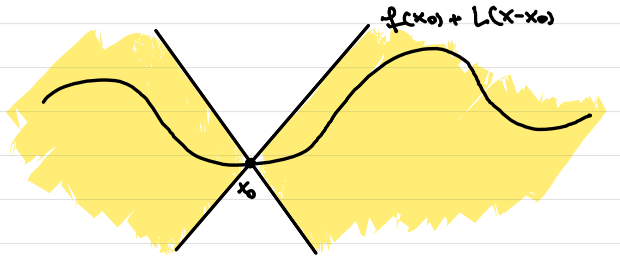
We see some more details about the rectifiability via convergence of blow-ups.

<u>Theorem</u>: let μ be a Radon measure on $\mathbb{R}^n$ which concentrates on $M \in \mathcal{B}(\mathbb{R}^n)$. If, $\forall x \in M$ $\exists$ k-dimensional plane $\pi_x \subseteq \mathbb{R}^n$ s.t.

$$\mu_{x,r} \xrightarrow[r \to 0^+]{*} \mathcal{H}^k \llcorner \pi_x,$$

then $\mu = \mathcal{H}^k \llcorner M$ and M is locally $\mathcal{H}^k$-rectifiable.

To prove it we will exploit a rectifiability criterion. This criterion resembles a property of Lipschitz functions in of one variable. Namely, f is Lipschitz $\iff$ for every $\tilde{x}$ its graph is contained inside the cone $\{x \in \mathbb{R}^2 : |x_2 - f(\tilde{x}_1)| \leq L |x_1 - \tilde{x}_1|\}$ where L is the Lipschitz constant of f.



Let first introduce the notation $p_\pi : \mathbb{R}^n \to \mathbb{R}^n$ to denote the orthogonal projection onto a linear subspace $\pi \subseteq \mathbb{R}^n$. Given $t > 0$, we define $K(\pi, t) := \{y \in \mathbb{R}^n : |p_{\pi^\perp} y| < t |p_\pi y|\}$. Intuitively, $K(\pi, t)$ is a conical neighborhood (of amplitude t) of π.

<u>Proposition</u> (rectifiability criterion): if $M \subseteq \mathbb{R}^n$ compact and $\exists$ π a k-dimensional plane, $\delta, t > 0$ s.t. $\forall x \in M$

$$M \cap B_\delta(x) \subseteq x + K(\pi, t)$$

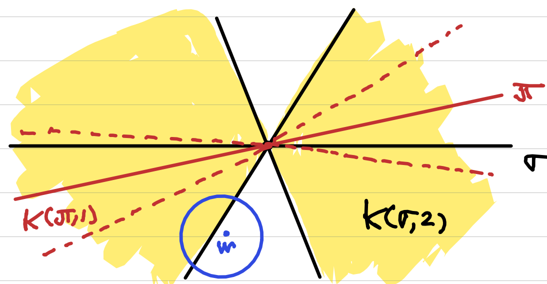
then M is $\mathcal{H}^k$-rectifiable.

We omit the proof of the rectifiability criterion (we refer to [Proposition 10.8, Maggi (2012)]).

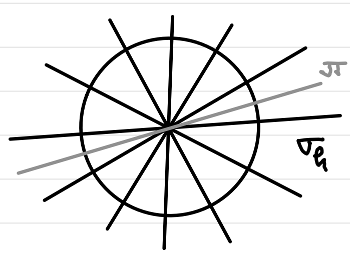
<u>Proof</u> (theorem): given π, σ two k-dimensional planes in $\mathbb{R}^n$ we define the distance $d(\pi,\sigma) := \sup\{ |p_\pi v - p_\sigma v| : v \in S^{n-1} \}$.

We will use the fact that, if $d(\pi,\sigma) < \lambda$ then

$$\forall\, w \in \mathbb{R}^n \setminus K(\sigma,2) \Rightarrow B_{\lambda|w|}(w) \cap K(\pi,1)\ \emptyset. \qquad (*)$$



We also take $\{\sigma_h\}_{h=1}^{N}$ k-dimensional planes "covering" the sphere, namely s.t. $\min_{h=1,\dots,N} d(\sigma_h,\pi) < \lambda$, for a certain λ to be fixed.



Step 1: we first show that, for $M' \subseteq M$ compact and

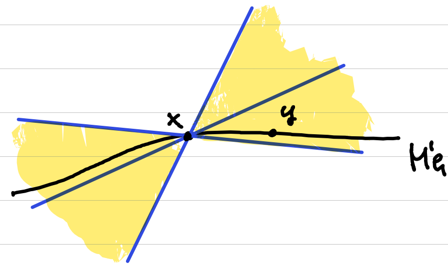
$$\lim_{r\to 0^+} \frac{\mu(B_r(x))}{r^k \omega_k} = 1, \qquad \lim_{r\to 0^+} \frac{\mu(B_r(x)\setminus(x+K(\pi_x,1)))}{r^k\omega_k} = 0$$

uniformly on M', then M' is $\mathcal{H}^k$-rectifiable. Indeed, this yields that $\forall \varepsilon>0\ \exists\, \delta>0$ s.t. $\forall\, |r|<2\delta\ \forall\, x\in M'$

$$\mu(B_r(x)) > (1-\varepsilon) r^k\omega_k, \qquad \mu(B_r(x)\setminus(x+K(\pi_x,1))) < \varepsilon r^k \omega_k.$$

Let $M'_q := \{x \in M' : \mathrm{dist}(\pi_q, \pi_x) < \lambda\}$. This is the set of points whose approximate tangent space is closer to π_q than to the other planes.

Take $x \in M'_q$, $y \in M'_q \cap B_\delta(x)$ but assume by contradiction that $y \in \mathbb{R}^n \setminus (x + K(\pi_q, 2))$



By property $(*)$, $B_{\lambda|y-x|}(y-x) \cap K(\pi_x, 1)$, thus

$$B_{\lambda|y-x|}(y) \subseteq B_{2|y-x|}(x) \setminus (x + K(\pi_x, 1))$$

but by the assumption

$$(1-\varepsilon)\omega_k \lambda^k |y-x|^k \leq \mu(B_{\lambda|y-x|}(y)) \leq \mu(B_{2|y-x|}(x) \setminus (x+K(\pi_x,1)))$$
$$\leq \varepsilon \omega_k 2^k |y-x|^k$$

which gives a contradiction for ε small enough. Hence we proved that $M'_q \cap B_\delta(x) \subseteq x + K(\pi_x, 2) \Rightarrow$ by the rectifiability criterion M'_q is $\mathcal{H}^k$-rectifiable $\Rightarrow$ so is M', as it is finite union of $\mathcal{H}^k$-rectifiable sets.

step 2: by the weak-$*$ convergence of the blow-ups, and since $\mathcal{H}^k \llcorner \pi_x(\partial B_1) = 0$, $\forall x \in M$

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{r^k \omega_k} = 1, \qquad \lim_{r \to 0^+} \frac{\mu(B_r(x) \setminus (x + K(\pi_x, 1)))}{r^k \omega_k} = 0.$$

Denoting $f_r(x) := \mu(B_r(x)) \cdot r^{-k} \omega_k^{-1}$, $g_r(x) := \mu(B_r(x) \setminus (x + K(\pi_x,1))) \cdot r^{-k} \omega_k^{-1}$ we have $f_r \to 1$, $g_r \to 0$ pointwise.

By Egoroff's theorem, $\forall R>0$ $\exists M' \subset M \cap B_R$ compact and $\mu((M\cap B_R)\setminus M') < 1/2$ s.t. $f_r \rightrightarrows 1$, $g_r \rightrightarrows 0$ on M'. So M' is $\mathcal{H}^k$-rectifiable by step 1.

We iterate this by taking $M'' \subseteq (M\cap B_R)\setminus M'$, $\mu((M\cap B_R)\setminus(M'\cup M'')) < 1/4$ and so on. By this, M is locally $\mathcal{H}^k$-rectifiable

step 3: by $f_r(x) \to 1$ as $r \to 0^+$ $\forall x \in M$, by Lebesgue-Besicovitch differentiation $\mu = \mathcal{H}^k \llcorner M$. $\boxtimes$