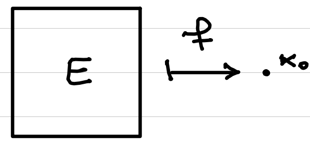
Lecture 4: 21.04.2025

In order to prove the Area formula in its general formulation, we provide (without a proof) the following two results (cf. for a proof see [Theorems 8.7, 8.8, Maggi (2012)].

<u>Lemma 1</u>: if $f:\mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a Lipschitz function, then

$$\mathcal{H}^k(f(E))=0, \qquad E=\{x\in\mathbb{R}^k : Jf(x)=0\}.$$

As we discussed at the end of the previous Lecture notes, Jf "measures" how volumes transform.
The lemma above tells us that, when $Jf=0$, we "loose" k-dimensional volume in the codomain. The most trivial example is the constant function $f(x)\equiv x_0 \in \mathbb{R}^n$.



<u>Exc</u>: given $T\in\mathbb{R}^{2\times 2}$ a matrix of rank 1, find $T(B_1)$ and verify that $\mathcal{H}^2(T(B_1))=0$.

In the next lemma we see that we can approximate Lipschitz function with piecewise affine ones, suitabley.

<u>Lemma 2</u> (Lipschitz linearization): let $f:\mathbb{R}^k\longrightarrow\mathbb{R}^n$ be a Lipschitz function and let $F=\{x\in\mathbb{R}^k : 0<Jf(x)<+\infty\}$. Then, $\forall t>1$ $\exists$ disjoint $\{F_h\}\subseteq\mathcal{B}(\mathbb{R}^k)$ s.t. $F=\bigcup_h F_h$ and $S_h\in GL(k)$ s.t.

$$t^{-1}|S_hx-S_hy|<|f(x)-f(y)|<t|S_hx-S_hy|,$$

$$t^{-1}|S_hv|<|\nabla f(x)v|<t|S_hv|, \quad \forall v\in\mathbb{R}^k$$

$$t^{-k}JS_h\leq Jf(x)\leq t^k JS_h,$$

$\forall x,y\in F_h$, $\forall h$.

<u>note</u>: intuitively $S_h \in \mathbb{R}^{k\times k}$ in the previous stement approximate the symmetrice part of ∇f in its polar decomposition

<u>Rmk</u>: by the change of variable $x' = S_h x$ on F_h, and by denoting $f_h := f \circ S_h^{-1}$, the first chain of inequalities in the statement above reads

$$t^{-1}|y'-x'| \leq |f_h(y') - f_h(x')| \leq t\,|y'-x'|, \quad \forall\, x',y' \in S_h(F_h).$$

We are now in position to see the proof of the Area formula.

<u>proof</u> (of the Area formula): as $\mathcal{H}^k(f(E)) \leq \mathrm{Lip}(f)^k \mathcal{L}^k(E)$ (see Lecture 2 and the addendum), the Area formula holds for null sets. Hence, by Rademacher's theorem (i.e. Lipschitz functions are a.e. differentiable) and Lemma 1, we can assume that $E \subseteq F = \{x \in \mathbb{R}^k : 0 < Jf(x) < +\infty\}$.

Let $t>1$, $\{F_h\}$ and S_h as in Lemma 2. By injectivity $f(E) = \bigcup_h f(F_h \cap E)$, where the union is disjoint. Then by the validity of the area formula in the linear case

$$\mathcal{H}^k(f(E)) = \sum_h \mathcal{H}^k(f(F_h\cap E)) = \sum_h \mathcal{H}^k(f_{|F_h} \circ S_h^{-1}(S_h(F_h\cap E)))$$

$$\leq \sum_h \mathrm{Lip}(f_{|F_h} \circ S_h^{-1})^k\, \mathcal{H}^k(S_h(F_h\cap E))$$

$$\overset{\text{Lemma 2}}{\leq} \sum_h t^k\, \mathcal{H}^k(S_h(F_h\cap E)) \overset{\text{linear}}{=} \sum_h t^k \int_{F_h\cap E} JS_h\, dx$$

$$\overset{\text{Lemma 2}}{\leq} \sum_h t^{2k} \int_{F_h\cap E} Jf(x)\, dx = \int_E Jf(x)\, dx.$$

By repeating the same steps in the chain of inequalities above, but using the opposite inequality provided by Lemma 2, we get the result. $\square$

Corollary: let $g : \mathbb{R}^n \to [-\infty, +\infty]$ Borel s.t. either $g \geq 0$ or $L^1(\mathbb{R}^n; \mathcal{H}^k \llcorner f(\mathbb{R}^k))$, then

$$\int_{f(\mathbb{R}^k)} g(y)\, d\mathcal{H}^k(y) = \int_{\mathbb{R}^k} g(f(x))\, Jf(x)\, dx.$$

Ex: for $k=1$, $\gamma : [0,a] \to \Gamma \subseteq \mathbb{R}^n$, injective and Lipschitz then the Area formula reads

$$\mathcal{H}^1(\Gamma) = \int_{[0,a]} |\gamma'(t)|\, dt = L(\Gamma)$$

recovering the result from Lecture 2.

Ex: for $k=n$, $E \subseteq \mathbb{R}^n$, $\phi : E \to \phi(E) \subseteq \mathbb{R}^n$ injective and Lipschitz and let $g \in L^1(\mathbb{R}^n)$ then the Area formula reads

$$\int_{\phi(E)} g(y)\, dy = \int_{\mathbb{R}^n} g(y)\, \chi_{\phi(E)}(y)\, dy$$

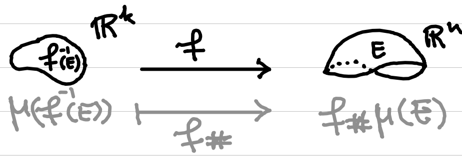
$$= \int_{\mathbb{R}^n} g(\phi(x))\, \chi_E(x)\, Jf(x)\, dx = \int_E g(\phi(x))\, Jf(x)\, dx$$

which is the change of variables formula.

1. Rectifiable sets

Thanks to the Area formula, it is now certified that $\mathcal{H}^k$ generalizes the notion of k-dimensional surface area. We now see how to generalize the notion of k-dimensional surfaces. These will be rectifiable sets.

Before discussing this, we recall the notion of push-forward of Radon measures, that will be useful in the following.



Given $f:\mathbb{R}^k \to \mathbb{R}^n$ and μ an outer measure on $\mathbb{R}^k$ we define the push-forward measure through f as $f_\#\mu : \mathcal{P}(\mathbb{R}^n) \to [0,+\infty]$,

$$f_\#\mu(E) = \mu(f^{-1}(E)).$$

If we have a measure defined on $\mathbb{R}^k$, we can push it forward to $\mathbb{R}^n$ by pulling back sets $E \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$, measure them with μ and then push it back forward, defining a measure on $\mathbb{R}^n$.

<u>Exc</u>: prove that $f_\#\mu$ is an outer measure.

An important property is that, if f is continuous and proper and μ is Radon $\Rightarrow$ $f_\#\mu$ is Radon, $\operatorname{spt}(f_\#\mu) = f(\operatorname{spt}(\mu))$ and, for any Borel $u:\mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} u(y)\, d f_\#\mu(y) = \int_{\mathbb{R}^k} u(f(x))\, d\mu(x).$$

The notion of push-forward can of course be given in more general measure (topological) spaces.

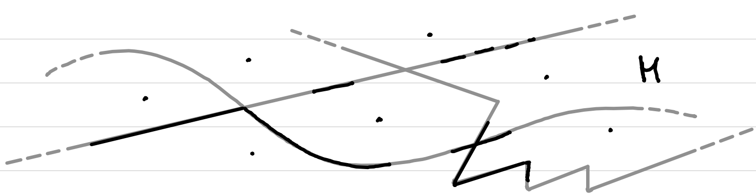
Def: an $\mathcal{H}^k$-measurable set $M \subseteq \mathbb{R}^n$ is called *countably $\mathcal{H}^k$-rectifiable* iff $\exists \{f_h : \mathbb{R}^k \to \mathbb{R}^n\}_h$ Lipschitz functions s.t.

$$\mathcal{H}^k(M \setminus \cup_h f_h(\mathbb{R}^k)) = 0.$$

If additionally $\mathcal{H}^k(M \cap K) < +\infty$ $\forall K \subseteq \mathbb{R}^n$ compact we say that M is *locally $\mathcal{H}^k$-rectifiable*.
If $\mathcal{H}^k(M) < +\infty$ we simply say *$\mathcal{H}^k$-rectifiable*.

$\mathcal{H}^k$-rectifiable sets generalize the notion of embedded surfaces. In words, by definition, $\mathcal{H}^k$-rectifiable sets are those sets that are contained in (possibly countable) union of Lipschitz images of k-dimensional "parameter" sets.

By the Area formula, as locally $M \subseteq f(E)$ for some $E \subseteq \mathbb{R}^k$, it is regular enough to have a notion of k-dimensional surface area encoded.



1.1. Decomposition of rectifiable sets

By Lemma 2, we can always split rectifiable sets into union of sufficiently regularly parametrized sets.

Def: given $f : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz and $E \subseteq \mathbb{R}^k$ Borel, bounded we say that $f(E)$ is a *regular Lipschitz image* iff

(i) f is injective and differentiable, with $Jf(x) > 0$ $\forall x \in E$
(ii) $\forall x \in E$, $\Theta_k(E)(x) = 1$ (cf. Lecture 3 for def. of $\Theta_k(E)$)
(iii) every $x \in E$ is a Lebesgue point of ∇f
(iv) $\exists \lambda > 0$ s.t. $|f(x) - f(y)| > \lambda |x - y|$, $\forall x, y \in E$

Rmk: (ii) is a measure theoretical equivalent of requesting that E is open.

We recall that, given $u \in L^1_{loc}(\mathbb{R}^k;\mathbb{R}^m)$, x is a Lebesgue point of u iff

$$\lim_{r\to 0^+} \frac{1}{|B_r|}\int_{B_r(x)} |u(y)-u(x)|\,dy = 0.$$

For L^1_{loc} functions, a.e. x is a Lebesgue point.

Theorem 1: let M be a (countably) $\mathcal{H}^k$-rectifiable set and let $t>1$. Then $\exists\, M_0 \in \mathcal{B}(\mathbb{R}^n)$, $\{f_h:\mathbb{R}^k\to\mathbb{R}^n\}_h$ Lipschitz, $\{E_h\}_h \subseteq \mathcal{B}_b(\mathbb{R}^n)$ s.t. $\forall h$, $f_h(E_h)$ are regular Lipschitz images and

$$M = M_0 \cup \bigcup_h f_h(E_h), \qquad \mathcal{H}^k(M_0)=0,$$

$$t^{-1}|x-y| \leq |f_h(x)-f_h(y)| \leq t|x-y|,$$

$$t^{-1}|v| \leq |\nabla f_h(x) v| \leq t|v|, \ \forall v\in\mathbb{R}^k,$$

$$t^{-k} \leq Jf_h(x) \leq t^k,$$

$\forall\, x,y \in E_h$.

proof (not discussed in class): by definition, $M = \tilde{M}_0 \cup \bigcup_h g_h(\mathbb{R}^k)$ for some g_h Lipschitz and $\mathcal{H}^k(\tilde{M}_0)=0$.

step 1: for any h, consider

$$G_h := \{x\in\mathbb{R}^k : f_h \text{ is diff.}, Jf_h(x)>0, x \text{ is Lebesgue of } \nabla g_h\}.$$

By Rademacher, the fact that a.e. x is a Lebesgue point, Lemma 1, and the lipschitzianity of g_h, $\mathcal{H}^k(g_h(\mathbb{R}^k\setminus G_h))=0$. So, writing $M_0' := \tilde{M}_0 \cup \bigcup_h g_h(\mathbb{R}^k\setminus G_h)$ we have

$$M = M_0' \cup \bigcup_h g_h(G_h), \qquad \mathcal{H}^k(M_0')=0.$$

Analogously, denoting $G_h^{(1)} = \{x \in G_h : \Theta_k(G_h)(x) = 1\}$, we know (cf. an example in Lecture 3) that $|G_h \Delta G_h^{(1)}| = 0$, writing $M_0'' = M_0' \cup \bigcup_h g_h(G_h \setminus G_h^{(1)})$, we get

$$M = M_0'' \cup \bigcup_h G_h^{(1)}, \qquad \mathcal{H}^k(M_0'') = 0.$$

So g_h and $G_h^{(1)}$ comply with (i)-(iii). We are missing that the sets are not bounded yet, and property (iv). We can always $G_h^{(1)}$ are Borel.

Step 2: as $g_1(G_1^{(1)}) = g_1(G_1^{(1)} \cap \bigcup_j B_j) = \bigcup_j g_1(G_1^{(1)} \cap B_j)$, we get

$$M = M_0'' \cup \bigcup_{j,h} g_h(G_h^{(1)} \cap B_j), \qquad \mathcal{H}^k(M_0'') = 0.$$

So up to changing the numbering, we can assume $G_h^{(1)}$ to be bounded.

Step 3: similarly as in the previous step, considering $g_1, G_1^{(1)}$, by Lemma 2, $\exists \{G_j'\}$ disjoint $G_1^{(1)} = \bigcup_j G_j'$, $S_j \in GL(k)$ s.t., for a given $t > 1$, $\forall x, y \in G_j'$, $\forall j$, the inequalities in Lemma 2 hold true.
Taking $F_j := S_j(G_j')$, $f_j := g_1 \circ S_j^{-1}$, we have $\forall x, y \in F_j$

$$|f_j(x) - f_j(y)| \sim_t |x-y|, \quad |\nabla f_j(x) v| \sim_t |v|, \ \forall v \in \mathbb{R}^k, \quad Jf(x) \sim_{t^k} 1,$$

where we use the notation $a \sim_t b \iff t^{-1} b < a < t b$. Again, up to a change in the numbering, since (iv) is automatically satisfied, we conclude the proof. $\boxtimes$

2. Approximate tangent space

As rectifiable sets generalize surfaces, we may expect to be able to define a sort of tangent space in (almost) every point.
This is possible, locally, by taking limits of blow-ups of the $\mathcal{H}^k$-measure.

Def: given $x \in \mathbb{R}^n$, $r > 0$, let $\Phi_{x,r} : \mathbb{R}^n \ni y \mapsto \frac{y-x}{r} \in \mathbb{R}^n$. Given μ a Radon measure on $\mathbb{R}^n$, we define its k-dimension blow-up measure at x as $\mu_{x,r} := (\Phi_{x,r})_\# \mu$.

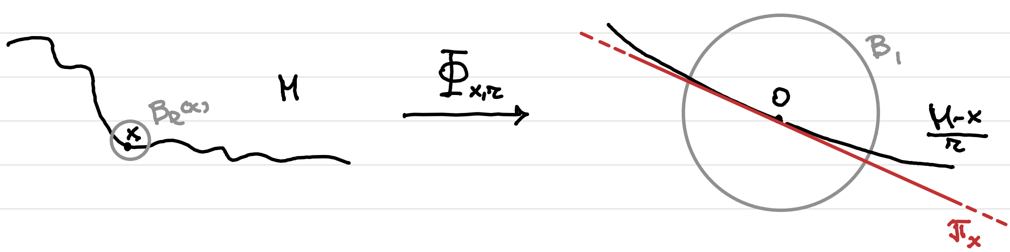
The blow-up measure at x, it is a sort of "zoom" in a neighbourhood of radius r of x.

Theorem (approximate tangent space): let M be (locally) $\mathcal{H}^k$-rectifiable. Then, for $\mathcal{H}^k$-a.e. $x \in M$ $\exists!$ k-dimensional plane $\pi_x \subseteq \mathbb{R}^n$ s.t.

$$(\mathcal{H}^k \llcorner M)_{x,r} \xrightarrow{*} \mathcal{H}^k \llcorner \pi_x, \quad \text{as } r \to 0^+.$$

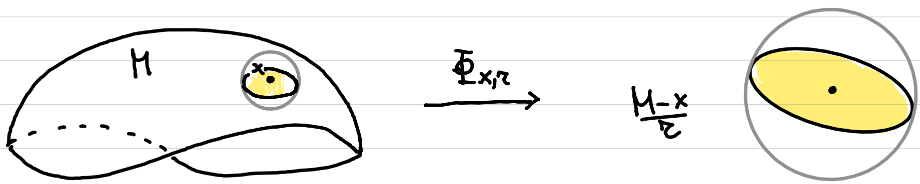
In particular, $D_{\mathcal{H}^k}(\mathcal{H}^k \llcorner M) \left(= \Theta_k(M)\right) = 1$ $\mathcal{H}^k$-a.e. on M, namely

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k(B_r(x) \cap M)}{r^k \omega_k} = 1, \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in M.$$



note: zooming-in close to (almost) every $x \in M$, M resembles a plane π_x, which is the support of the weak-$*$ limit of the blow-ups.

__note__: we will write $\pi_x = T_x M$ and we call it the approximate tangent plane to M at x.



__note__: the second part of the statement tells us that, zooming-in, M resembles a k-dimensional ball.

In the case of regular Lipschitz images, the computation of the approximate tangent space is easier and coincides with the C^1 case, namely is given by the derivatives of the parametrization.

__Lemma__: let $f:\mathbb{R}^k \to \mathbb{R}^n$ Lipschitz, $E \in \mathcal{B}_b(\mathbb{R}^n)$, and $f(E)=M$ be a regular Lipschitz image. Then

$$\pi_{f(z)} = \nabla f(z)(\mathbb{R}^n).$$

__Proof__: we test $(\mathcal{H}^k \llcorner M)_{x,r}$ on $\varphi \in C_c(\mathbb{R}^n)$, so

$$\int_{\mathbb{R}^n} \varphi(x')\, d(\mathcal{H}^k \llcorner M)_{x,r}(x') = \frac{1}{r^k}\int_M \varphi(\Phi_{x,r}(y))\, d\mathcal{H}^k(y)$$

$$= \frac{1}{r^k}\int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) \qquad (*)$$

$$\overset{AF}{=} \frac{1}{r^k}\int_E \varphi\left(\frac{f(w)-f(z)}{r}\right) Jf(w)\, dw$$

$$= \int_{\mathbb{R}^k} u_r(w')\, dw'$$

where we used the Area formula and, in the last identity, the change of variable $w' = \frac{w-z}{r}$, and

$$u_r(w') := \chi_E(z+rw')\,\varphi\left(\frac{f(z+rw')-f(z)}{r}\right) Jf(z+rw').$$

By differentiability of f and continuity of φ

$$\varphi\left(\frac{f(z+rw')-f(z)}{r}\right) \xrightarrow[r\to 0^+]{} \varphi(\nabla f(z)w'), \ \forall w' \in \mathbb{R}^k,$$

as every point is a Lebesgue point for ∇f ($\Rightarrow$ for Jf)

$$\int_{B_R} |Jf(z+rw') - Jf(z)|\, dw' = \frac{1}{|B_{rR}|}\int_{B_{rR}(z)} |Jf(y)-Jf(z)|\,dy \xrightarrow[r\to 0^+]{} 0$$

and analogously for $\chi_{\tilde{E}}(z+rw')$ to 1 (by property (i) of regular Lipschitz images). So, up to subsequences,

$$u_r(w') \xrightarrow[r\to 0^+]{} u_0(w') := \varphi(\nabla f(z)w')\,Jf(z), \quad \text{for a.e. } w' \in \mathbb{R}^k.$$

Moreover $\|u_r\|_{L^\infty} \leq \|\varphi\|_\infty \operatorname{Lip}(f)^k$, so if we prove that u_r has uniformly bounded support, we can conclude that u_r converges L^1 strongly to u_0 by dominated convergence.

For r sufficiently small, $w' \in \mathbb{R}^k$ s.t.

$$u_r(w') \neq 0 \iff \varphi\left(\frac{f(z+rw')-f(z)}{r}\right) \neq 0 \Rightarrow \frac{f(z+rw')-f(z)}{r} \in \operatorname{spt}(\varphi).$$

As φ has compact support, $\exists R>0$ s.t. $\operatorname{spt}(\varphi) \subseteq B_R$. Thus, by (iv) in the definition of regular Lipschitz image

$$u_r(w') \neq 0 \Rightarrow \lambda|w'| \leq \left|\frac{f(z+rw')-f(z)}{r}\right| \leq R$$

which implies that $\operatorname{spt}(u_r) \subseteq B_{R/\lambda}$, for r sufficiently small. Hence $u_r \to u_0$ in L^1 (since this independent of the subseq.) as $r \to 0^+$.

Taking the limit in equation (*) at the beginning of the proof, and using the area formula to $w' \mapsto \nabla f(z)w'$ we get

$$\begin{aligned}
\lim_{r\to 0^+} \int_{\mathbb{R}^k} \varphi(y)\, d(\mathcal{H}^k \llcorner M)_{x,r} &= \lim_{r\to 0^+} \int_{\mathbb{R}^k} u_r(w')\, dw' \\
&= \int_{\mathbb{R}^k} u_0(w')\, dw' \\
&= \int_{\mathbb{R}^k} \varphi(\nabla f(z) w') \nabla f(z)\, dw' \\
&\overset{AF}{=} \int_{\nabla f(z)(\mathbb{R}^k)} \varphi(x')\, d\mathcal{H}^k(x')
\end{aligned}$$

which concludes the proof. $\square$