_{\varepsilon,t}(s) := \begin{cases} 1 & t+\varepsilon \le s \\ \frac{s-t}{\varepsilon} & t \le s < t+\varepsilon \\ 0 & s < t \end{cases}$$



and notice that $\psi'_{\varepsilon,t}(s) = \frac{1}{\varepsilon}\chi_{(t,t+\varepsilon)}(s)$.
Then, by Cauchy-Schwartz, for any $T \in C^\infty_c(A;\mathbb{R}^n)$ we get

$$\frac{m(t+\varepsilon)-m(t)}{\varepsilon} \geq 1/\varepsilon \int_{A\cap\{t<u<t+\varepsilon\}} |\nabla u(x)|\,dx$$

$$\geq -1/\varepsilon \int_{A\cap\{t<u<t+\varepsilon\}} \nabla u(x)\cdot T(x)\,dx.$$

Since, by the chain rule $\nabla\psi_{\varepsilon,t}(u(x)) = \psi'_{\varepsilon,t}(u(x))\nabla u(x)$, we have

$$= -1/\varepsilon \int_A \chi_{(t,t+\varepsilon)}(u(x))\nabla u(x)\cdot T(x)\,dx$$

$$= -\int_A \nabla\psi_{\varepsilon,t}(u(x))\cdot T(x)\,dx = \int_A \psi_{\varepsilon,t}(u(x))\,\mathrm{div}T(x)\,dx.$$

By sending $\varepsilon \to 0^+$ we get, for a.e. t, and taking the sup over T (by density of $C^\infty_c(A;\mathbb{R}^n) \subseteq C^1_c(A;\mathbb{R}^n)$) we get

$$P(\{u>t\};A) \leq m'(t), \quad \text{for a.e. } t\in\mathbb{R}$$

which yields the result. ☑

<u>Rmk</u>: a more general formula, from which the connection with Fubini's theorem is even more evident, is the following. Given $g \in L^1_{loc}(\mathbb{R}^n)$, $u:\mathbb{R}^n \to \mathbb{R}$ Lipschitz, and $A \subseteq \mathbb{R}^n$ open, it holds that

$$\int_A g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}}\int_{A\cap\{u=t\}} g(y)\,d\mathcal{H}^{n-1}(y)\,dt.$$

An immediate consequence of the coarea formula is that super-level sets of (sufficiently regular) functions are sets of finite perimeter.

Rmk: let $u:\mathbb{R}^n \to \mathbb{R}$ a Lipschitz function, then $\nabla u \in L^1_{loc}(\mathbb{R}^n)$ so the function $t \mapsto P(\{u>t\};A)$ for A bounded is integrable, so it is finite a.e., thus for a.e. t, the super-level set $\{u>t\}$ is of locally finite perimeter.

Exc: using the coarea formula with $u(x)=|x-x_0|$, prove that, for every $E \subseteq \mathbb{R}^n$ Borel it holds that

$$|E \cap B_R(x_0)| = \int_0^R \mathcal{H}^{n-1}(E \cap \partial B_t(x_0))\,dt.$$

Rather than a generalization of Fubini, we will use the coarea formula to gain information on the boundary of superlevel sets.

We have seen that (cf. Proposition 1 of L6) that the perimeter is lower-semicontinuous, so if $E_h \xrightarrow{loc} E \Rightarrow P(E;A) \leq \underline{\lim}_h P(E_h;A)$, but in general the perimeters of a converging sequence may not approximate the perimeter of the limit.

It will be important to be able to build sequences that approximate the perimeter too. Thanks to the coarea formula we can do it by cutting superlevel sets of a mollification of the characteristic function of the set we want to approximate.
We give only ideas of the proof (for details [Chapter 13.2, Maggi (2012)]).

Theorem (approximation by smooth sets): $E \subseteq \mathbb{R}^n$ measurable is of locally finite perimeter $\Leftrightarrow$ $\exists$ sequence $\{E_h\}$ of open sets with smooth boundary in $\mathbb{R}^n$, and $\{\varepsilon_h\}$, $\varepsilon_h \to 0^+$ s.t.

$$E_h \xrightarrow{loc} E, \qquad \sup_h P(E_h;B_R) < \infty \quad \forall R>0,$$

$$|\mu_{E_h}| \xrightarrow{*} |\mu_E|, \qquad \partial E_h \subseteq \partial E + B_{\varepsilon_h}.$$

In particular, $P(E_h;F) \to P(E;F)$ whenever $P(E;\partial F)=0$.
Moreover, if $|E|<\infty \Rightarrow E_h \to E$; if $P(E)<\infty \Rightarrow P(E_h) \to P(E)$.

<u>note</u> (Morse-Sard lemma): we will use the following deep result, $u \in C^\infty(\mathbb{R}^n)$ then $\{u=t\}$ is smooth for a.e. $t \in \mathbb{R}$.
Again, for a proof see [Chapter 13.3, Maggi (2012)].

<u>proof</u> (ideas): we take $u_h := \rho_{\varepsilon_h} * \chi_E$. By the regularization result of last lecture, $u_h \to \chi_E$ in L^1_{loc}.

Defining $E_h^t := \{u_h > t\}$, these are smooth sets for a.e. $t \in (0,1)$, by Morse-Sard lemma. Notice that for $t<0$, $E_h^t = \mathbb{R}^n$, for $t>1$, $E_h^t = \emptyset$. By the layer-cake formula $E_h^t \xrightarrow{loc} E$ for a.e. $t \in (0,1)$.

By lower-semicontinuity of the perimeter

$$P(E;A) \leq \varliminf_h P(E_h^t;A), \quad \text{for a.e. } t \in (0,1).$$

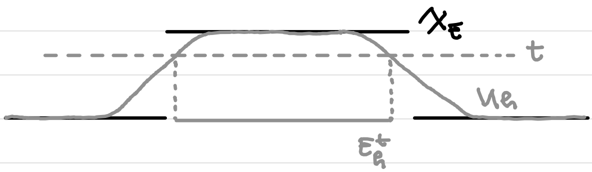
This can be upgraded by the coarea formula. Indeed, by Fatou's lemma and by regularization

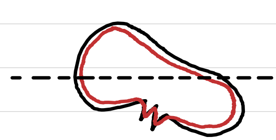
$$P(E;A) \leq \int_0^1 \varliminf_h P(E_h^t;A)\,dt \leq \varliminf_h \int_0^1 P(E_h^t;A)\,dt$$

$$= \lim_h \int_A |\nabla u_h(x)|\,dx$$

$$= P(E;A).$$

So $t \mapsto P(E_h^t;A)$ are non-negative function with converging averages, so they converge strong in $L^1(0,1)$, and up to sub-sequences a.e.. ☐





Of course, apart from building sets approximating the perimeter of a set, the fact that the approximating sets can be taken smooth is of great importance, as we see later.

A nice observation, as a consequence of the previous theorem, is that the "distributional" perimeter (that we define in Lecture 5) is the "best" possible notion of perimeter that extend the notion of surface area of boundary of regular sets to general Lebesgue measurable sets.

<u>Rmk</u> (relaxed perimeter): We discussed several times that, for C^1 regular sets, a notion of perimeter was already at our disposal. With the aim of extending this notion to every Lebesgue set, we denote

$$\sigma(E) := \begin{cases} \mathcal{H}^{n-1}(\partial E), & \text{if } \partial E \in C^1 \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the distributional perimeter $P(E) \leq \sigma(E)$ and they actually coincide for C^1 sets. Moreover, P is lower-semicontinuous for set convergence, so $P(E) \leq \sigma^{rel}(E)$, where σ^{rel} denotes its lower-semicontinuous envelope.
By the previous theorem we say that, $\forall E \subseteq \mathbb{R}^n$ s.t. $P(E) < +\infty$ $\exists E_h$ regular s.t. $\sigma(E_h) \to P(E)$.
In other words, $P(E) = \sigma^{rel}(E)$, so the distributional perimeter is the lower-semicontinuous envelope of σ.

<u>note</u>: by piecewise-affine approximation of u_h as in the proof above, we can approximate with sets with polyhedral boundaries

<u>Rmk</u>: if $|E| < +\infty$ and $P(E) < +\infty$, we can approximate E with a sequence of bounded sets with smooth boundary E_h with $|E_h| \to |E|$ and $P(E_h) \to P(E)$.

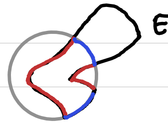
Indeed, for every $R>0$,

$$P(E) \geq P(E; B_R) + P(E; \mathbb{R}^n \setminus \overline{B}_R).$$

Since $P(E) < +\infty$, taking the limit as $R \to +\infty$ we get

$$\lim_{R\to\infty} P(E; \mathbb{R}^n \setminus \overline{B}_R) \leq P(E) - \lim_{R\to\infty} P(E; B_R) = 0.$$

Moreover one can prove that, for a.e. $R>0$,



$$P(E \cap B_R) = P(E; B_R) + \mathcal{H}^{n-1}(E \cap \partial B_R).$$

By the exercise before, as a consequence of the coarea formula, $R \mapsto \mathcal{H}^{n-1}(E \cap \partial B_R)$ is $L^1(0,+\infty)$ since

$$\int_0^{+\infty} \mathcal{H}^{n-1}(E \cap \partial B_R)\, dR = |E| < +\infty.$$

Hence $\exists \{R_h\}_h$ s.t. $\mathcal{H}^{n-1}(E \cap \partial B_{R_h}) \longrightarrow 0$ as $R_h \to +\infty$, so by the previous property

$$|P(E \cap B_{R_h}) - P(E)| = P(E; \mathbb{R}^n \setminus B_{R_h}) + \mathcal{H}^{n-1}(E \cap \partial B_{R_h}) \longrightarrow 0.$$

By approximating each $E \cap B_{R_h}$ with sets with smooth boundary we get the desired result.

2. The isoperimetric problem

We have now (almost) everything in place to attack the isoperimetric problem variationally.

<u>Theorem</u> (isoperimetric problem): if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $|E| < +\infty$, then

$$P(E) \geq n \omega_n^{1/n} |E|^{\frac{n-1}{n}},$$

and the equality holds true $\iff |E \Delta B_r(x)| = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$.

Ideally, we would like to apply the Direct Method to the infimum problem

$$\inf \{ P(E) : E \subseteq \mathbb{R}^n, |E| = V \}.$$

Unfortunately, when set in the whole space, the perimeter is not globally precompact since we may have loss of mass escaping at infinity, so that limits of infimizing sequence may have volume $< V$. To work around this, we first consider a "constraint" problem.

2.1. The constraint problem

We restrict the isoperimetric problem to uniformly bounded sets. In this setting, we can apply the Direct Method.

<u>Lemma 1</u>: let $V > 0$ and let $R > \left(\frac{V}{\omega_n}\right)^{1/n}$ and consider

$$\mathcal{J}_R(V) := \inf \{ P(E) : E \subseteq B_R \text{ with } |E| = V \}.$$

Then $\exists\, E_0 \subseteq B_R$ s.t. $P(E_0) = \mathcal{J}_R(V)$.

proof: as the class of test sets contains the ball of volume V, the infimum problem is not trivial.

Let $E_h \subseteq B_R$, $|E_h|=V$, s.t. $P(E_h) \longrightarrow \mathcal{J}_R(V)$ as $h \to +\infty$ be an infimizing sequence.
By the coercivity theorem of last lecture, $\exists \{E_{h'}\}_{h'} \subseteq \{E_h\}_h$ and $E_0 \subseteq B_R$ s.t. $E_{h'} \longrightarrow E_0$. By lower semicontinuity of the perimeter (i.e. Proposition 1 of last lecture)

$$\mathcal{J}_R(V) \leq P(E_0) \leq \varliminf_{h'} P(E_{h'}) = \mathcal{J}_R(V)$$

which proves the claim. $\boxtimes$

Exc (the relative isoperimetric problem): analogously as in the proof above prove existence of minimizers for the relative isoperimetric problem.
Namely, for any $A \subseteq \mathbb{R}^n$ open with $|A| < +\infty$, for any $V < |A|$ the infimum problem

$$\inf \{ P(E;A) : E \subseteq A \ , |E| = V \}$$

admits a minimizer.

We can further reduce the minimization problem by analyzing only smooth sets, thanks to the approximation result from before.

Lemma 2: for every V and R as in Lemma 1, it holds

$$\mathcal{J}_R(V) = \mathcal{J}_R^{reg}(V) := \inf \{ P(E) : E \subseteq B_R, |E| = V, \partial E \text{ is smooth} \}.$$

proof: let E_0 be a minimizer of $\mathcal{J}_R(V)$. Then, by approximation there exists a sequence of smooth sets s.t. $E_h \longrightarrow E_0$ and $P(E_h) \longrightarrow P(E_0)$.

It is easy to check that, up to intersecting with B_R and make sure that this does not create (too much perimeter), we can assume that $E_h \subseteq B_R$, recalling also that $E_h \subseteq B_{R+\varepsilon_h}$ for some $\varepsilon_h \to 0$.

If, $|E_h| = V$ for a subsequence then there is nothing else to prove.

Assume that, for some h, $|E_h| > V$ and denote $V_h := |E_h| - V$. Then consider $F_h := V/V_h \, E_h$. By the scaling properties of volume and the dominated convergence theorem.

$$|F_h \Delta E| \leq |F_h \Delta V/V_h E| + |V/V_h E \Delta E|$$

$$= \left(\frac{V}{V_h}\right)^n |E_h \Delta E| + \| \chi_E(\tfrac{V_h}{V}\cdot) - \chi_E \|_{L^1} \to 0.$$

More easily, for the perimeter, $P(F_h) = (V/V_h)^{n-1} P(E_h) \to P(E_0)$.

If instead $|E_h| < V$ and $V_h := V - |E_h|$ along a subsequence, then take $\forall t > 0$, $F_h^t := E_h \cup B_t(x_h)$ with $x_h \notin B_R$ s.t. B_R and $B_t(x_h)$ are tangent, and take $G_h^t := \frac{R}{R+t} F_h^t$. As the function

$$v_h : t \mapsto |G_h^t| = \frac{R^n}{(R+t)^n} (|E_h| + \omega_n t^n)$$

is continuous, $v(0) = |E_h|$ and $v(t) \to \omega_n R^n$ as $t \to +\infty$, $\exists t_h$ s.t. $v_h(t_h) = V$, and $t_h \to 0$ as $h \to 0$.
So taking $G_h := G_h^{t_h}$ we have $|G_h| = V$ and, working as above

$$|G_h \Delta E| = \left|\frac{R}{R+t_h} E_h \Delta E\right| + \left(\frac{R}{R+t_h}\right)^n |B_{t_h}| \to 0$$

and $P(G_h) = \left(\frac{R}{R+t_h}\right)^{n-1} (P(E_h) + n\omega_n t_h^{n-1}) \to P(E)$ as $h \to +\infty$.
Since $G_h \subseteq B_R$ by construction we are done. $\square$