Lecture 5: 27.04.2026

By splitting rectifiable sets into regular Lipschitz images, we can prove that the approximate tangent space $\exists$ $\mathcal{H}^k$-a.e.

<u>Note</u>: in the proof below we need the following technical fact: given $M_0 \subseteq M \subseteq \mathbb{R}^n$ with $\mathcal{H}^k(M) < \infty$, then

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k(B_r(x) \cap (M \setminus M_0))}{r^k \omega_k} = 0, \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in M_0.$$

Indeed, for $\delta > 0$ let $F_\delta := \{x \in M_0 : D^+_{\mathcal{H}^k}(\mathcal{H}^k \llcorner (M \setminus M_0))(x) \geq \delta\}$, so for every $x \in F_\delta$ $\exists\, r = r(x) < \delta/2$ s.t. $\mathcal{H}^k(B_r(x) \cap (M \setminus M_0)) > \delta/2\, r^k \omega_k$. So let $\mathcal{F} := \{B_{r(x)}(x) ,\ x \in F_\delta\}$. By Besicovitch covering $\exists\, \mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)}$ disjoint, countable subcoverings. As the balls are disjoint

$$\mathcal{H}^k_\delta(F_\delta) \leq \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} \omega_k r^k \leq 2/\delta \sum_{i=1}^{\xi(n)} \Big(\sum_{B \in \mathcal{F}_i} \mathcal{H}^k(B \cap (M \setminus M_0)) \Big)$$

$$\leq 2/\delta\ \xi(n)\ \mathcal{H}^k(F_\delta \setminus M_0) = 0,$$

so $\mathcal{H}^k_\delta(F_\delta) = 0$ $\forall \delta$ which proves the claim.

For this we could not simply use Lebesgue-Besicovitch differentiation (as done in an example in Lecture 3) since $\mathcal{H}^k$ is not a Radon mesure on $\mathbb{R}^n$.

<u>proof</u> (Theorem 2): by Theorem 1 of last lecture, $M = M_0 \cup \bigcup_h f_h(E_h)$ where $f_h(E_h)$ are regular Lipschitz images and $\mathcal{H}^k(M_0) = 0$.

By the last lemma of Lecture 4, $\forall h$, for $M_h = f_h(E_h)$, $\forall x = f(z)$, $\pi_x = \nabla f(z)(\mathbb{R}^k)$ and

$$(\mathcal{H}^k \llcorner M_h)_{x,r} \xrightarrow[r \to 0^+]{*} \mathcal{H}^k \llcorner \pi_x.$$

From the note above, for $\mathcal{H}^k$-a.e. $x \in M$, $\exists h$ s.t. $x \in M_h$ and $\forall \varphi \in C_c(\mathbb{R}^n)$

$$\lim_{r\to 0^+} \int_{\mathbb{R}^n} \varphi(x')\, d(\mathcal{H}^k \llcorner M)_{x,r}(x') = \lim_{r\to 0^+} \int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$= \lim_{r\to 0^+} \int_{M_h} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$= \lim_{r\to 0^+} \int_{\mathbb{R}^n} \varphi(x')\, d(\mathcal{H}^k \llcorner M_h)_{x,r}(x')$$

$$= \int_{\nabla f(z)(\mathbb{R}^k)} \varphi(x')\, d\mathcal{H}^k(x'),$$

which proves the weak-$*$ convergence as claimed.

Moreover, as $\mathcal{H}^k \llcorner \pi_x(\partial B_1) = \mathcal{H}^k(\pi_x \cap \partial B_1) = 0$, since $\partial B_1 \cap \pi_x \simeq \simeq S^{k-1}$, by the properties of the weak-$*$ convergence

$$\lim_{r\to 0^+} \frac{\mathcal{H}^k(M \cap B_r(x))}{r^k \omega_k} = \lim_{r\to 0^+} \frac{(\mathcal{H}^k \llcorner M)_{x,r}(B_1)}{\omega_k} = \frac{\mathcal{H}^k \llcorner \pi_x(B_1)}{\omega_k} = 1. \qquad \square$$

One last remark is that also the converse result holds true, namely, if $\mathcal{H}^k \llcorner M$ has an approximate tangent plane $\mathcal{H}^k$ a.e. then M is $\mathcal{H}^k$-rectifiable.

This will be useful in studying the structure of sets of finite perimeter, but we do not see the proof [you find it in an addendum of lecture 5]

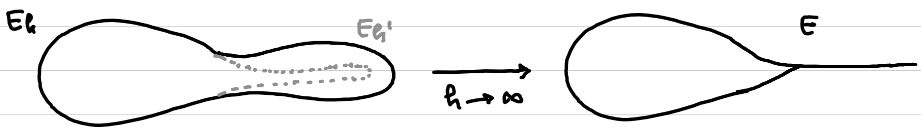
Theorem: let μ be a Radon measure on $\mathbb{R}^n$ which concentrates on $M \in \mathcal{B}(\mathbb{R}^n)$. If, $\forall x \in M$ $\exists$ k-dimensional plane $\pi_x \subseteq \mathbb{R}^n$ s.t.

$$\mu_{x,r} \xrightarrow[r\to 0^+]{*} \mathcal{H}^k \llcorner \pi_x,$$

then $\mu = \mathcal{H}^k \llcorner M$ and M is $\mathcal{H}^k$-rectifiable.

1. Sets of finite perimeter

Our objective is to use variational methods to solve geometrical problems (e.g. the isoperimetric problem). For this we need to be able to treat sequences of sets.



Along sequences of sets, some well defined quantities, like the perimeter, may degenerate in the limit.
In the sequence $E_h \to E$ in the picture above, even if the perimeter of E_h for each h is a clear notion, how to define the perimeter of their limit E?

For this we need to generalize the notion of perimeter.

Def: a set $E \subseteq \mathbb{R}^n$ Lebesgue measurable is a set of locally finite perimeter iff, for every $K \subseteq \mathbb{R}^n$ compact,

$$C(K) := \sup\left\{\int_E \operatorname{div} T(x)\,dx \,:\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), \|T\|_\infty \le 1, \operatorname{spt}(T) \subseteq K\right\}.$$

If $C(K)$ is uniformly bounded $\forall K \subseteq \mathbb{R}^n$ compact (i.e. $\exists\, C>0 : C(K) \le C$) then we say that E is a set of finite perimeter.

We immediately provide a characterization of sets of (locally) finite perimeter.

Proposition 1 (characterization): E is of locally finite perimeter $\iff$ $\exists$ an $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ s.t.

$$\int_E \operatorname{div} T(x)\,dx = \int_{\mathbb{R}^n} T(x)\cdot d\mu_E(x), \quad \forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n).$$

Moreover, E is of finite perimeter $\iff |\mu_E|(\mathbb{R}^n) < +\infty$.

Rmk (notation): we call μ_E the Gauss-Green measure of E and $\forall\, F \subseteq \mathbb{R}^n$, we call

$$P(E;F) := |\mu_E|(F) \qquad \text{and} \qquad P(E) := |\mu_E|(\mathbb{R}^n)$$

the relative perimeter of E in F and the perimeter of E, respectively.
Notice that, by Riesz Theorem, $A \subseteq \mathbb{R}^n$ is open

$$P(E;A) = |\mu_E|(A) = \sup\left\{ \int_{\mathbb{R}^n} \varphi \cdot d\mu_E : \varphi \in C_c(A;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}$$

$$= \sup\left\{ \int_E \operatorname{div} T(x)\, dx : T \in C_c^1(A;\mathbb{R}^n),\ \|T\|_\infty \leq 1 \right\}.$$

Proof (Proposition 1): we prove the iff part.

($\Rightarrow$) consider the operator $L : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ defined as

$$LT := \int_E \operatorname{div} T(x)\, dx.$$

This is linear and, for every compact $K \subseteq \mathbb{R}^n$, by definition

$$|LT| \leq C(K)\, \|T\|_\infty, \quad \forall\, T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt}(T) \subseteq K.$$

Since $C_c^1(\mathbb{R}^n;\mathbb{R}^n) \subseteq C_c(\mathbb{R}^n;\mathbb{R}^n)$ is dense in the uniform norm, we can extend uniquely L, via Hahn-Banach theorem, to a linear operator $\tilde{L} : C_c(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \mathbb{R}$ still with $|\tilde{L}\varphi| \leq C(K)\|\varphi\|_\infty$ for every $\varphi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$.
By Riesz Theorem $\exists\ \mu_E$ $\mathbb{R}^n$-valued Radon measure s.t.

$$\tilde{L}\varphi = \int_{\mathbb{R}^n} \varphi(x) \cdot d\mu_E(x).$$

Since, for $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, $LT = \tilde{L}T$ we get the desired result.

($\Leftarrow$) if such a μ_E exists then

$$C(K) = \sup\left\{ \int_E \operatorname{div} T(x)\, dx : T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|T\|_\infty \le 1,\ \operatorname{spt}(T) \subseteq K \right\}$$

$$\le \sup\left\{ \int_{\mathbb{R}^n} \varphi \cdot d\mu_E : \varphi \in C_c(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1,\ \operatorname{spt}(\varphi) \subseteq K \right\} < +\infty$$

as μ_E is locally finite.

Finally, from the characterization, $C(K) \le C$ $\forall K \subseteq \mathbb{R}^n$ compact $\Leftrightarrow$ $|\mu_E|(\mathbb{R}^n) < +\infty$. $\boxtimes$

1.1. The divergence theorem

So far, the definitions of sets of finit perimeter and of perimeter are a bit "mysterious" and are not clearly related to a notion of "surface area of the boundary", as in the regular case.

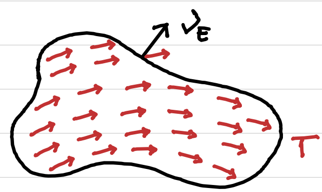
To better explain sets of finite perimeter, it is usefue to recall the divergence theorem, in its classical formulation.

Given $T \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$ compact with C^1 boundary then

$$\int_E \operatorname{div} T(x)\, dx = \int_{\partial E} T \cdot \nu_E \, d\sigma$$

where ν_E is the outer normal vector to ∂E and $d\sigma$ is the infinitesimal surface area.

Loosely speaking, the divergence theorem relates a "bulk" quantity, the integral of the divergence of a vector field, with a "boundary" quantity, namely its outward flux through the boundary.



It is a convenient way to define the boundary of a Lebesgue set. Indeed, for $E \subseteq \mathbb{R}^n$, ∂E has in many cases measure 0, but Lebesgue measurable sets are identified up to set of measure zero.

Since an integration in the bulk "does not see" modifications of zero Lebesgue measure, a notion of boundary defined via bulk integration will be well defined for every set in the same Lebesgue class.

<u>Rmk</u> (generalized divergence theorem): let $f : A \subseteq \mathbb{R}^{n-1} \to \partial E \subseteq \mathbb{R}^n$ be a C^1-parametrization of ∂E (i.e. $Jf > 0$).
Then, the classical integral on surface can be written, with the area formula, as follows

$$\int_{\partial E} F \cdot \nu_E \, d\sigma = \int_A F(f(x')) \cdot \nu_E(f(x')) \, Jf(x') \, dx'$$

$$\overset{AF}{=} \int_{f(A)} F \cdot \nu_E \, d\mathcal{H}^{n-1} = \int_{\partial E} F \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

So one can prove [Chaper 9, Maggi (2012)] a "generalized" version of the divergence theorem.
For every $E \subseteq \mathbb{R}^n$ compact with Lipschitz regular boundary, for every $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ with $\mathrm{spt}(T) \subseteq E$ it holds that

$$\int_E \operatorname{div} T(x) \, dx = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

<u>Exc</u>: write it explicitely for the case of curves in $\mathbb{R}^2$ and of surfaces in $\mathbb{R}^3$.

1.2. <u>Examples and discussion</u>

(i)
- from what discussed above, $E \subseteq \mathbb{R}^n$ with C^1 boundary is a set of locally finite perimeter, since

$$\int_E \operatorname{div} T(x) \, dx = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial E \cap K).$$

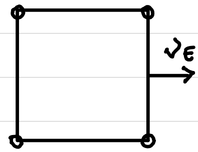
In particular, $\mu_E = \nu_E (\mathcal{H}^{n-1} \llcorner \partial E)$ and $P(E; F) = |\mu_E|(F) = \mathcal{H}^{n-1}(\partial E \cap F)$.

(ii)

- if $E \subseteq \mathbb{R}^n$ has Lipschitz boundary is of locally finite perimeter. By Rademacher and the Area formula ν_E is defined $(\mathcal{H}^{n-1} \llcorner \partial E)$-a.e. and $\mu_E = \nu_E (\mathcal{H}^{n-1} \llcorner \partial E)$ and $P(E;F) = \mathcal{H}^{n-1}(\partial E \cap F)$.

A particular case are polyhedral sets, i.e. $\nu_E \in \{\nu_i\}_{i=1}^N \subseteq \mathbb{S}^{n-1}$ $(\mathcal{H}^{n-1} \llcorner \partial E)$-a.e.

Even if ν_E is not always well-defined (e.g. in the vertices) it is $\mathcal{H}^{n-1}$-a.e. on ∂E.



<u>Note</u>: the cases from above, namely ∂E (at least) Lipschitz regular, are treated "by hands" with the (generalized) divergence theorem.

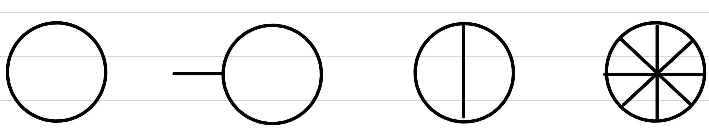
To have a <u>complete</u> understanding of sets of finite perimeter we will need De Giorgi's Structure Theorem that states that $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$, where $\partial^* E$ is a sort of generalization of the notion of ∂E.

<u>Rmk</u>: if E is of locally finite perimeter and $F \subseteq \mathbb{R}^n$ s.t. $|E \Delta F| = 0 \iff F$ is of locally finite perimeter and $\mu_E = \mu_F$, as they induce the same linear operator, i.e.

$$\int_E \operatorname{div} T(x)\, dx = \int_F \operatorname{div} T(x)\, dx.$$

As a consequence of this, we can always assume that sets of finite perimeter are Borel regular.

As we commented above, this remark tells us that the perimeter is not affected by modifications of zero Lebesgue measure.



All the sets in the picture have the same perimeter.

Exc: given $I := (-\pi,\pi) \cap \mathbb{Q}$ and consider $\ell_j = \{(\cos\theta_j, \sin\theta_j)t \,;\, t \in \mathbb{R}\}$ for any $\theta_j \in I$.
Let $E \subseteq \mathbb{R}^2$ be $E = B \setminus \bigcup_{j=1}^{\infty} \ell_j$. What is $P(E)$?

Some useful properties of the perimeter, that are easily obtained by definition are left to prove as an excercise.

Exc (some useful properties): prove that

- $P(x+\lambda E) = \lambda^{n-1} P(E)$, $\forall x \in \mathbb{R}^n$, $\lambda > 0$
- $\mu_{\mathbb{R}^n \setminus E} = -\mu_E$, thus $P(E;F) = P(\mathbb{R}^n \setminus E; F)$ $\forall F \subseteq \mathbb{R}^n$.

When talking about sets of finite perimeter, it is impossible not to mention functions of bounded variations.
It turns out that an equivalent definition of sets of finite perimeter can be given via distributions.

Rmk (functions of bounded variations): $u \in L^1_{loc}(\mathbb{R}^n)$ is called a function of locally bounded variations, and $u \in BV_{loc}(\mathbb{R}^n)$, iff

$$\sup\left\{ \int_{\mathbb{R}^n} u(x)\, \operatorname{div} T(x)\, dx \,:\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|T\|_\infty \leq 1,\ \operatorname{spt}(T) \subseteq K \right\} < +\infty$$

for every $K \subseteq \mathbb{R}^n$ compact.
A characterization of BV function (the analog of Proposition 1) is that their distributional derivative, i.e.

$$Du : C^\infty_c(\mathbb{R}^n) \to \mathbb{R}, \quad \langle Du, \varphi \rangle := -\int_{\mathbb{R}^n} u(x)\, D\varphi(x)\, dx,$$

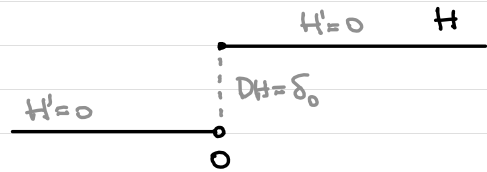
can be extended to an $\mathbb{R}^n$-valued Radon measure. So that $BV_{loc}(\mathbb{R}^n) := \{ u \in L^1_{loc}(\mathbb{R}^n) : Du \in \mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^n) \}$, where $\mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued Radon measures.

Loosely speaking, BV-functions are functions whose derivative is a (Radon) measure.

The simplest example possible is the Heaviside function, namely

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, \qquad \langle DH, \varphi \rangle = \int_0^{+\infty} -\varphi'(x)\,dx = \varphi(0) = \delta_0 \varphi .$$

So $DH = \delta_0$, which is a Radon measure, so $H \in BV_{loc}(\mathbb{R})$.



With this approach, clearly E is of locally finite perimeter $\iff \chi_E \in BV_{loc}(\mathbb{R}^n)$. Moreover, $D\chi_E = -\mu_E$ since $\forall \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi(x) \cdot dD\chi_E(x) = -\int_{\mathbb{R}^n} \chi_E(x) \operatorname{div}\varphi(x)\,dx$$

$$= -\int_E \operatorname{div}\varphi(x)\,dx = -\int_{\mathbb{R}^n} \varphi(x) \cdot d\mu_E(x).$$

In particular, $Per(E;F) = |D\chi_E|(F)$.

Loosely speaking, sets of finite perimeter are those sets whos characteristic function is BV.