We give here more details on the proof of the quantitative isoperimetric inequality for axis-symmetric (and n-symmetric) sets. This result was already proved by Hall in 1992, but its proof has been simplified by Fusco, Maggi, and Pratelli in 2008. We will follow their strategy.

We start by proving two lemmas for which we show we can reduce to study the deviation of E and B on the central slice.

<u>Lemma</u> 7: there exist $\delta(n,e), C_1(n,e) > 0$ s.t. $\forall\, E \in X_{e,\delta}$ axis-symmetric (with axis e_1), n-symmetric, and s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0,$$

then $\exists\, \bar{x} \in [0, \sqrt{3}/2]$ s.t. given $x' \geq 0$ defined by

$$\int_0^{x'} v_B(t)\, dt = \int_0^{\bar{x}} v_E(t)\, dt$$

then

$$|v_E(\bar{x}) - v_B(x')| \geq 1/C_1(n,e)\, |(E \Delta B) \cap Z|.$$

<u>proof</u>: as E is axis-symmetric and n-symmetric

$$|(E\Delta B) \cap Z| = 2 \int_0^{\sqrt{3}/2} |v_E(t) - v_B(t)|\, dt,$$

thus $\exists\, x_0 \in [0, \sqrt{3}/2]$ s.t. $|(E\Delta B)\cap Z| \leq \sqrt{2}\, |v_E(x_0) - v_B(x_0)|$. We now find a slice complying the volume constraint as in the claim. We assume, without loss of generality, that $v_B(x_0) > v_E(x_0)$ and work subdividing the casistic.

<u>case 1</u>: consider first the case

$$\int_0^{x_0} v_B(t)\,dt \geq \int_0^{x_0} v_E(t)\,dt.$$

We take $\bar{x} := x_0$. By the condition above and continuity of the integral $\exists\ 0 \leq x' \leq \bar{x}$ as in the statement. Since v_B is a decreasing function

$$v_B(x') - v_E(\bar{x}) \geq v_B(x_0) - v_E(x_0) \geq \frac{1}{\sqrt{2}} |(E \Delta B) \cap Z|$$

and the result is proven.

<u>case 2</u>: assume instead that

$$\int_0^{x_0} v_E(t)\,dt > \int_0^{x_0} v_B(t)\,dt \geq \int_0^{x_0} v_E(t)\,dt - \frac{1}{2^{\frac{n+2}{2}}(n-1)} |(E \Delta B) \cap Z|.$$

Again we consider $\bar{x} := x_0$ and $x' > \bar{x}$ as in the statement. By qualitative isoperimetric stability, up to take δ small enough, by Lemma 1 $|E \Delta B| \leq 2^n \alpha(E)$ so x' is as close as $\bar{x}$ as we want. Let δ be such that $v_B \geq \omega_{n-1} 2^{n/2}$ for $|t| \leq x'$. Hence

$$|\bar{x} - x'| \leq \frac{2^{n/2}}{\omega_{n-1}} \int_{\bar{x}}^{x'} v_B(t)\,dt = \frac{2^{n/2}}{\omega_{n-1}} \left(\int_0^{x'} v_B(t)\,dt - \int_0^{\bar{x}} v_B(t)\,dt \right)$$

$$= \frac{2^{n/2}}{\omega_{n-1}} \left(\int_0^{x_0} v_E(t) - v_B(t)\,dt \right) \leq \frac{1}{2(n-1)\omega_{n-1}} |(E \Delta B) \cap Z|$$

Noticing that $0 \geq v_B' \geq -(n-1)\omega_{n-1}$ we get

$$v_B(x') = v_B(\bar{x}) + \int_{\bar{x}}^{x'} v_B'(t)\,dt \geq v_B(\bar{x}) - (n-1)\omega_{n-1} |x' - \bar{x}|$$

$$\geq v_B(\bar{x}) - \frac{1}{2} |(E \Delta B) \cap Z| \geq v_E(\bar{x}) + \left(\sqrt{2} - \frac{1}{2}\right) |(E \Delta B) \cap Z|$$

which again yields the desired result.

Case 3: we are left to consider the case

$$\int_0^{x_0} \mathcal{V}_E(t)-\mathcal{V}_B(t)\,dt > \frac{1}{2^{\frac{n+2}{2}}(n-1)}|(E\Delta B)\cap Z|.$$

As there exists at least one point larger than the average, let $\bar{x}$ be the largest point in $[0,x_0]$ s.t.

$$\mathcal{V}_E(\bar{x})-\mathcal{V}_B(\bar{x}) \geq \frac{\sqrt{2}}{2^{\frac{n+2}{2}}(n-1)}|(E\Delta B)\cap Z|.$$

Hence, for every $t>\bar{x}$ the opposite inequality holds, so

$$\int_{\bar{x}}^{x_0} \mathcal{V}_E(t)-\mathcal{V}_B(t)\,dt \leq (x_0-\bar{x})\frac{1}{2^{\frac{n+2}{2}}(n-1)}|(E\Delta B)\cap Z|$$

$$\leq \frac{1}{2^{\frac{n+2}{2}}(n-1)}|(E\Delta B)\cap Z|.$$

So with x' as in the statement, since $\int_0^{\bar{x}} \mathcal{V}_E(t)-\mathcal{V}_B(t)\,dt>0$ then $x'\geq\bar{x}$ and then

$$\mathcal{V}_B(x')\leq\mathcal{V}_B(\bar{x})\leq\mathcal{V}_E(\bar{x})-\frac{1}{2^{\frac{n+2}{2}}(n-1)}|(E\Delta B)\cap Z|.$$

which concludes the proof. $\boxtimes$

With the next result we show that we can reduce to the case with $\bar{x}=x'=0$.

Lemma 8: let $\delta(n,\varepsilon)$ and E be as in the statement of Lemma 7. Then $\exists\, E'\in X_{\varepsilon,\delta}$ s.t.

$$\mathcal{H}^{n-1}(\{x\in\partial^*E' : \nu_{E'}(x)=\pm e_1\})=0$$

and

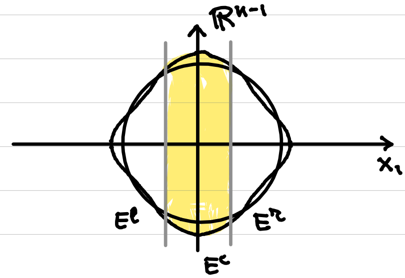
$$P(E')\leq P(E), \qquad |\mathcal{V}_{E'}(0)-\mathcal{V}_B(0)|\geq \tfrac{1}{C_1}|\mathcal{V}_E(\bar{x})-\mathcal{V}_B(x')|,$$

for some $C_1>0$ depending on n.

proof: given $\bar{x}$ as in Lemma 7, we split E into three parts $E = E^\ell \cup E^c \cup E^r$

$$E^\ell := E \cap ((-\infty, -\bar{x}] \times \mathbb{R}^{n-1}), \quad E^r := E \cap ([\bar{x}, +\infty) \times \mathbb{R}^{n-1}), \quad E^c := E \cap ((-\bar{x}, \bar{x}) \times \mathbb{R}^{n-1}).$$

The idea is to replace E^c with a portion of a ball having the same slice of E in $\bar{x}$. In this way the volume of the slice in 0 is explicitely determined by that of E in $\bar{x}$.



Let $r \in \mathbb{R}$ and consider B_{1+r} the ball of radius $1+r$ centered in zero. Let $x_c > 0$ be s.t.

$$|B_{1+r} \cap ((-x_c, x_c) \times \mathbb{R}^{n-1})| = |E^c| \quad \text{and} \quad v_{B_{1+r}}(x_c) = v_E(\bar{x}).$$

We use the notation $B^c := B_{1+r} \cap ((-x_c, x_c) \times \mathbb{R}^{n-1})$, $B^\ell := B_{1+r} \cap ((-\infty, -x_c] \times \mathbb{R}^{n-1})$, $B^r := B_{1+r} \cap ([x_c, +\infty) \times \mathbb{R}^{n-1})$.

Note that $r \in \mathbb{R}$ and $x_c > 0$ are uniquely determined. We then define

$$E' := (E^\ell + (\bar{x} - x_c)e_1) \cup B^c \cup (E^r + (x_c - \bar{x})e_1).$$

Obviously $|E'| = |E|$ and by isoperimetric inequality

$$P((B^\ell + (x_c - \bar{x})e_1) \cup E^c \cup (B^r + (\bar{x} - x_c)e_1)) \geq P(B^\ell \cup B^c \cup B^r)$$

which implies that $P(E^c) \geq P(B^c) \Rightarrow P(E') \leq P(E)$.

We are left to show that $|v_{E'}(0) - v_B(0)| \geq 1/C_1 |v_E(\bar{x}) - v_B(x')|$. For simplicity and without loss of generality we assume $v_E(\bar{x}) > v_B(x')$. In this case $r > 0$ and $x_c < x'$ by the fact the $|E \cap ((-\bar{x}, \bar{x}) \times \mathbb{R}^{n-1})| = |B \cap ((-x', x') \times \mathbb{R}^{n-1})|$.

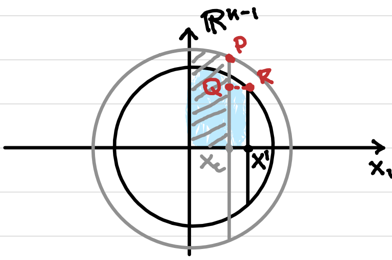
Since E' coincides with B_{1+r} in $(-x_c,x_c)\times\mathbb{R}^{n-1}$ we have

$$|v_{E'}(0)-v_B(0)| = \omega_{n-1}((1+r)^{n-1}-1) \geq (n-1)\omega_{n-1} r$$

by convexity of $t\mapsto t^{n-1}$. Hence, it is sufficient to show that $|v_E(\bar{x})-v_B(x')|\leq C_1 r$. This can be inferred by elementary geometrical considerations since $v_E(\bar{x})$ is a slice of a ball of radius $1+r$ and $v_B(x')$ is a (close) slice of a ball of radius 1.

Working as in the proof of Lemma 7, by qualitative stability of the isoperimetric inequality, x', x_c close to $\bar{x}\in[0,\sqrt{3}/2]$ so that $v_B(t)\geq 2^{-\frac{n+1}{2}}\omega_{n-1}$ for $x_c\leq t\leq x'$.

By definition of x' and x_c, $|B^c| = |B\cap((-x',x')\times\mathbb{R}^{n-1})|$ and hence we get $|(B_{1+r}\setminus B)\cap((0,x_c)\times\mathbb{R}^{n-1})| = |B\cap((x_c,x')\times\mathbb{R}^{n-1})|$.



As the first set is a section of the annulus $B_{1+r}\setminus B$, has measure (at most) of order r, i.e. $\exists\, \alpha_n>0$ such that $|(B_{1+r}\setminus B)\cap((0,x_c)\times\mathbb{R}^{n-1})|\leq\alpha_n r$.
Regarding the second we have

$$|B\cap((x_c,x')\times\mathbb{R}^{n-1})| = \int_{x_c}^{x'} v_B(t)\,dt \geq 2^{-\frac{n+1}{2}}\omega_{n-1}|x_c-x'|.$$

From this we have $|x_c-x'|\leq\tilde{\alpha} r$ for a suitable dimensional constant $\tilde{\alpha}$.

Now, by construction $v_{E'}(x_c)=\omega_{n-1}(\mathcal{H}^1([P,x_c e_1]))^{n-1}$ and $v_B(x')=\omega_{n-1}\cdot(\mathcal{H}^1([Q,x_c e_1])^{n-1}$ with P and Q as in the figure. Hence, by convexity and inverse triangle inequality

$$|v_{E'}(x_c)-v_B(x')|\leq C_1|\mathcal{H}^1([Q,x_c e_1])-\mathcal{H}^1([P,x_c e_1])|\leq C_1\mathcal{H}^1([P,Q])$$

but $\mathcal{H}^1([P,Q])\leq\tilde{C}_1 r$ since the rectangle of vertices P,Q,R is not degenerate and $\mathcal{H}^1([Q,R])=|x_c-x'|\leq\tilde{\alpha} r$. This concludes the proof. $\square$

Theorem 3: Let $E \subset X_\varepsilon$ be axis-symmetric (with axis e_1), n-symmetric, and s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0.$$

Then, for some constant $C_1(n,\varepsilon) > 0$ it holds that

$$|(E \Delta B) \cap Z| \leq C_1(n,\varepsilon)\sqrt{\delta P(E)}.$$

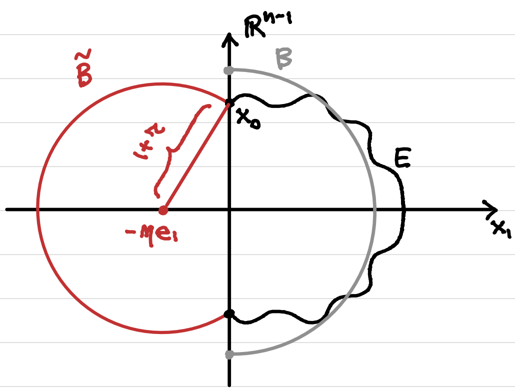
proof: from Lemma 7 and 8, it is sufficient to prove that
$|V_E(0) - V_B(0)|^2 \leq C_1 \delta P(E)$.

Step 1: as the perimeter of a general set is complicated to compute, we consider a ball $\tilde{B}$ which has same slice of E in 0 and same volume in the halfspace π_1^-.
Notice that, analogously as in the proof of Lemma 8, as a consequence of the isoperimetric inequality (and by n-symmetry) we have $P(\tilde{B} \cap \pi_1^-) \leq P(E \cap \pi_1^-) = \frac{1}{2} P(E)$. We then reduce to study

$$|V_E(0) - V_B(0)|^2 = |V_{\tilde{B}}(0) - V_B(0)|^2 \leq C_1\left(P(\tilde{B} \cap \pi_1^-) - P(B \cap \pi_1^-)\right),$$

$$\leq 2C_1\left(P(E) - P(B)\right) \leq (2n\omega_n) C_1 \delta P(E).$$

Since this estimate involves only balls, we can make explicit computations.



Let $r, \eta \in \mathbb{R}$ be such that $\tilde{B} = B_{1+r}(\eta e_1)$.
We denote by $\varepsilon := \frac{V_E(0) - V_B(0)}{\omega_{n-1}}$.

Denoting as x_0 any point in $\partial^* E \cap \pi_1$, then $|x_0|$ is the radius of $E_0 = \tilde{B}_0$, i.e. $|x_0| = (V_E(0)/\omega_{n-1})^{\frac{1}{n-1}}$.
Moreover, as $x_0 \in \partial\tilde{B}$ as well we have $|x_0 + \eta e_1| = 1+r$.

By Pythagora's Theorem $|x_0| = \sqrt{(1+r)^2 - \eta^2}$. As $\sqrt{}_B(0) = \omega_{n-1}$ we have

$$1+\varepsilon = \sqrt{}_E(0) = \left((1+r)^2 - \eta^2\right)^{\frac{n-1}{2}}.$$

We notice that, by Fubini

$$|B \cap \pi_1^-| = \int_{-1}^{0} \sqrt{}_B(t)\,dt = \omega_{n-1} \int_{-1}^{0} (1-t^2)^{\frac{n-1}{2}}\,dt$$

$$|\tilde{B} \cap \pi_1^-| = \int_{\eta-(1+r)}^{0} \sqrt{}_{\tilde{B}}(t)\,dt = \omega_{n-1} \int_{\eta-(1+r)}^{0} \left((1+r)^2 - (t-\eta)^2\right)^{\frac{n-1}{2}}\,dt$$

$$= \omega_{n-1} \int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-1}{2}}\,dt$$

where we used the change of variable $t-\eta \mapsto t$ in the second identity. By the conditions $|\tilde{B} \cap \pi_1^-| = \frac{|B|}{2}$ we get

$$\int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-1}{2}}\,dt = \int_{-1}^{0} (1-t^2)^{\frac{n-1}{2}}\,dt.$$

We can write η and r in terms of ε by the above arguments collected in the following system

$$\begin{cases} \left((1+r)^2 - \eta^2\right)^{\frac{n-1}{2}} = 1+\varepsilon \\ \displaystyle\int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-1}{2}}\,dt = \int_{-1}^{0} (1-t^2)^{\frac{n-1}{2}}\,dt. \end{cases}$$

Notice that r and η are uniquely determined as we have two variables solving two equations.

Step 2: by Taylor expansion (up to the second order) we write the explicit dependence of η and r from ε.
Expanding the first equation we get

$$\left((1+r)^2 - \eta^2\right)^{\frac{n-1}{2}} = \left(1 + (2r + r^2 - \eta^2)\right)^{\frac{n-1}{2}}$$

$$= 1 + \frac{(n-1)}{2}(2r + r^2 - \eta^2) + \frac{(n-1)(n-3)}{8}(2r + r^2 - \eta^2)^2 + o(r^2 + \eta^4)$$

$$= 1 + (n-1)r + \left(\frac{(n-1)}{2} + \frac{(n-1)(n-3)}{2}\right)r^2 - \frac{(n-1)}{2}\eta^2 + o(r^2 + \eta^2)$$

$$= 1 + (n-1)r + \frac{(n-1)(n-2)}{2}r^2 - \frac{n-1}{2}\eta^2 + o(r^2 + \eta^2).$$

We now expand the second equation of the system. Notice first that

$$\frac{d}{dr}\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt = -((1+r)^2-t^2)^{\frac{n-1}{2}}\Big|_{t=1+r}^{=0}$$

$$+\int_{-(1+r)}^{\eta}(n-1)((1+r)^2-t^2)^{\frac{n-3}{2}}(1+r)\,dt,$$

$$\frac{d^2}{dr^2}\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt = \frac{d}{dr}\int_{-(1+r)}^{\eta}(n-1)((1+r)^2-t^2)^{\frac{n-3}{2}}(1+r)\,dt$$

$$=\int_{-(1+r)}^{\eta}(n-1)(n-3)((1+r)^2-t^2)^{\frac{n-5}{2}}(1+r)^2\,dt +$$

$$+(n-1)\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-3}{2}}\,dt$$

and that

$$\frac{d}{d\eta}\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt = ((1+r)^2-\eta^2)^{\frac{n-1}{2}},$$

$$\frac{d^2}{d\eta^2}\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt = \frac{d}{d\eta}((1+r)^2-\eta^2)^{\frac{n-1}{2}} = -(n-1)((1+r)^2-\eta^2)^{\frac{n-3}{2}}\eta.$$

$$\frac{d^2}{d\eta\, dr}\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt = \frac{d}{dr}((1+r)^2-\eta^2)^{\frac{n-1}{2}} = (n-1)((1+r)^2-\eta^2)^{\frac{n-3}{2}}(1+r)$$

Hence, denoting $Q := \int_{-1}^{0}(1-t^2)^{\frac{n-3}{2}}\,dt > 0$, we get

$$\int_{-(1+r)}^{\eta}((1+r)^2-t^2)^{\frac{n-1}{2}}\,dt$$

$$=\int_{-1}^{0}(1-t^2)^{\frac{n-1}{2}}\,dt + (n-1)Q\,r + \eta + \frac{1}{2}\Big(\int_{-1}^{0}(n-1)(n-3)(1-t^2)^{\frac{n-5}{2}}\,dt +$$

$$+\int_{-1}^{0}(n-1)(1-t^2)^{\frac{n-3}{2}}\,dt\Big)r^2 + (n-1)\eta r + o(r^2+\eta^2).$$

Noticing that, by integration by parts,

$$\int_{-1}^{0}(1-t^2)^{\frac{n-3}{2}}\,dt = \int_{-1}^{0}(n-3)(1-t^2)^{\frac{n-5}{2}}t^2\,dt$$

$$=\int_{-1}^{0}(n-3)(1-t^2)^{\frac{n-5}{2}}\,dt - \int_{-1}^{0}(n-3)(1-t^2)^{\frac{n-3}{2}}\,dt$$

which yields

$$(n-2)\int_{-1}^{0}(1-t^2)^{\frac{n-3}{2}}\,dt = (n-3)\int_{-1}^{0}(1-t^2)^{\frac{n-5}{2}}\,dt.$$

From this computation we can further simplify the coefficient of r^2, that is

$$\frac{1}{2}\left(\int_{-1}^{0}(n-1)(n-3)(1-t^2)^{\frac{n-5}{2}}\,dt+\int_{-1}^{0}(n-1)(1-t^2)^{\frac{n-3}{2}}\,dt\right)$$

$$=\frac{1}{2}\left(\int_{-1}^{0}(n-1)(n-2)(1-t^2)^{\frac{n-3}{2}}\,dt+\int_{-1}^{0}(n-1)(1-t^2)^{\frac{n-3}{2}}\,dt\right)$$

$$=\frac{1}{2}(n-1)^2\int_{-1}^{0}(1-t^2)^{\frac{n-3}{2}}\,dt=\frac{1}{2}(n-1)^2 a.$$

Collecting the calculations from above we have

$$\begin{cases}(n-1)r+\frac{(n-1)(n-2)}{2}r^2-\frac{n-1}{2}\eta^2+o(r^2+\eta^2)=\varepsilon\\[2mm](n-1)ar+\eta+\frac{(n-1)^2}{2}ar^2+(n-1)\eta r+o(r^2+\eta^2)=0.\end{cases}$$

<u>step 3</u>: to solve the system above (end of step 2) we first consider the first orders, that is

$$\begin{cases}r=\frac{1}{n-1}\varepsilon+o(r+\eta)\\[2mm]\eta=-a\varepsilon+o(r+\eta),\end{cases}$$

from which we also have $o(r+\eta)=O(\varepsilon)$, $o(r^2+\eta^2)=o(\varepsilon^2)$. To have a second order expansion we write $r=\frac{1}{n-1}\varepsilon+r'$ and $\eta=-a\varepsilon+\eta'$ in the system at end of step 2. Reminding that the terms $\varepsilon r', \varepsilon\eta', (r')^2, (\eta')^2$ are all $o(\varepsilon^2)$ we get

$$\begin{cases}\varepsilon=\varepsilon+(n-1)r'+\frac{(n-1)(n-2)}{2}\left(\frac{1}{(n-1)^2}\varepsilon^2+o(\varepsilon^2)\right)-\frac{n-1}{2}\left(a^2\varepsilon^2-o(\varepsilon^2)\right)+o(\varepsilon^2)\\[2mm]0=a\varepsilon+(n-1)ar'-a\varepsilon+\eta'+\frac{(n-1)^2}{2}a\left(\frac{1}{(n-1)^2}\varepsilon^2+o(\varepsilon^2)\right)+(n-1)(-a\varepsilon^2+o(\varepsilon^2))+o(\varepsilon^2).\end{cases}$$

From this we obtain finally

$$\begin{cases}r=\frac{1}{n-1}\varepsilon+\frac{1}{2(n-1)}\left((n-1)a^2-\frac{n-2}{n-1}\right)\varepsilon^2+o(\varepsilon^2)\\[2mm]\eta=-a\varepsilon+\frac{a}{2}\left(1-(n-1)a^2+\frac{n-2}{n-1}\right)\varepsilon^2+o(\varepsilon^2).\end{cases}$$

Step 4: We now write the perimeter of $\tilde{B}$ in terms of r and η obtaining a lower bound in terms of ε^2.

By Lemma 4, since $\tilde{B}$ is axis-symmetric (with respect to e_1) we have, since $V_{\tilde{B}}(t) = \omega_{n-1}((1+r)^2-(t+\eta)^2)^{\frac{n-1}{2}}$ and

$$V_{\tilde{B}}'(t) = -(n-1)\,\omega_{n-1}\left(((1+r)^2-(t+\eta)^2\right)^{\frac{n-3}{2}}(t+\eta),$$

$$p_{\tilde{B}}(t) = (n-1)\,\omega_{n-1}\left((1+r)^2+(t+\eta)^2\right)^{\frac{n-2}{2}},$$

so that $\qquad ((1+r)^2-\eta^2)^{\frac{n-3}{2}}(1+r)$

$$\begin{aligned}
P(\tilde{B};\pi_1^-) &= \int_{-(1+r)-\eta}^{0} \sqrt{(p_{\tilde{B}}(t))^2+(V_{\tilde{B}}'(t))^2}\,dt \\
&= (n-1)\,\omega_{n-1}\int_{-(1+r)}^{\eta} \left(((1+r)^2-t^2)^{n-2}+((1+r)^2-t^2)^{n-3}t^2\right)^{\frac{1}{2}}dt \\
&= (n-1)\,\omega_{n-1}\underbrace{\int_{-(1+r)}^{\eta} ((1+r)^2-t^2)^{\frac{n-3}{2}}(1+r)\,dt}_{=:\,\xi(r,\eta)}.
\end{aligned}$$

We expand the integral up to the second order. With computations similar to those in Step 2 we have

$$\partial_r\xi(0,0) = \int_{-1}^{0}(n-3)(1-t^2)^{\frac{n-5}{2}}+(1-t^2)^{\frac{n-3}{2}}\,dt = (n-1)a,$$

$$\begin{aligned}
\partial_r^2\xi(0,0) &= (n-3)\int_{-1}^{0}(n-5)(1-t^2)^{\frac{n-7}{2}}+3(1-t^2)^{\frac{n-5}{2}}\,dt \\
&= (n-3)\int_{-1}^{0}(n-1)(1-t^2)^{\frac{n-5}{2}}\,dt = (n-1)\int_{-1}^{0}(n-2)(1-t^2)^{\frac{n-3}{2}}\,dt \\
&= (n-1)(n-2)a,
\end{aligned}$$

$$\partial_\eta\xi(0,0)=1,\quad \partial^2_{\eta r}\xi(0,0)=(n-2),\quad \partial^2_\eta\xi(0,0)=0.$$

Hence

$$\xi(r,\eta) = \xi(0,0)+(n-1)ar+\eta+\frac{(n-1)(n-2)}{2}ar^2+(n-2)\eta r+o(r^2+\eta^2).$$

Replacing the expression of r and m in terms of ε we get

$$\zeta(r,m) = Q + (n-1)Q\left(\frac{1}{n-1}\varepsilon + \frac{1}{2(n-1)}\left((n-1)Q^2 - \frac{n-2}{n-1}\right)\varepsilon^2 + o(\varepsilon^2)\right)$$

$$- Q\varepsilon + \frac{Q}{2}\left(1-(n-1)Q^2 + \frac{n-2}{n-1}\right)\varepsilon^2 + o(\varepsilon^2)$$

$$+ \frac{(n-1)(n-2)}{2} Q \left(\frac{1}{n-1}\varepsilon + o(\varepsilon)\right)^2$$

$$+ (n-2)(-Q\varepsilon + o(\varepsilon))\left(\frac{1}{n-1}\varepsilon + o(\varepsilon)\right)$$

$$= Q + \cancel{Q\varepsilon} + \cancel{(n-1)\frac{Q^3}{2}\varepsilon^2} - \cancel{\frac{n-2}{n-1}\frac{Q}{2}\varepsilon^2}$$

$$- \cancel{Q\varepsilon} + \frac{Q}{2}\varepsilon^2 - \cancel{(n-1)\frac{Q^3}{2}\varepsilon^2} + \cancel{\frac{n-2}{n-1}\frac{Q}{2}\varepsilon^2}$$

$$+ \frac{n-2}{2(n-1)}Q\varepsilon^2 - \frac{n-2}{n-1}Q\varepsilon^2 + o(\varepsilon^2)$$

$$= Q + \frac{n-1+n-2-2n+4}{2(n-1)}Q\varepsilon^2 + o(\varepsilon^2)$$

$$= Q + \frac{1}{2(n-1)}Q\varepsilon^2 + o(\varepsilon^2).$$

Notice that this quantity is quadratic in ε for every $n \geq 2$. Putting the pieces together

$$\delta P(E) = \frac{P(E) - P(B)}{n\omega_n} = 2\,\frac{P(E;\pi_1^-) - P(B;\pi_1^-)}{n\omega_n}$$

$$\geq 2\,\frac{P(\tilde{B};\pi_1^-) - P(B;\pi_1^-)}{n\omega_n} = \frac{2(n-1)\omega_{n-1}}{n\omega_n}\left(\zeta(r,m) - Q\right)$$

$$= \frac{\omega_{n-1}}{n\omega_n} Q\varepsilon^2 + o(\varepsilon^2).$$

By taking ε sufficiently small, i.e. $\varepsilon < \varepsilon_n$ so that $o(\varepsilon^2)$ can be absorbed by the quadratic term we get

$$\delta P(E) \geq \frac{\omega_{n-1}}{2n\omega_n} Q\,\varepsilon^2 = \frac{Q}{2n\omega_n\omega_{n-1}}\left|v_E(0) - v_B(0)\right|^2$$

Which concludes the proof. $\square$

1. Cut-off argument

We are just left to give details of the proof of the fact that we can assume, without loss of generality, that the set E is uniformly bounded.

Lemma 9: there exist $l = l(n)$, $C_1 = C_1(n)$ s.t. if $E \in X$, $\exists E' \in X_\ell$ s.t.

$$\alpha(E) \leq \alpha(E') + C_1 \delta P(E), \qquad \delta P(E') \leq C_1 \delta P(E).$$

proof: as done in the proof of the last theorem of Lecture IV.4, we can assume $\nu_E(x) \neq e_j$, $\mathcal{H}^{n-1} \llcorner \partial^* E$ - a.e. for all $j = 1, \dots, n$. Hence, by Lemma 4 v_E is absolutely continuous.

Define, $E_t^- := \{x \in E : x_1 < t\}$. The it holds that, $\forall t \in \mathbb{R}$

$$P(E_t^-) \leq P(E; (-\infty, x_1) \times \mathbb{R}^{n-1}) + v_E(t), \quad P(E \setminus E_t^-) \leq P(E; (x_1, +\infty) \times \mathbb{R}^{n-1}) + v_E(t).$$

Indeed, these are equalities for smooth sets (by the condition on ν_E). By approximating E with smooth sets E_j s.t. $P(E_j; \Omega) \to P(E; \Omega)$ for every Ω open, by lower-semicontinuity of the perimeter (and since $v_{E_j}(t) \to v_E(t)$ $\forall t \in \mathbb{R}$) we have the claim (fill the details as an Exc).

The set E_t^- is our truncated set. Since truncation may reduce the volume, we need a suitable rescaling argument.
Let $g : \mathbb{R} \to \mathbb{R}^+$ be defined as $g(t) := |E_t^-| \omega_n^{-1}$. This is a non-decreasing, C^1 function with $g'(t) = v_E(t) \cdot \omega_n^{-1}$ (Exc). By monotonicity, $\exists a, b \in \mathbb{R}$ s.t. $\{t \in \mathbb{R} : 0 < g(t) < 1\} = (a, b)$. For any $t \in (a, b)$ it holds that $|g(t)^{-1/n} E_t^-| = \omega_n$.
By isoperimetric inequality

$$P(E_t^-) = P(g(t)^{\frac{1}{n}}(g(t)^{-\frac{1}{n}} E_t^-)) = g(t)^{\frac{n-1}{n}} P(g(t)^{-\frac{1}{n}} E_t^-) \geq g(t)^{\frac{n-1}{n}} P(B).$$

Working analogously exploiting that $|E\setminus F_t^-| = |E|-|F_t^-| = \omega_n - g(t)\omega_n$ we also have $P(E\setminus F_t^-) \geq (1-g(t))^{\frac{n-1}{n}} P(B)$. Hence

$$P(E) + 2\nu_E(t) \geq P(B)\left(g(t)^{\frac{n-1}{n}} + (1-g(t))^{\frac{n-1}{n}}\right), \quad \forall t \in (a,b).$$

Since $\delta P(E) = \frac{P(E)}{P(B)} - 1$, this also tells us that

$$\nu_E(t) \geq \tfrac{1}{2} P(B)\left(g(t)^{\frac{n-1}{n}} + (1-g(t))^{\frac{n-1}{n}} - 1 - \delta P(E)\right).$$

Denoting $\psi(t) := t^{\frac{n-1}{n}} + (1-t)^{\frac{n-1}{n}} - 1$, by explicit one-dimensional calculations, we get $\psi(t) \geq 2(2^{1/n}-1)t$ for $|t|<1/2$ and that, assuming $\delta P(E) < 1/2\,\psi(1/2)$, $\exists\, a<t_1<t_2<b$ s.t. $\psi(g(t)) \geq 2\delta P(E)$ for $t\in(t_1,t_2)$ and $g(t_1) = 1-g(t_2) \leq \delta P(E)(2^{1/n}-1)^{-1}$ (fill the details as an Exc). By these properties we get, for $t_1<t<t_2$,

$$\nu_E(t) \geq \frac{1}{2} P(B)\left(\psi(g(t)) - \delta P(E)\right)$$

$$= \frac{1}{4} P(B)\psi(g(t)) + \frac{1}{4} P(B)\left(\psi(g(t)) - 2\delta P(E)\right)$$

$$\geq \frac{n\omega_n}{4}\psi(g(t)).$$

By this we can prove that, $|t_2-t_1|$ is uniformly bounded

$$t_2 - t_1 \leq \int_{t_1}^{t_2} \frac{4n}{\omega_n}\frac{\nu_E(t)}{\psi(g(t))}\,dt = \frac{4}{n}\int_{t_1}^{t_2} \frac{g'(t)}{\psi(g(t))}\,dt$$
$$= \frac{4}{n}\int_{g(t_1)}^{g(t_2)} \frac{1}{\psi(s)}\,ds < \frac{4}{n}\int_0^1 \frac{1}{\psi(s)}\,ds =: \kappa(n)$$

as $1/\psi$ is integrable both in 0 and 1, where $\kappa(n)>0$ is some dimensional constant.

We now look for good t where to cut-off our set keeping the isoperimetric deficit controlled. For this purpose let

$$T_1 = \max\left\{t\in(a,t_1]: \nu_E(t) \leq \frac{n\omega_n \delta P(E)}{2}\right\}, \quad T_2 = \min\left\{t\in[t_2,b): \nu_E(t) \leq \frac{n\omega_n\delta P(E)}{2}\right\},$$

which are well-defined since $\nu_E(a) = \nu_E(b) = 0$.

As by continuity $v_E(T_1) = v_E(T_2) = \frac{1}{2} n\omega_n \delta P(E)$ and recalling that $g(t_1) \leq \delta P(E)(2^{1/n}-1)^{-1}$ we get

$$t_1 - T_1 = \frac{2}{n\delta P(E)} \int_{T_1}^{t_1} g'(t)\, dt \leq \frac{2}{n\delta P(E)} g(t_1) \leq \frac{2}{n(2^{1/n}-1)},$$

and analogously for $T_2 - t_2$, which yields that $T_2 - T_1 \leq \alpha(n) + \frac{4}{n(2^{1/n}-1)} =: \beta(n)$.

We set $\tilde{E} = E \cap ((T_1, T_2) \times \mathbb{R}^{n-1})$. By definition of T_1, T_2 and the properties of the relative perimeters of the truncations we get

$$|\tilde{E}| \geq |B| \left(1 - 2\frac{\delta P(E)}{2^{1/n}-1}\right), \qquad P(\tilde{E}) \leq P(B) + n\omega_n \delta P(E).$$

Readjusting the volume of $\tilde{E}$ as follows we conclude the argument.

Assume (without loss of generality) that $\delta P(E) < \frac{1}{4}(2^{1/n}-1)$ and let $\sigma := (|B|/|\tilde{E}|)^{1/n}$.

Define $E' := \sigma \tilde{E}$. Then clearly $|E'| = \omega_n$ and E' is contained in $(T_1', T_2') \times \mathbb{R}^{n-1}$ where T_1' and T_2' are given by T_1 and T_2 after the dilation via $x \mapsto \sigma x$.

We only need to check that $\delta P(E')$ and $\alpha(E')$ are controlled.

By the estimate on $|\tilde{E}|$, we get $1 \leq \sigma \leq 1 + C_0 \delta P(E)$ for a suitable constant $C_0 > 0$. Hence, by the control on $P(\tilde{E})$,

$$P(E') = \sigma^{n-1} P(\tilde{E}) \leq \sigma^{n-1}(P(E) + P(B)\delta P(E))$$

$$= \sigma^{n-1} P(B)(1 + 2\delta P(E)) \leq P(B)(1 + C_1(n)\delta P(E))$$

for a constant $C_1(n) \geq 2 \vee C_0(n)$. This gives

$$\delta P(E') \leq C_1(n)\, \delta P(E).$$

Finally, again by the estimate on $|\hat{E}|$, letting $p \in \mathbb{R}^n$ be a point s.t. $\alpha(E') = |E' \Delta (B+p)|$, we get

$$\alpha(E) \leq |E \Delta (B + p/r)|$$

$$\leq |E \Delta \tilde{E}| + |\hat{E} \Delta (B_{1/r} + p/r)| + |(B_{1/r} + p/r) \Delta (B + p/r)|$$

$$= |E \Delta \tilde{E}| + r^{-n} \alpha(E') + |B \setminus B_{1/r}|$$

$$\leq C(n) \delta P(E) + \alpha(E') + C(n)(1-r) \leq \alpha(E') + C(n) \delta P(E)$$

for a suitable $C(n)$.

Repeating this argument in every coordinates we conclude the proof. $\square$