Lecture 6: 26.04.2025

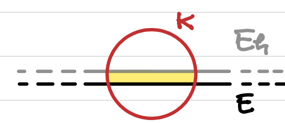
1. Lower-semicontinuity of the perimeter

We define a suitable notion of convergence of sets that will be the topology which we will use to apply the Direct Method to prove $\exists$ of minimizers for the isoperimetric problem.

The correct topology is given by $E_h \xrightarrow{loc} E$ iff $|K \cap (E_h \Delta E)| \to 0$ $\forall K \subseteq \mathbb{R}^n$ compact. This coincides with $\chi_{E_h} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$.
We write $E_h \longrightarrow E$ iff $|E_h \Delta E| \to 0$.

Ex.: there might be sets with infinite measure that converge locally but not globally, for instance

$$E_h = \{x \in \mathbb{R}^2 : x_2 > 1/h\} \xrightarrow{loc} E = \{x \in \mathbb{R}^2 : x_2 > 0\},$$



but also bounded sets "escaping to infinity", that do not converge globally, but locally they escape from every compact, so they (locally) converge to the empty set, e.g. $E_h = B_1(x_h)$ with $x_h \in \mathbb{R}^n$, $|x_h| \to +\infty$ as $h \to +\infty$.

With respect to this convergence, the perimeter is a lower semicontinuous functional.

Proposition 1 (lsc of P): Let $E_h \subseteq \mathbb{R}^n$ be of locally finite perimeter and

$$E_h \xrightarrow{loc} E, \quad \overline{\lim_{h \to \infty}}\, P(E_h; K) < +\infty \quad \forall K \subseteq \mathbb{R}^n \text{ compact.}$$

Then E is of locally finite perimeter and

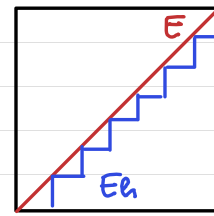
$$\mu_{E_h} \xrightarrow{*} \mu_E, \qquad P(E;A) \leq \underline{\lim_{h \to \infty}}\, P(E_h; A) \quad \forall A \subseteq \mathbb{R}^n \text{ open.}$$

It is worth noticing that the perimeter is not continuous. The most standard phenomena in which we may have (local) loss of perimeter in the limit are oscillation and annihilation. They are shown in the following two examples, respectively.

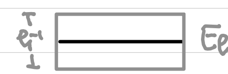
<u>Exc</u>($\sqrt{2}<2$): let $f_h(t) := \frac{\lfloor ht \rfloor}{h}$ where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ denotes the floor function, namely $\lfloor s \rfloor := \max\{k \in \mathbb{Z} : k \leq s\}$.

Take $E_h := \{x \in (0,1)^2 : x_1 < f_h(x_2)\}$.
Prove that $E_h \longrightarrow E := \{x \in (0,1)^2 : x_1 < x_2\}$.
By using that $\sqrt{2} < 2$, prove that

$$\lim_h P(E_h; (0,1)^2) > P(E; (0,1)^2).$$



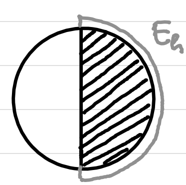
<u>Ex</u> (strict inequality): $E_h = (0,1) \times (0, h^{-1})$, then $|E_h| = h^{-1} \longrightarrow 0$, so we have $E_h \longrightarrow \phi$ but $P(E_h) = 2 + 2h^{-1} \longrightarrow 2$.
Hence $P(E) = 0 < 2 = \lim_h P(E_h)$.



Another thing to remark is that, we have lower semicontinuity of relative perimeters on open sets. Inside closed sets we may have problems if some perimeter is concentrating on the boundary.

<u>Ex</u>: take $E_h = B_{1+h^{-1}} \cap \{x_1 > 0\}$. Then $E_h \longrightarrow E := B_1 \cap \{x_1 > 0\}$. Taking $K = \overline{B}_1$ we have

$$P(E; K) = 1 + \pi > 1 = \lim_{h \to \infty} P(E_h; K)$$



which violates the lower semicontinuity.
This is because, as we saw in example (ii) of last lecture, as E and E_h are Lipschitz sets $P(E;K) = \mathcal{H}^1(\partial E \cap K)$, $P(E_h;K) = \mathcal{H}^1(\partial E_h \cap K)$ and

$$\partial E \cap K = \{0\} \times [0,1] \cup (\{x_1 \geq 0\} \cap \partial B_1), \qquad \partial E_h \cap K = \{0\} \times [0,1], \quad \forall h.$$

proof: for any $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $spt(T) \subseteq A$, by the strong L^1 convergence of the characteristic functions

$$\int_E \operatorname{div} T(x)\,dx = \lim_{h\to\infty} \int_{E_h} \operatorname{div} T(x)\,dx \leq \varliminf_{h\to\infty} P(E_h;A).$$

Taking the sup over T, E is of locally finite perimeter and P is lower-semicontinuous.

By the characterization of sets of locally finite perimeter via the Gauss-Green measure, $\forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $spt(T) \subseteq K$

$$\int_{\mathbb{R}^n} T \cdot d\mu_{E_h} = \int_{E_h} \operatorname{div} T(x)\,dx \xrightarrow[h\to\infty]{} \int_E \operatorname{div} T(x)\,dx = \int_{\mathbb{R}^n} T\cdot d\mu_E$$

and $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ by density of $C^1_c(\mathbb{R}^n;\mathbb{R}^n) \subseteq C_c(\mathbb{R}^n;\mathbb{R}^n)$. $\boxtimes$

2. Topological boundary and Gauss-Green measure

As we have seen in last lecture, modifying a set of finite perimeter with a null set does not modify the perimeter, though this may heavily modify its topological boundary.

With the following result we see that, a better notion of boundary for sets of finite perimeter is the support of the Gauss-Green measure.

To prove it, the following observation will turn useful.

Exc: prove that $\int_E \operatorname{div} T(x)\,dx = \int_{\mathbb{R}^n} T\cdot d\mu_E$, $\forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ holds $\iff \int_E \nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi\, d\mu_E$, $\forall\, \varphi \in C^1_c(\mathbb{R}^n)$.

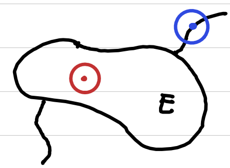
Proposition 2: E of locally finite perimeter, the

$$spt(\mu_E) = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < r^n\omega_n, \forall\, r>0\} \subseteq \partial E.$$

Moreover $\exists\, F \in \mathcal{B}(\mathbb{R}^n)$ s.t. $|E \Delta F| = 0$ and $spt(\mu_F) = \partial F$.

proof: We denote $S := \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < r^n \omega_n, \forall r > 0\}$. For every $x \in S$, $B_r(x) \cap E \neq \emptyset$, $B_r(x) \cap (\mathbb{R}^n \setminus E) = \emptyset \Rightarrow x \in \partial E$. We now prove that $\operatorname{spt}(\mu_E) = S$.

($\subseteq$) if $x \notin S$ then $\exists\, r > 0$ s.t. either $|E \cap B_r(x)| = 0$ or $r^n \omega_n$.
If $|E \cap B_r(x)| = 0$ then $\forall \varphi \in C_c(B_r(x))$



$$\int_{\mathbb{R}^n} \varphi \, d\mu_E = \int_E \nabla \varphi(x)\, dx = 0$$

so $|\mu_E|(B_r(x)) = 0 \Rightarrow x \notin \operatorname{spt}(\mu_E)$.
Analogousely, if $|E \cap B_r(x)| = r^n \omega_n$ then, by the divergence theorem we get

$$\int_{\mathbb{R}^n} \varphi \, d\mu_E = \int_E \nabla \varphi(y)\, dy = \int_{B_r(x)} \nabla \varphi(y)\, dy \overset{\text{DIV}}{=} 0.$$

So again $|\mu_E|(B_r(x)) = 0$ and $x \notin \operatorname{spt}(\mu_E)$.

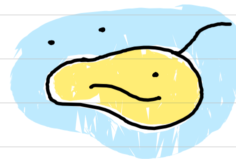
($\supseteq$) $x \notin \operatorname{spt}(\mu_E) \Rightarrow \exists\, r > 0$ s.t. $|\mu_E|(B_r(x)) = 0$, namely

$$0 = \int_{\mathbb{R}^n} \varphi \, d\mu_E = \int_{\mathbb{R}^n} \chi_E(y) \nabla \varphi(y)\, dy, \quad \forall \varphi \in C_c(B_r(x)).$$

In particular, $D\chi_E \equiv 0$ in $B_r(x)$, hence by the fundamental lemma of Calculus of Variations, χ_E is constant in $B_r(x)$. As it is a characteristic function either $\chi_E = 0$ on $B_r(x)$, i.e. $|E \cap B_r(x)| = 0$, or $\chi_E = 1$ on $B_r(x)$, i.e. $|E \cap B_r(x)| = r^n \omega_n$. In both cases $x \notin S$.

We are left to prove the last part of the statement. For this we may assume $E \in \mathcal{B}(\mathbb{R}^n)$, by the remark that null sets do not modify the perimeter and μ_E.

We define the sets $A_0 := \{x \in \mathbb{R}^n : \exists r>0 \text{ s.t. } |E\cap B_r(x)| = 0\}$ and $A_1 := \{x\in\mathbb{R}^n : \exists r>0 \text{ s.t. } |E\cap B_r(x)| = r^n\omega_n\}$. It is immediate that these are open sets.



Consider $\mathcal{F}_0 = \{B_{r(x)}(x) : x\in A_0,\ r(x)>0 \text{ s.t. } |E\cap B_{r(x)}(x)|=0\}$. By Besicovitch covering theorem $\exists\, B_{r_k}(x_k)\in\mathcal{F}_0$ s.t. $A_0 \subseteq \bigcup_k B_{r_k}(x_k)$.
Hence $|E\cap A_0| \le \sum_k |E\cap B_{r_k}(x_k)| = 0$, by σ-subadditivity. Analogously, one can prove $|A_1\setminus E|=0$.

We define $F := (A_1\cup E)\setminus A_0$, then $|E\Delta F| = |(A_1\setminus E)\setminus A_0| + |(E\cap A_0)\setminus A_1| = 0$.
Since $\mathbb{R}^n\setminus(A_0\cup A_1) = S = \mathrm{spt}(\mu_E) = \mathrm{spt}(F) \subseteq F$ by the first part of the proof.
Moreover, $A_1\subseteq F$ and it is open so $A_1 \subseteq \mathring{F}$. Similarly $\mathbb{R}^n\setminus A_0 \supseteq \overline{F}$, thus $\partial F = \overline{F}\setminus\mathring{F} \subseteq \mathbb{R}^n\setminus(A_0\cup A_1) = \mathrm{spt}(\mu_F)$. So $\mathrm{spt}(\mu_F) = \partial F$. $\boxtimes$

3. Regularization and coercivity

Identifying sets of finite perimeter with BV-regular characte-ristic functions, we can exploit regularizations results via mollification. Eventually, we can obtain coercivity of the peri-meter.

Consider $\rho \in C_c^\infty(B_1)$ with $0 \le \rho \le 1$ and $\|\rho\|_{L^1} = 1$. For $\varepsilon > 0$ sufficiently small we define $\rho_\varepsilon(z) := \varepsilon^{-n} \rho(z/\varepsilon)$.

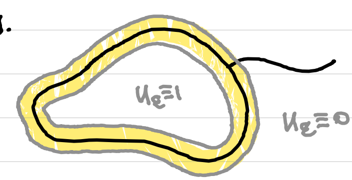
<u>Rmk</u>: given $E \subseteq \mathbb{R}^n$ of locally finite, we define $u_\varepsilon := \rho_\varepsilon * \chi_E$, namely

$$\rho_\varepsilon * \chi_E(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-z)\chi_E(z)\,dz = \int_{E \cap B_\varepsilon(x)} \rho_\varepsilon(x-z)\,dz.$$

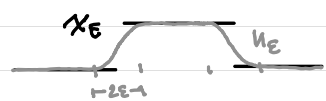
By the standard theory of approximation in Lebesgue spaces, $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, $u_\varepsilon \longrightarrow \chi_E$ in L^1_{loc}. Moreover, we can infer the fol-lowing properties (cf. [Chapter 12.3, Maggi (2012)]).

(i) $\operatorname{spt}(u_\varepsilon) \subseteq E + B_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \varepsilon\}$.
Additionally

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } |B_\varepsilon(x) \cap E| = \varepsilon^n \omega_n \\ 0 & \text{if } |B_\varepsilon(x) \cap E| = 0. \end{cases}$$



(ii) for sufficiently regular E
$\operatorname{spt}(\nabla u_\varepsilon) \subseteq \partial E + B_\varepsilon$ and



$$\nabla u_\varepsilon(x) \sim -\tfrac{1}{\varepsilon}\, \nu_E(x_E)\, \chi_{\partial E + B_\varepsilon}(x)$$

where x_E is the projection of x onto ∂E. Hence

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon(x)|\,dx \sim \tfrac{1}{\varepsilon}\, \mathcal{L}^n(\partial E + B_\varepsilon) \sim \tfrac{1}{\varepsilon}\, \mathcal{H}^{n-1}(\partial E)\cdot \varepsilon = P(E).$$

This heuristic considerations can be actually formalized, see [Proposition 12.20, Maggi (2012)] for details.

<u>Proposition</u>: if E is of locally finite perimeter, then $u_\varepsilon := \rho_\varepsilon * \chi_E$ is s.t. $u_\varepsilon \to \chi_E$ L^1_{loc} and

$$-\nabla u_\varepsilon \mathcal{L}^n \xrightarrow{*} \mu_E, \quad |\nabla u_\varepsilon| \mathcal{L}^n \xrightarrow{*} |\mu_E|.$$

Conversely, if E is s.t. $\overline{\lim_{\varepsilon \to 0^+}} \int_K |\nabla u_\varepsilon| dx < \infty$ $\forall K \subseteq \mathbb{R}^n$ compact then is of locally finite perimeter.

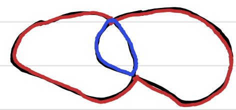
<u>note</u>: this is in fact true for any $u \in BV_{loc}(\mathbb{R}^n)$, namely $u_\varepsilon := \rho_\varepsilon * u$ then $u_\varepsilon \to u$ in L^1_{loc} and

$$\nabla u_\varepsilon \mathcal{L}^n \xrightarrow{*} Du, \qquad |\nabla u_\varepsilon| \mathcal{L}^n \xrightarrow{*} |Du|.$$

One useful estimate, that can be easily obtained via regularization is the following. It is left here as an excercise, for a proof see [Lemma 12.22, Maggi (2012)].

<u>Exc</u>: show that, if E, F are of locally finite perimeter then for every $A \subseteq \mathbb{R}^n$ open

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; A) + P(F; A)$$



In particular, $E \cap F$ and $E \cup F$ are of locally finite perimeter.

An easy example of the strict inequality is for $E = (0,1)^2$ and $F = (1,2) \times (0,1)$.

2.1. <u>Compactness</u>

As in Sobolev spaces, a control of the gradient implies strong compactness (in L^1) also for BV functions.

The geometric equivalent is that set with uniformly finite perimeter are compact in the set convergence.

<u>Theorem</u>: $\{E_h\}$ sets of finite perimeter with, for some $R>0$,

$$\sup_h P(E_h) < +\infty, \qquad E_h \subseteq B_R \quad \forall h.$$

Then $\exists\, E \subseteq \mathbb{R}^n$ of finite perimeter s.t. (up to subsequences)

$$E_h \longrightarrow E, \qquad \mu_{E_h} \xrightarrow{*} \mu_E, \qquad E \subseteq B_R.$$

You find a nice geometrical proof in [Theorem 12.26, Maggi (2012)]. For convenience we see an alternative proof which is based on approximation of BV-functions.

In the proof we will need the following uniform control on translations of BV-functions.

<u>Note</u>: $v \in C^\infty_c(\mathbb{R}^n)$, $\forall h \in \mathbb{R}^n$ $\|v(\cdot+h)-v\|_{L^1} \leq |h|\, \|\nabla v\|_{L^1}$. Indeed

$$\int_{\mathbb{R}^n} |v(x+h)-v(x)|\,dx = \int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{dt} v(x+th)\,dt \right| dx$$

$$= \int_{\mathbb{R}^n} \left| \int_0^1 \nabla v(x+th)\cdot h\,dt \right| dx \leq |h|\, \|\nabla v\|_{L^1}$$

By approximation, $u_\varepsilon \in C^\infty_c(\mathbb{R}^n)$ and
$$\|u(\cdot-h)-u\|_{L^1} = \lim_{\varepsilon\to 0} \|u_\varepsilon(\cdot+h)-u_\varepsilon\|_{L^1} \leq C|h| \lim_{\varepsilon\to 0} |Du_\varepsilon|(\mathbb{R}^n) = C|h|\,|Du|(\mathbb{R}^n)$$

<u>proof</u>: by assumption $\|\chi_{E_h}\|_{L^1(B_R)} + |D\chi_{E_h}|(B_R) \leq C_1$ $\forall h$, for some constant $C_1>0$, and considere $\rho_\varepsilon * \chi_{E_h} \in C^\infty_c(B_{R+1})$ for $0<\varepsilon<1$.

For ε fixed, the sequence $\{\rho_\varepsilon * \chi_{E_h}\}_h$ is uniformly Lipschitz,

$$\|\nabla(\rho_\varepsilon * \chi_{E_h})\|_{L^\infty} = \sup_{x\in\mathbb{R}^n} \left| \int_{E_h} \nabla\rho_\varepsilon(x-z)\,dx \right| \leq \|\nabla \rho_\varepsilon\|_{L^1} \leq C_1/\varepsilon.$$

By Ascoli-Arzelà $\rho_\varepsilon * \chi_{E_h}$ is totally bounded, i.e. $\forall \delta>0$ $\exists$ $N=N(\varepsilon,\delta)$, $\{\varphi_i^\varepsilon\}_{i=1}^N \subseteq C_c(B_{R+1})$ s.t.

$$\{\rho_\varepsilon * \chi_{E_h}\}_h \subseteq \bigcup_{i=1}^N B_\delta^{\|\cdot\|_\infty}(\varphi_i^\varepsilon), \ \forall \delta>0.$$

The inclusion above is equivalent to say that, $\forall h$ $\exists i$ s.t.

$$\|\rho_\varepsilon * \chi_{E_h} - \varphi_i^\varepsilon\|_{L^\infty(B_{R+1})} < \delta$$

which implies that

$$\|\rho_\varepsilon * \chi_{E_h} - \varphi_i^\varepsilon\|_{L^1(B_{R+1})} \leq |B_{R+1}|\delta$$

that is, calling $C_1' = |B_{R+1}|$ the sequence $\{\rho_\varepsilon * \chi_{E_h}\}_h$ is also totally bounded in $L^1(B_{R+1})$, i.e.

$$\{\rho_\varepsilon * \chi_{E_h}\}_h \subseteq \bigcup_{i=1}^N B_{C_1'\delta}^{\|\cdot\|_{L^1}}(\varphi_i^\varepsilon).$$

For every fixed h, by the observation above and Fubini

$$\|\chi_{E_h} - \rho_\varepsilon * \chi_{E_h}\|_{L^1} = \int_{B_{R+1}} \left| \int_{B_\varepsilon} (\chi_{E_h}(x) - \chi_{E_h}(x-z))\rho_\varepsilon(z)\,dz \right| dx$$

$$\leq \int_{B_\varepsilon} \sup_{|z|<\varepsilon} \|\chi_{E_h} - \chi_{E_h}(\cdot+z)\|_{L^1} \rho_\varepsilon(z)\,dz$$

$$\leq \varepsilon\, |D\chi_E|(B_R) \leq C_1 \varepsilon.$$

Hence, $\forall \delta>0$, choosing $\varepsilon=\delta$, from what we proved above $\exists$ N_δ and $\{\varphi_i^\delta\}_{i=1}^{N_\delta} \subseteq C_c(B_{R+1})$ s.t.

$$\{\chi_{E_h}\}_h \subseteq \bigcup_{i=1}^{N_\delta} B_{C_1''\delta}^{\|\cdot\|_{L^1}}(\varphi_i^\delta),$$

namely $\{\chi_{E_h}\}_h$ is totally bounded in L^1, so precompact, i.e. up to subsequences $\chi_{E_h} \to u \in L^1(B_R)$ and a.e.. By a.e. convergence $u(x) \in \{0,1\}$ a.e. in B_R, so $\exists$ $E \subseteq B_R$ s.t. $u = \chi_E$.
The full result follows by Proposition 1 of today. ☑

Rmk: the same result holds also locally.

If $\{E_h\}_h$ are of locally finite perimeter (not necessarily uniformly bounded) s.t. $\sup_h P(E_h; K) < +\infty$ $\forall K \subseteq \mathbb{R}^n$ compact, then up to subsequences $E_h \xrightarrow{loc} E$ and $\mu_{E_h} \xrightarrow{*} \mu_E$.

Ex: an example of local (not global) compactness is e.g. again the case in which some mass escape to infinity. Take $E_h = B_1 \cup B_1(x_h)$ with $|x_h| \to \infty$, this never converges globally, but locally it converges to B_1.