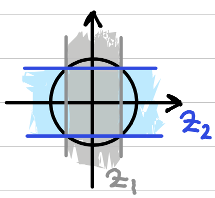
We first need the following technical result saying that (up to a change of order in the coordinates) the symmetric difference between E n-symmetric and B does not "concentrates" close to the poles.

Lemma 5: given $Z := \{x \in \mathbb{R}^n : |x_1| \leq \frac{\sqrt{2}}{2}\}$, and E n-symmetric with $|E| = \omega_n$, then (up to a rotation)

$$|E \Delta B| \leq 4\, |(E \Delta B) \cap Z|.$$

proof: defining $Z_j := \{x \in \mathbb{R}^n : |x_j| \leq \frac{\sqrt{2}}{2}\}$, we have $B \subseteq Z_1 \cup Z_2$.



Hence, $B \setminus E = ((B \setminus E) \cap Z_1) \cup ((B \setminus E) \cap Z_2)$. Thus, up to switch the roles of the coordinates we have

$$|(B \setminus E) \cap Z_1| \geq \frac{|B \setminus E|}{2}$$

which yields

$$|E \Delta B| = 2|E \setminus B| \leq 4\, |(B \setminus E) \cap Z_1| \leq 4\, |(E \Delta B) \cap Z_1|,$$

where again we used that $|E| = |B|$. $\boxtimes$

In the next technical lemma we show that, given a set with sufficiently small isoperimetric deficit, its Schwarz symmetral cannot have "spikes".

Lemma 6: for every $\rho > 0$ $\exists$ $\delta > 0$ s.t. $\forall$ $E \in \mathcal{X}_{e,\delta}$, n-symmetric, and s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0,$$

it holds that

$$\| v_E - v_B \|_{L^\infty(\mathbb{R})} \leq \rho.$$

The ideas of the proof are intuitively clear. As v_E and v_B are continuous, given $\rho_0 := \|v_E - v_B\|_{L^\infty(\mathbb{R})}$, then there exists a $\bar{t}$ s.t. $|v_E(\bar{t}) - v_B(\bar{t})| = \rho_0$ and a small interval $I_{\rho_0} \subseteq \mathbb{R}$ of length $\mathcal{H}^1(I_{\rho_0}) < \varepsilon \ll \rho_0$ s.t.

$$|v_E(t^\pm) - v_B(\bar{t})| = \rho_0/2$$

where $t^\pm$ are the extrema of I_{ρ_0}. By axis-symmetry and Schwarz inequality we get

$$\delta P(E) \geq \delta P(E^*) = \frac{P(E^*) - P(B)}{n\omega_n} \overset{!}{\geq} \frac{P(E^*; I_{\rho_0} \times \mathbb{R}^{n-1}) - P(B; I_{\rho_0} \times \mathbb{R}^{n-1})}{n\omega_n}$$

$$\geq \frac{\rho_0 \omega_{n-1} - C_1\varepsilon}{n\omega_n} \geq \frac{\rho_0\,\omega_{n-1}}{2n\omega_n},$$

fact

where we took $\varepsilon > 0$ sufficiently small, and where we used the least that, by continuity and axis-symmetry, $P(E^*; I_{\rho_0} \times \mathbb{R}^{n-1})$ is at the lenght of the curve connecting the extreme rotated around the axis x_1. Hence, taking $\delta P(E) < \delta$ with $\delta < \frac{\rho\,\omega_{n-1}}{2n\omega_n}$ then yields that $\rho_0 < \rho$.

In this heuristic argument, the reduction to relative perimeters is not obvious and must be treated carefully. We present the full-detailed proof below.

proof: by Lemma 1, since E^* is still n-symmetric and $v_E = v_{E^*}$ we get

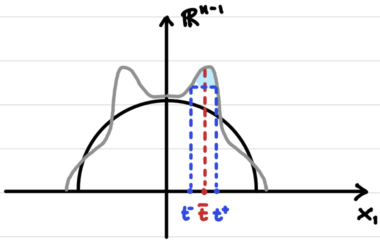
$$\|v_E - v_B\|_{L^1(\mathbb{R})} = \|v_{E^*} - v_B\|_{L^1(\mathbb{R})} = |E^* \Delta B| \overset{L1}{\leq} 2^n \alpha(E^*).$$

By isoperimetric stability $\forall \varepsilon > 0\ \exists\, \delta(\varepsilon, \rho) > 0$ s.t. if $\delta P(E^*) \leq \delta P(E) \leq \delta(\varepsilon)$ then $\alpha(E^*) < \rho\varepsilon/2^{n+2}$. Letting $I_\rho := \{t \in \mathbb{R} : |v_E(t) - v_B(t)| > \rho/4\}$ we get

$$\frac{\rho\varepsilon}{4} > \|v_E - v_B\|_{L^1(\mathbb{R})} > \int_{I_\rho} |v_E(t) - v_B(t)|\, dt > \rho/4\, \mathcal{H}^1(I_\rho)$$

thus $\mathcal{H}^1(I_\rho) < \varepsilon$.

Assume by contradiction that $\|v_E - v_B\|_{L^\infty(\mathbb{R})} > \rho$. Since $v_E, v_B \in W^{1,1}(\mathbb{R})$, namely absolutely continuous, then $\exists\, \bar{t} \in \mathbb{R}$ s.t. $|v_E(\bar{t}) - v_B(\bar{t})| > \rho$.



By uniform continuity of v_B, taking $\varepsilon > 0$ sufficientely small $|t-\bar{t}| < \varepsilon$ we have

$$|v_B(t) - v_B(\bar{t})| < \rho/4.$$

Hence, by continuity of v_E and the fact that $\mathcal{H}^1(I_\rho) < \varepsilon$ there exist $\bar{t}-\varepsilon < t^- < \bar{t} < t^+ < \bar{t}+\varepsilon$ such that $v_E(t^-) = v_E(t^+)$, for $t \in (t^-, t^+)$ and

$$|v_E(t^\pm) - v_B(\bar{t})| = \rho/2.$$

By cutting the spike off, we construct a test set F which allows us to compare the perimeter contribution of the spike, namely allowing us to pass to estimate $\delta P(E)$ with the isoperimetric deficit localized in $I_\rho \times \mathbb{R}^{n-1}$.

So, let F be defined as $F = \{(t,y) \in \mathbb{R}^n : |y| \le (\frac{v_F(t)}{\omega_{n-1}})^{\frac{1}{n-1}}\}$ with $v_F(t) = v_E(t)$ $\forall\, t \notin (t^-, t^+)$ and

$$v_F(t) = \begin{cases} v_E(t) & t \notin (t^-, t^+) \\ v_E(t^\pm) & t \in (t^-, t^+). \end{cases}$$

Since $E \subseteq [-\ell, \ell]^n$ (and hence $F \subseteq [-\ell, \ell]^n$) we get

$$|F| > |E^*| - 2^n \ell^{n-1} \varepsilon,$$

$$P(F) = P(E^*) + P(F; \{x \in \mathbb{R}^n : t^- < x_1 < t^+\}) - P(E^*; \{x \in \mathbb{R}^n : t^- < x_1 < t^+\}).$$

By axis-symmetry and the formulation of perimeter of axis-symmetric sets

$$P(F; \{x \in \mathbb{R}^n : t^- < x_1 < t^+\}) = (t^+ - t^-)\mathcal{H}^{n-2}(\partial B^{n-1}_{E_{t^+}}) \le \varepsilon\, \sigma_n |v_E(t^\pm)|^{\frac{n-2}{n-1}} \le \varepsilon\, C_{n,\ell}$$

where $B^{n-1}_{E_{t^+}}$ is the $(n-1)$-dimensional ball of volume $v_E(t^+)$ and where we used that $p_{E^*}(t^+) = \mathcal{H}^{n-2}(\partial E^*_{t^+}) = \mathcal{H}^{n-1}(\partial B^{n-1}_{E_{t^+}}) = (n-1)\omega_{n-1}^{\frac{1}{n-1}} (v_E(t^+))^{\frac{n-2}{n-1}}$. In the estimate above $\sigma_n := (n-1)\omega_{n-1}^{\frac{1}{n-1}}$ and $C_{n,\ell} = \sigma_n (2\ell)^{n-2}$ coming from the fact that $E \in X_\ell$.

$$P(E^*; \{x \in \mathbb{R}^n : t^- < x_1 < t^+\}) = \int_{t^-}^{t^+} \sqrt{p_{E^*}(t)^2 + v_E'(t)^2}\, dt \geq \int_{t^-}^{t^+} |v_E'(t)|\, dt \geq \rho.$$

This implies $P(F) \leq P(E^*) + C_{n,\ell}\,\varepsilon - \rho$. By the isoperimetric inequality and by concavity of $x \mapsto (1+|x|)^{\frac{n-1}{n}}$

$$P(E^*) + C_{n,\ell}\,\varepsilon - \rho \geq P(F) \geq n\omega_n^{1/n} (\overset{=|B|}{|E^*|} - 2^n \ell^{n-1} \varepsilon)^{\frac{n-1}{n}} \geq P(B) - \tilde{C}_{n,\ell}\,\varepsilon.$$

$$= n\omega_n^{\frac{1}{n}} |B| \left(1 - \frac{2^n \ell^{n-1} \varepsilon}{\omega_n}\right)^{\frac{n-1}{n}} \geq P(B) - \tilde{C}_{n,\ell}\,\varepsilon$$

where $\tilde{C}_{n,\ell} = (n-1)\omega_n^{\frac{1}{n}} 2^n \ell^{n-1}$, having used that every concave function lays below its tangent lines.
Recollecting everything, by choosing $\varepsilon < \rho / 2(C_{n,\ell} + \tilde{C}_{n,\ell})$ and $\delta < \frac{\rho}{2n\omega_n}$ we get

$$\delta \geq \delta P(E) > \frac{P(E^*)}{P(B)} - 1 \geq \frac{\rho}{2P(B)} > \delta$$

which gives a contradiction. $\boxtimes$

With these two techical lemmas at hand, we are ready to prove the following result with which we control the deviation of a set from its Schwarz symmetral by the square-root of the isoperimetric deficit.

This allows us to reduce the proof of the quantitative isoperimetric inequality to axis-symmetric sets.

Theorem 2: Let $E \in \mathcal{X}_e$ be n-symmetric, of finite perimeter, and such that $\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0$. If either

(i) $n = 2$, or

(ii) $n \geq 3$, and $\forall\, F \subseteq \mathbb{R}^{n-1}$ it holds $\alpha_{n-1}(F) \leq C_{n-1}\sqrt{\delta P_{n-1}(F)}$,

then

$$|E \Delta B| \leq 4|(E^* \Delta B) \cap Z| + C_{n,e}\sqrt{\delta P(E)}$$

for some constant $C_{n,e} > 0$ depending on n and e only.

<u>proof</u>: We start by noticing that, as $|E \Delta B| \leq 2\omega_n$ we can assume that $\delta P(E) \leq \delta$, where δ is given by Lemma 6 with $\rho = \omega_{n-1} 2^{-\frac{n+1}{2}}$. Note that, as ρ depends on n only, also δ is a dimensional constant. Notice also that, by this choice Lemma 6 holds true, so as for $|t| \leq \frac{\sqrt{2}}{2}$, $\omega_{n-1} 2^{-\frac{n-1}{2}} \leq v_B(t) \leq \omega_{n-1}$ then

$$\omega_{n-1} 2^{-\frac{n+1}{2}} = \omega_{n-1} 2^{-\frac{n-1}{2}} - \rho \leq v_E(t) \leq \omega_{n-1} + \rho \leq 2\omega_{n-1}, \quad |t| \leq \frac{\sqrt{2}}{2}.$$

In particular $p_E(t) \geq p_{E^*}(t) \geq \sigma_n (v_E(t))^{\frac{n-2}{n-1}} \geq \sigma_n \left(\omega_{n-1} 2^{-\frac{n+1}{2}}\right)^{\frac{n-2}{n-1}} =: \tau_n$.

As a last preliminary consideration, we observe that, by Lemma 5 and using Fubini

$$|E \Delta B| \leq 4|(E \Delta B) \cap Z| = 4\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \mathcal{H}^{n-1}(E_t \Delta B_t)\,dt$$

$$= 4\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta B_t)\,dt + 4\int_{J_M} \mathcal{H}^{n-1}(E_t \Delta B_t)\,dt$$

where $I_M := \{t \in (-\sqrt{2}/2, \sqrt{2}/2) : |v_E'(t)| \leq M\}$ and $J_M := (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \setminus I_M$ for some constant $M > 0$ to be determined later.

We now control the two terms on the right-hand side separately.

Step 1: we claim that, $\forall\, t \in I_M$ it holds that

$$\mathcal{H}^{n-1}(E_t \Delta E_t^*) \leq C_1(n,\ell)\sqrt{p_E(t) - p_{E^*}(t)},$$

for both cases (i) and (ii).

Indeed, for $n=2$, for any t, $\mathcal{H}^1(E_t \Delta E_t^*) \leq 2\ell$. If t is s.t. $p_E(t) = p_{E^*}(t)$ by symmetry of E, E_t and E_t^* are the same interval, so also $\mathcal{H}^1(E_t \Delta E_t^*) = 0$.
If instead t is s.t. $p_E(t) \neq p_{E^*}(t)$ then, since E_t^* is an interval, E_t is union of (at least) two intervals so $p_E(t) - p_{E^*}(t) \geq 2$ and $\mathcal{H}^1(E_t \Delta E_t^*) \leq \sqrt{2}\ell\sqrt{p_E(t) - p_{E^*}(t)}$.

For $n \geq 3$, since $E_t^* = B^{n-1}_{E_t}$ and E_t is $(n-1)$-symmetric, by Lemma 1 and the hypothesis in point (ii) we have

$$\begin{aligned}
\mathcal{H}^{n-1}(E_t \Delta E_t^*) \leq 2^{n-1}\alpha_{n-1}(E_t) &\leq C_{1,n-1}\sqrt{\frac{\mathcal{H}^{n-2}(\partial E_t) - \mathcal{H}^{n-2}(\partial B^{n-1}_{E_t})}{\mathcal{H}^{n-2}(\partial B^{n-1}_{E_t})}} \\
&= C_{1,n-1}\sqrt{\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}} \\
&\leq \frac{C_{1,n-1}}{\sqrt{\tau_{n-1}}}\sqrt{p_E(t) - p_{E^*}(t)}
\end{aligned}$$

where we have used the non-degeneracy of $p_{E^*}(t)$.

Step 2: in the previous step we have proved a sort of "slicing version" of the quantitative isoperimetric inequality, assuming it to be true in one dimension less, and provinving it "by hands" for $n=2$. We exploit this to prove the statement in $I_M \times \mathbb{R}^{n-1}$, namely in regions in which the profile of E^* is not "too steep".
By triangle inequality

$$\begin{aligned}
\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta B_t)\,dt &\leq \int_{I_M} \mathcal{H}^{n-1}(E_t \Delta E^*_t)\,dt + \int_{I_M} \mathcal{H}^{n-1}(E_t^* \Delta B_t)\,dt \\
&\leq \int_{I_M} \mathcal{H}^{n-1}(E_t \Delta E^*_t)\,dt + |(E^* \Delta B) \cap Z|.
\end{aligned}$$

By Schwarz inequality

$$P(E)-P(E^*) \geq \int_{I_M} \sqrt{p_E(t)^2+(\sqrt{_E}'(t))^2} - \sqrt{p_{E^*}(t)^2+(\sqrt{_E}'(t))^2}\, dt$$

$$= \int_{I_M} \frac{p_E(t)^2 - p_{E^*}(t)^2}{\sqrt{p_E(t)^2+(\sqrt{_E}'(t))^2}+\sqrt{p_{E^*}(t)^2+(\sqrt{_E}'(t))^2}}\, dt$$

Recalling that $|\sqrt{_E}'| \leq M$ in I_M, and that $p_E(t) \geq p_{E^*}(t) \geq \tau_n > 0$, as we noticed at the beginning of the proof, by subadditivity of the squareroot we find a constant $C_1 = C_1(n,\ell,M)$ s.t. $\forall t \in I_M$

$$\frac{\sqrt{p_E(t)^2+(\sqrt{_E}'(t))^2}+\sqrt{p_{E^*}(t)^2+(\sqrt{_E}'(t))^2}}{p_E(t)+p_{E^*}(t)} \leq \frac{p_E(t)+p_{E^*}(t)+2M}{p_E(t)+p_{E^*}(t)} \leq 1+\frac{2M}{c_n \rho^{\frac{n-2}{n-1}}} \leq C_1.$$

Hence, by step 1 and the isoperimetric inequality (i.e. $P(E^*) \geq P(B)$)

$$n\omega_n \delta P(E) \geq P(E)-P(E^*) \geq \frac{1}{C_1}\int_{I_M} p_E(t)-p_{E^*}(t)\, dt$$

$$\geq \frac{\tau_n}{C_1 c_{n-1}^2}\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta E_t^*)^2\, dt$$

$$\geq \frac{\tau_n}{C_1 c_{n-1}^2}\cdot\frac{1}{\mathcal{H}^1(I_M)}\left(\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta E_t^*)\, dt\right)^2$$

having also used Jensen's inequality. Since $\mathcal{H}^1(I_M) \leq \sqrt{2}$ we have

$$\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta E^*_t)\, dt \leq \sqrt{\frac{n\omega_n \cdot C_1}{\tau_n}}\, c_{n-1}\, 2^{1/4}\sqrt{\delta P(E)}$$

and in particular, for suitable $C_{n,\ell,M} > 0$

$$\int_{I_M} \mathcal{H}^{n-1}(E_t \Delta B_t)\, dt \leq C_{n,\ell,M}\sqrt{\delta P(E)} + |(E^* \Delta B) \cap Z|.$$

Step 3: when $t \in J_M$ intuitively (for M large enough) the pro-file of E^* is very "steep" so E is far from being close to a ball. In this case, we even have a better (linear) control.

We notice first that $\mathcal{H}^{n-1}(E_t \Delta B_t) \leq (2\ell)^{n-1}$, so

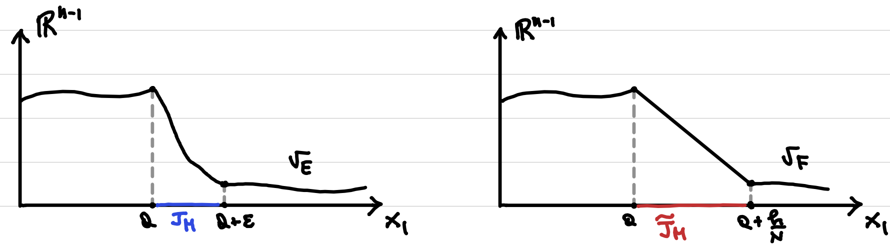
$$\int_{J_M} \mathcal{H}^{n-1}(E_t \Delta B_t)\, dt \leq (2\ell)^{n-1}\mathcal{H}^1(J_M).$$

We reduced to prove $\mathcal{H}^1(J_M) \leq \frac{K}{M}\delta P(E^*)$ for some $K > 0$.

By approximation (see the Remark below) we can assume that $v_E \in \mathcal{C}^1(\mathbb{R})$ and $J_M = (a, a+\varepsilon)$ is an interval, for some $a \in \mathbb{R}$ and $\varepsilon > 0$. Also, as $|v_E'| > M$, we can assume without loss of generality that $v_E' < 0$.

We define $h := v(a) - v(a+\varepsilon)$ and $\tilde{J}_M := (a, a+\frac{h}{N})$ for some $N \in \mathbb{N}$ to be determined and define $F := \{(t,z) \in \mathbb{R}^n : |z| \leq (\frac{v_F(t)}{\omega_{n-1}})^{\frac{1}{n-1}}\}$ with

$$v_F(t) := \begin{cases} v_E(t) & t < a \\ v_E(a) - N(t-a) & a < t < a + h/N \\ v_E(t - \frac{h}{N} + \varepsilon) & t > a + h/N. \end{cases}$$



Note that, by $v_E' < -M$, $h > M\varepsilon$. Notice also that, as v_F is a convex combination of v_E than maximum and minimum are (at most/least the same) namely

$$\omega_{n-1} 2^{-\frac{n+1}{2}} < v_F(t) < 2\,\omega_{n-1}, \quad |t| \leq \sqrt{3}/2.$$

Working similarly as done in Lemma 6, we use F to compare the perimeters of E and B in $J_M \times \mathbb{R}^{n-1}$, eventually exploiting the isoperimetric inequality.

We start controlling the volumes as

$$|E^* \cap (J_M \times \mathbb{R}^{n-1})| \leq 2\omega_{n-1} \mathcal{H}^1(J_M) = 2\omega_{n-1}\varepsilon,$$

$$|F \cap (\tilde{J}_M \times \mathbb{R}^{n-1})| \geq 2^{-\frac{n+1}{2}} \omega_{n-1} \mathcal{H}^1(\tilde{J}_M) = 2^{-\frac{n+1}{2}} \omega_{n-1} \frac{h}{N}.$$

Hence, since $|F| = |E^*| - |E^* \cap (J_M \times \mathbb{R}^{n-1})| + |F \cap (\tilde{J}_M \times \mathbb{R}^{n-1})|$ we get

$$|F| \geq |E^*| + 2\omega_{n-1}\left(2^{\frac{1-n}{2}} \frac{h}{N} - \varepsilon\right) \geq |E^*| + 2\omega_{n-1}\left(\frac{2^{\frac{1-n}{2}}}{N} - \frac{1}{M}\right) h \geq |E^*| + \frac{\omega_{n-1}}{2^{\frac{n+1}{2}} N} h,$$

by choosing $M \geq \frac{1}{2^{\frac{n+1}{2}} N}$, having used that $h \geq M\varepsilon$.

Regarding the perimeter, $P(F)=P(E^*)-P(E^*;J_M\times\mathbb{R}^{n-1})+P(F;\tilde{J}_M\times\mathbb{R}^{n-1})$. By Lemma 4, denoting $\tilde{\tau}_n:=\sigma_n^2(2\omega_{n-1})^{2\cdot\frac{n-2}{n-1}}$

$$
\begin{aligned}
P(F;\tilde{J}_M\times\mathbb{R}^{n-1}) &= \int_{\tilde{J}_M}\sqrt{\sigma_n^2(v_F(t))^{\frac{2n-4}{n-1}}+(v_F'(t))^2}\,dt \\
&\le \int_{\tilde{J}_M}\sqrt{\tilde{\tau}_n+N^2}\,dt \le \mathcal{H}^1(\tilde{J}_M)N\sqrt{1+\frac{\tilde{\tau}_n}{N^2}} \\
&\le h\left(1+\frac{\tilde{\tau}_n}{2N^2}\right)
\end{aligned}
$$

where we have used that $\mathcal{H}^1(\tilde{J}_M)=h/N$ by construction and that $\sqrt{1+t}\le 1+\frac{t}{2}$ for $t\ge 0$ by concavity.
We also immediately have $P(E^*;J_M\times\mathbb{R}^{n-1})\ge h$ just by continuity of v_E. So $P(F)\le P(E^*)+h\frac{\tilde{\tau}_n}{2N^2}$.

By the isoperimetric inequality and concavity of $t\mapsto(1+t)^{\frac{n-1}{n}}$

$$
\begin{aligned}
P(F) &\ge n\omega_n^{1/n}|F|^{\frac{n-1}{n}} \ge n\omega_n^{1/n}\left(\overset{=|B|}{|E^*|}+\frac{\omega_{n-1}}{2^{\frac{n+1}{2}}N}h\right)^{\frac{n-1}{n}} \ge P(B)\left(1+\frac{\omega_{n-1}}{2^{\frac{n+1}{2}}N\omega_n}h\right)^{\frac{n-1}{n}} \\
&\ge P(B)+C_n\frac{h}{N}
\end{aligned}
$$

for a suitable constant $C_n>0$. This implies immediately

$$
\begin{aligned}
\delta P(E^*) &= \frac{P(E^*)}{P(B)}-1 \ge \frac{P(F)}{P(B)}-1-\frac{\tilde{\tau}_n h}{2N^2P(B)} \ge \frac{C_n h}{NP(B)}-\frac{\tilde{\tau}_n h}{2N^2P(B)} \\
&= \left(C_n-\frac{\tilde{\tau}_n}{2N}\right)\frac{h}{P(B)N} \ge \tilde{C}_n h
\end{aligned}
$$

taking $N\ge\frac{\tilde{\tau}_n}{C_n}$ and $\tilde{C}_n=\frac{C_n^2}{2n\omega_n\tilde{\tau}_n}$. We have proved that

$$
\mathcal{H}^1(J_M)=\varepsilon\le\frac{h}{M}\le\frac{1}{\tilde{C}_n M}\delta P(E^*).
$$

which was our goal. We also stress that M depends on N and n and, in turn, N depends on n, so all the constants involved are dimensional (or at most depending also on ℓ).

Finally, by the very beginning of the proof, Step 2 and this estimate we obtain

$$|E \Delta B| \leq 4\, C_{n,\ell,M} \sqrt{\delta P(E)} + 4|(E^* \Delta B) \cap Z| + 4(2\ell)^{n-1} \frac{1}{\tilde{c}_{n,M}} \delta P(E)$$

which concludes the proof as $\delta P(E) < \delta$ and all the constants depend on n and ℓ only. $\boxtimes$

<u>Rmk</u>: the assumption, in the proof above (Step 3) that $v_E \in C^1$ and that J_M is an interval it is not restrictive.

Indeed, if $v_E \in C^1$ and J_M is not an interval, as J_M is at most a countable union of disjoint intervals and by additivity of perimeter and volume we can simply repeat the above argument for every connected component.

If v_E is not C^1, we can find function $v_h \in C^1$, $v_h \to v_E$ in $W^{1,1}$ by approximation in Sobolev, and such that $v_h \geq 0$, $\int_{\mathbb{R}} v_h = \int_{\mathbb{R}} v_E$. Defining $E_h := \{(t,z) : |z| \leq \left(\frac{v_h(t)}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\}$ we have $|E_h| = \omega_n$, $E_h \to E^*$ and $P(E_h) \to P(E^*)$.
Applying the arguments from above on each E_h the result for general E follows by taking the limit as $h \to \infty$ and ob--serving that $\mathcal{H}^1(J_h) \to \mathcal{H}^1(J_M)$ where $J_h := \{|t| \leq \frac{\sqrt{3}}{2} : |v_h'(t)| \geq M\}$ as a consequence of strong convergence of the derivative.

1. The axially-symmetric case

Thanks to Theorem 2, we reduced to prove the quantitative iso-perimetric inequality for axis-symmetric (and n-symmetric) sets.

This case essentially reduces to one-dimensional, complicated but elementary computations. This case was already proved by Hall in 1992. In [Fusco-Maggi-Pratelli (2008)] the proof has been simplified by means of further reductions and with clever geometrical techniques. Their strategy is the follows:

(i) compare the deviation from the unit ball with the deviation on two (carefully selected) slices

$$|E \Delta B| \lesssim |v_E(\bar{x}) - v_B(x')|;$$

(ii) prove that we can assume that these slices are the "central" ones, namely passing through the origin

$$|E \Delta B| \lesssim |v_E(\bar{x}) - v_B(x')| \lesssim |v_E(0) - v_B(0)|;$$

(iii) control the deviation on the central slice with the iso-perimetric deficit with an explicit computation

$$|v_E(0) - v_B(0)| \lesssim \sqrt{\delta P(E)}.$$

We will see ideas on the proof of this result, with all the details given in the addendum to this lecture.

As a last remark, the most novel improvement from Fusco, Maggi, and Pratelli has been to prove the reduction to axis-symmetric sets controlling the isoperimetric deficit with sharp power and Fraenkel asymmetry comparing it with the deviation from the ball centered in the origin (namely Theorems 1 and 2).

We start by giving (the statements) of two technical lemmas which allow us to reduce to study the difference between the central slices of E and B. You find all the details of the proof in the addendum.

<u>Lemma</u> 7: there exist $\delta(n,e), C(n,e) > 0$ s.t. $\forall E \in X_{e,\delta}$ axis-symmetric (with axis e_1), n-symmetric, and s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0,$$

then $\exists \bar{x} \in [0, \sqrt{3}/2]$ s.t. given $x' \geq 0$ defined by

$$\int_0^{x'} v_B(t)\, dt = \int_0^{\bar{x}} v_E(t)\, dt$$

then

$$|v_E(\bar{x}) - v_B(x')| \geq 1/C(n,e)\, |(E \Delta B) \cap Z|.$$

<u>Lemma</u> 8: let $\delta(n,e)$ and E be as in the statement of Lemma 7. Then $\exists E' \in X_{2e,\delta}$ s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E' : \nu_{E'}(x) = \pm e_1\}) = 0$$

and

$$P(E') \leq P(E), \qquad |v_{E'}(0) - v_B(0)| \geq 1/C_1\, |v_E(\bar{x}) - v_B(x')|,$$

for some $C_1 > 0$ depending on n.

We now present the quantitative isoperimetric inequality for axis-symmetric (and n-symmetric) sets, providing some ideas of the proof. Again, a detailed proof can be found in the addendum.

Theorem 3: Let $E \in \mathcal{X}_e$ be axis-symmetric (with axis e_1), n-symme-
-tric, and s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0.$$

Then, for some constant $C_1(n,e) > 0$ it holds that

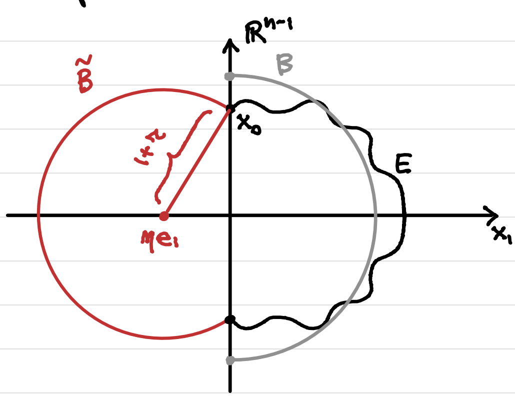
$$|(E \Delta B) \cap Z| \leq C_1(n,e)\sqrt{\delta P(E)}.$$

proof (ideas): from Lemma 7 and 8, it is sufficient to prove that $|V_E(0) - V_B(0)|^2 \leq C_1 \delta P(E)$.

Step 1: as the perimeter of a general set is complicated to compute, we consider a ball $\tilde{B}$ which has same slice of E in 0 and same volume in the halfspace π_1^-. So we reduce to study (by the isoperimetric inequality, i.e. $P(\tilde{B} \cap \pi_1^-) \leq P(E \cap \pi_1^-)$)

$$|V_E(0) - V_B(0)|^2 = |V_{\tilde{B}}(0) - V_B(0)|^2 \leq C_1(P(\tilde{B} \cap \pi_1^-) - P(B \cap \pi_1^-)).$$

Since this estimate involves only balls, we can make explicit computations.



Let $r, \eta \in \mathbb{R}$ be such that $\tilde{B} = B_{1+r}(\eta e_1)$. They are uniquely determined.

We denote by $\varepsilon := \frac{|V_E(0) - V_B(0)|}{\omega_{n-1}}$.

Denoting as x_0 any point in $\partial^* E \cap \pi_1$, then $|x_0|$ is the radius of $E_0 = \tilde{B}_0$, i.e. $|x_0| = \left(\frac{V_E(0)}{\omega_{n-1}}\right)^{\frac{1}{n-1}}$.

By Pythagora $|x_0| = \sqrt{(1+r)^2 - \eta^2}$. By this and $|\tilde{B} \cap \pi_1^-| = |E \cap \pi_1^-|$ we get a system of equation determining η and r in terms of ε,

$$\begin{cases} 1+\varepsilon = \left((1+r)^2 - \eta^2\right)^{\frac{n-1}{2}} \\ \int_{-1}^{0} (1-t^2)^{\frac{n-1}{2}}\, dt = \int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-1}{2}}\, dt. \end{cases}$$

Expanding this expression via Taylor at second order of ε, given $Q := \int_{-1}^{0} (1-t^2)^{\frac{n-3}{2}}\, dt > 0$ we have

$$\begin{cases} r = \frac{1}{n-1}\varepsilon + \frac{1}{2(n-1)}\left((n-1)Q^2 - \frac{n-2}{n-1}\right)\varepsilon^2 + o(\varepsilon^2) \\ \eta = -Q\varepsilon + \frac{Q}{2}\left(1 + \frac{n-2}{n-1} - (n-1)Q^2\right)\varepsilon^2 + o(\varepsilon^2). \end{cases}$$

Step 2: we can now write the perimeter of $\tilde{B}$ in terms of r and η, so in terms of ε up to a third order error. So, for instance by coarea formula for $\partial\tilde{B} \cap \pi_1^-$,

$$\begin{aligned} P(\tilde{B} \cap \pi_1^-) &= \int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-2}{2}} (n-1)\omega_{n-1} \frac{1+r}{\sqrt{(1+r)^2 - t^2}}\, dt \\ &= (n-1)\omega_{n-1} \int_{-(1+r)}^{\eta} \left((1+r)^2 - t^2\right)^{\frac{n-3}{2}} (1+r)\, dt \\ &= (n-1)\omega_{n-1} \int_{-1}^{0} (1-t^2)^{\frac{n-3}{2}}\, dt + \omega_{n-1} Q \varepsilon^2 + o(\varepsilon^2) \\ &= \frac{1}{2} P(B) + \omega_{n-1} Q \varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

This finally yields

$$\delta P(E) \geq \frac{2P(\tilde{B} \cap \pi_1^-) - P(B)}{n\omega_n} \geq \frac{Q}{n\omega_n} \frac{|V_E(0) - V_B(0)|^2}{\omega_{n-1}} + o(\varepsilon^2)$$

which eventually implies the result. $\square$

2. The (sharp) quantitative isoperimetric inequality

One last techinal ingredient we need is a truncation argu-
-ment to be able to reduce ourselves to uniformly bounded sets (find the proof in the addendum).

Lemma 9: there exist $l = l(n)$, $C_1 = C_1(n)$ s.t. if $E \in X$, $\exists E' \in X_l$ s.t.

$$\alpha(E) \leq \alpha(E') + C_1 \delta P(E), \qquad \delta P(E') \leq C_1 \delta P(E).$$

We can now conclude the proof of the result in full generality.

Theorem (the sharp quantitative isoperimetric inequality): let $n \geq 2$, there exists a constant $C_1(n) > 0$ such that for every Borel set $E \subseteq \mathbb{R}^n$ with $0 < |E| < \infty$ it holds that

$$\alpha(E) \leq C_1(n) \sqrt{\delta P(E)}.$$

proof: as observed many times in this course, it is sufficient to prove the result for $|E| = \omega_n$ and $\delta P(E) \leq 1$.

Assume first that $E \in X_l$. There are at most countably many directions $\nu \in \mathbb{S}^{n-1}$ s.t. $\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \nu\}) > 0$.
Indeed, let $N \subseteq \mathbb{S}^{n-1}$ be the set of all such directions. For any $j \in \mathbb{N}$ we let $N_j := \{\nu \in N : \mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \nu\}) > 1/j\}$. Then $N = N_1 \cup \bigcup_{j \geq 2} (N_j \setminus N_{j-1})$

$$\mathcal{H}^{n-1}(\partial^* E) \geq \sum_{\nu \in N_j} \mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \nu\}) \geq \frac{\# N_j}{j},$$

thus $\# N_j \leq j \cdot P(E) \leq 2n\omega_n j$, so N is countable as countable union of finite sets.
So, up to rotations (which do not affect δP and α) we can assume

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0.$$

By Theorem 1, since the "mirroring" procedure preserves the property that $\nu_E \neq \pm e_1$ $\mathcal{H}^{n-1} \llcorner \partial^* E$-a.e., $\exists\, F \in X_{3\ell}$, n-symmetric s.t. $\mathcal{H}^{n-1}(\{x \in \partial^* F : \nu_F(x) = \pm e_1\}) = 0$ and

$$\alpha(E) \leq C_0(n,\ell)\, \alpha(F), \qquad \delta P(F) \leq 2^n \delta P(E).$$

<u>step 1</u> (induction base): let $n=2$. By Theorem 2 (i) and 3

$$\alpha(F) \leq |F \Delta B| \overset{\text{Thm2}}{\leq} 4|(F^* \Delta B) \cap Z| + C_1(n,\ell)\sqrt{\delta P(F)}$$

$$\overset{\text{Thm3}}{\leq} (4 C_2(n,\ell) + C_1(n,\ell))\sqrt{\delta P(F)}.$$

By the properties of F

$$\alpha(E) \leq C_0(n,\ell)\, \alpha(F) \leq C_0(n,\ell)(4C_2(n,\ell) + C_1(n,\ell))\sqrt{\delta P(F)}$$

$$\leq 2^{\frac{n}{2}} C_0(n,\ell)(4C_2(n,\ell) + C_1(n,\ell))\sqrt{\delta P(E)},$$

which proves the claim for $E \subseteq \mathbb{R}^2$, $E \in X_\ell$.

<u>step 2</u> (inductive step): by Theorem 2 (ii) and 3 with the same passages as above, working iteratively on the dimension the claim is proved for every $n \geq 2$, $E \in X_\ell$.

Finally, for any $E \subseteq \mathbb{R}^n$ $\exists\, E' \in X_{\ell(n)}$ and

$$\alpha(E) \leq \alpha(E') + C_3(n,\ell(n))\, \delta P(E)$$

$$\leq C_4(n,\ell(n))\sqrt{\delta P(E')} + C_3(n,\ell(n))\, \delta P(E)$$

$$\leq (C_4(n,\ell(n))\, C_3(n,\ell(n))^{1/2} + C_3(n,\ell(n)))\sqrt{\delta P(E)},$$

as $\delta P(E) \leq 1$. The result is proved. $\boxtimes$