Lecture 3: 20.04.2026

Rmk: one last property of the Hausdorff measure, that we will often use is that, the n-dimensional Hausdorff measure on $\mathbb{R}^n$ coincides with $\mathcal{L}^n$. Namely,

$$\forall E \subseteq \mathbb{R}^n, \ \forall \delta \in (0,\infty], \ |E| = \mathcal{H}^n(E) = \mathcal{H}^n_\delta(E).$$

See the addendum to the notes of Lecture 2 for a proof.

Today's reference:

- Maggi (2012), Chapters 4, 5, 8

In this course we will use (the support of) Radon measures as geometrical objects. We do this via integration.

Radon measure are characterized by their values on compact and open sets. So integration wrt a Radon measure will be characterized by testing on $\mathcal{C}_c$.
As integration is a linear, bounded (in some sense) operator, we will use this to generalize the notion of measure to one taking vectors as values, and to drag some analytical tools from operators to measures.

1. Vector valued Radon measures

To any Radon measure we can associate a linear operator

Rmk: given μ a Radon measure, let

$$L: \mathcal{C}_c(\mathbb{R}^n) \longrightarrow \mathbb{R}, \qquad L\varphi := \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x).$$

Then L is a linear, (locally) bounded, positive operator.

By (locally) bounded we mean that maps sets $\|\cdot\|_\infty$-bounded with bounded supports into bounded sets. This coincides with the fact that $\forall K \subseteq \mathbb{R}^n$ compact

$$\sup\{L\varphi : \|\varphi\|_\infty \leq 1,\ \mathrm{spt}(\varphi) \subseteq K\} < +\infty.$$

Being linear and bounded gives continuity wrt the following convergence;

$$u_j \to u \quad \text{iff} \quad \exists K \subseteq \mathbb{R}^n \text{ compact s.t. } \mathrm{spt}(u_j) \subseteq K \text{ and } u_j \rightrightarrows u.$$

Also the converse implication holds true, to any linear, locally bounded operator we can associate a Radon measure.

Theorem (Riesz): Let $L: \mathcal{C}_c(\mathbb{R}^k; \mathbb{R}^m) \longrightarrow \mathbb{R}$ linear and (locally) bounded. Then the set map $|L|: \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, +\infty]$

$$|L|(E) := \inf\{\sup\{L\varphi : \|\varphi\|_\infty \leq 1,\ \mathrm{spt}(\varphi) \subseteq A\} : A \supseteq E \text{ open}\}$$

is a Radon measure. Moreover $\exists\, g: \mathbb{R}^n \to \mathbb{R}^n$, $|L|$-measurable, $|g| = 1$ $|L|$-a.e. and s.t.

$$L\varphi = \int_{\mathbb{R}^n} \varphi(x) \cdot g(x)\, d|L|(x).$$

Def: we call a linear, (locally) bounded operator $\mu : C_c(\mathbb{R}^n;\mathbb{R}^m) \longrightarrow \mathbb{R}$ an $\mathbb{R}^m$-valued Radon measure.

The Radon measure $|\mu|$ and the function g as in Riesz's Theorem are called its total variation measure and its polar decomposition, respectively.

In particular we will write

$$\int_E \varphi(x)\cdot d\mu(x) = \int_E \varphi(x)\cdot g(x)\, d|\mu|(x).$$

note: an $\mathbb{R}$-valued Radon measure is called a signed Radon measure. We call (positive) Radon measures the standard ones.

Rmk (users guide): to each $\mathbb{R}^m$-valued Radon measure (as defined above) we can associate the set map $\mu : \mathcal{B}_b(\mathbb{R}^n) \longrightarrow \mathbb{R}^m$, where $\mathcal{B}_b(\mathbb{R}^n) := \{E \in \mathcal{B}(\mathbb{R}^n) : E \text{ bounded}\}$, s.t.

$$\mu(E) = \int_E g(x)\, d|\mu|(x) = \left(\int_E g_1(x)\, d|\mu|(x), \dots, \int_E g_m(x)\, d|\mu|(x)\right)$$

for which

(i) $\mu(\emptyset) = 0$ (ii) $\mu(E) = \sum_j \mu(E_j)$, $E = \cup_j E_j$

Exc: given μ a (positive) Radon measure and $u \in L^1_{loc}(\mathbb{R}^n, \mu; \mathbb{R}^m)$ we denote as $u\mu$ the $\mathbb{R}^m$-valued Radon measure

$$u\mu(E) := \int_E u(x)\, d\mu(x) \in \mathbb{R}^m.$$

Find $|u\mu|$ and g.

Riesz's Theorem implies that $C_c(\mathbb{R}^n;\mathbb{R}^m)$ and Radon measures are in duality (in the sense of locally convex spaces) with $\langle \varphi, \mu \rangle = \int \varphi \cdot d\mu$.

1.1. Topology and compactness

We collect here some notions and results we will work with in the sequel.

<u>Def</u>: let μ_h, μ be $\mathbb{R}^m$-valued Radon measures. Then we say the μ_h (locally) weak-* converges to μ, and write $\mu_h \xrightarrow{*} \mu$ iff

$$\lim_{h \to +\infty} \int_{\mathbb{R}^n} \varphi(x) \cdot d\mu_h(x) = \int_{\mathbb{R}^n} \varphi(x) \cdot d\mu(x), \quad \forall \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$$

For "good" sets, weak-* convergence translates into a pointwise convergence of the set-maps.

<u>Rmk</u>: if $\mu_h \xrightarrow{*} \mu \Rightarrow \forall E \in \mathcal{B}_b(\mathbb{R}^n)$ s.t. $|\mu|(\partial E) = 0$ then

$$\mu(E) = \lim_{h \to +\infty} \mu_h(E).$$

Given $A \subseteq \mathbb{R}^n$ open, $|\mu|(A)$ can be viewed as a norm on the space of Radon measures restricted on A. Hence, by Banach-Steinhaus one can infer the following property, namely that the total variation (norm) is lower semicontinuous wrt weak-* convergence.

<u>Proposition</u>: let μ_h, μ be $\mathbb{R}^m$-valued Radon measures s.t. $\mu_h \xrightarrow{*} \mu$ and let $A \subseteq \mathbb{R}^n$ be open. Then

$$|\mu|(A) \leq \underline{\lim}_{h \to +\infty} |\mu_h|(A).$$

Finally, we also recall the following (local) compactness.

<u>Proposition</u>: given $\{\mu_h\}_h$ $\mathbb{R}^m$-valued Radon measures s.t.

$$\sup_h |\mu_h|(K) < +\infty, \quad \forall K \subseteq \mathbb{R}^n \text{ compact},$$

then $\exists\, \mu$ an $\mathbb{R}^m$-valued Radon measure s.t. $\mu_h \xrightarrow{*} \mu$ up to a subsequence.

2. Besicovitch differentiation

In this course we will work with objects of different intrinsic dimensions; sets $E \subseteq \mathbb{R}^n$ with $|E| > 0$ have dimension n, whereas a k-surface $M \subseteq \mathbb{R}^n$ has dimension k, thus $|M| = 0$, and we will see (at the end of the lecture) that $\mathcal{H}^k(M) > 0$.

We now introduce a strong tool to compare two measures or sets, even if they have different intrinsic dimensions.

2.1. Besicovitch covering

This is a fundamental tool that allows us to pass from local arguments to global ones, without missing any set of measure zero.
We present the result without proof (you find it in Maggi, Section 5.1).

Theorem (Besicovitch's covering): for any $n \geq 1$, $\exists\, \xi(n) \in \mathbb{N}$, $\xi(n) > 0$. Let $\mathcal{F}$ be a family of closed balls and $C \subseteq \mathbb{R}^n$ the set of centers.
If either C is bounded or

$$\sup\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}\} < +\infty$$

then $\exists\, \mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)} \subseteq \mathcal{F}$ s.t.
at most countable

(i) $\mathcal{F}_i$ consists of disjoint balls, $\forall i$

(ii) $$C \subseteq \bigcup_{i=1}^{\xi(n)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}$$

note: contrary to Vitali's covering, the number of overlaps of the balls is uniformly controlled. This allows to obtain global properties (from local ones) without a scaling argument, that with Vitali would be needed.

One way in which we can apply Besicovitch covering is shown in the next remark. If e.g. we know that μ has some property on balls, this is inferred also on the whole C.

<u>Rmk</u>: let μ be a (positive) Radon measure and $C \subseteq \mathbb{R}^n$ a bounded set. Let $\mathcal{F} := \{B_r(x) : x \in C, r > 0\}$.
Then, $\exists\, \mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)}$ as in the Thm s.t.

$$C = \bigcup_{i=1}^{\xi(n)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B} \cap C$$

and hence

$$\mu(C) = \mu\Big(\bigcup_{i=1}^{\xi(n)} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B} \cap C\Big) \leq \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} \mu(\overline{B} \cap C)$$
$$\leq \xi(n) \sum_{\overline{B} \in \mathcal{F}_{i_0}} \mu(\overline{B} \cap C)$$

for some $i_0 = 1 \_\_ \xi(n)$.

2.1. <u>Lebesgue-Besicovitch differentiation</u>

As we said before, when working with objects with different dimensions, we want keep track of this precisely.

<u>Def</u>: given μ and ν two (positive) Radon measures, we say that ν is absolutely continuous wrt ν

(i) $\nu \ll \mu$ iff $\forall E \in \mathcal{B}(\mathbb{R}^n)$ s.t. $\mu(E) = 0 \Rightarrow \nu(E) = 0$.

We say that μ and ν are mutually singular

(ii) $\mu \perp \nu$ iff $\exists\, E \in \mathcal{B}(\mathbb{R}^n)$ s.t. $\mu(\mathbb{R}^n \setminus E) = 0 = \nu(E)$

If ν and μ are vector valued we use their total variation.

Given two (positive) Radon measures ν, μ on $\mathbb{R}^n$, $\forall x \in \operatorname{spt}\mu$

$$D_\mu^+ \nu(x) = \overline{\lim_{r\to 0^+}} \frac{\nu(B_r(x))}{\mu(B_r(x))}, \qquad D_\mu^- \nu(x) = \underline{\lim_{r\to 0^+}} \frac{\nu(B_r(x))}{\mu(B_r(x))}.$$

When $D_\mu^+\nu = D_\mu^-\nu$ we define the μ-density of ν at x as

$$D_\mu \nu(x) := \lim_{r\to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

<u>Note</u>: this can be defined analogously with closed balls.
If ν is vector-valued, $D_\mu \nu$ is defined identically when the limit exists.

We have the following "local version of Radon–Nykodim", saying that the density of ν wrt μ can be computed by checking how ν and μ behaves on balls.

<u>Theorem</u> (Lebesgue–Besicovitch differentiation): $D_\mu \nu$ is well-defined μ-a.e. on $\mathbb{R}^n$, $D_\mu \nu \in L^1_{loc}(\mathbb{R}^n, \mu)$ and Borel measurable. Moreover

$$\nu = (D_\mu \nu)\mu + \nu_\mu^s \quad \text{on } \mathcal{M}(\mu)$$

where $(D_\mu\nu)\mu \ll \mu$, and ν_μ^s concentrates on

$$Y = (\mathbb{R}^n \setminus \operatorname{spt}(\mu)) \cup \{x \in \operatorname{spt}(\mu) : D_\mu^+|\nu|(x) = \infty\},$$

in particular $\nu_\mu^s \perp \mu$.

$(D_\mu \nu)\mu$ and ν_μ^s are called, respectively, the absolutely continuous and singular part of ν wrt μ.

<u>Exc</u>: given $\nu = g|\nu|$, find $D_{|\nu|}\nu$.
Given another μ, find $D_\mu \nu$ in terms of $D_\mu|\nu|$ and g.

Ex: $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, if the following limit exists

$$\vartheta_n(E)(x) := \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{r^n \omega_n}$$

it is called the n-dimensional density of E at x.
By applying Lebesgue-Besicovitch differentiation Theorem with $\nu = \mathcal{L}^n \llcorner E$ and $\mu = \mathcal{L}^n$ we immediately get that $\vartheta_n(E)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$ and $\mathcal{L}^n \llcorner E = \vartheta_n(E)\, \mathcal{L}^n$.
Since $\mathcal{L}^n \llcorner E = \chi_E \mathcal{L}^n$

$$\vartheta_n(E) \equiv 1 \text{ a.e. on } E \qquad \vartheta_n(E) = \text{ a.e. on } \mathbb{R}^n \setminus E.$$

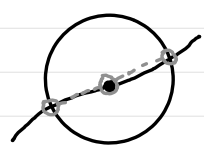
For every $t \in [0,1]$, $E^{(t)} := \{x \in \mathbb{R}^n : \vartheta_n(E)(x) = t\}$

$$|E \,\Delta\, E^{(1)}| = 0 \qquad\qquad |(\mathbb{R}^n \setminus E) \,\Delta\, E^{(0)}| = 0.$$

Exc: given $E = [-1/2, 1/2]^2 \setminus \{(0,0)\}$, find $\vartheta_E(x)$ for <u>every</u> $x \in \mathbb{R}^2$.

Ex: on $\mathbb{R}^n$, let $\Gamma \subseteq \mathbb{R}^n$ be a simple, continuous curve and take $\mu = \mathcal{L}^n$, $\nu = \mathcal{H}^1 \llcorner \Gamma$. For every $x \in \Gamma$

$$\lim_{r \to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap \Gamma)}{r^n \omega_n} \geq \lim_{r \to 0^+} \frac{2r}{r^n \omega_n} = +\infty.$$



Here we simply used the $L(B_r(x) \cap \Gamma)$ is larger than the length of the 2 segments connecting x with $\Gamma \cap \partial B_r(x)$, for r small.
As Γ is closed (by continuity), $\Gamma = \mathrm{spt}(\nu)$ so $\forall x \notin \Gamma$, $D_\mu \nu(x) = 0$.
So $\mu \perp \nu$ and ν concentrates on

$$\Gamma = \{x \in \mathbb{R}^n : D^+_\mu \nu(x) = +\infty\}.$$

3. Area formula

To prove isoperimetric inequality in full generality we need to generalize the notion of surface area.

Given $1 \le k \le n-1$, $f \in C^1(\mathbb{R}^k;\mathbb{R}^n)$, $A \subseteq \mathbb{R}^k$ open set and f s.t.

$$Jf(x) := \sqrt{\det(\nabla f(x)^t \, \nabla f(x))} > 0 \qquad \forall\ x \in A$$

then $f(A) = M$ is a k-dimensional (parametrized) surface and its surface area is

$$\int_A Jf(x)\, dx$$

<u>note</u>: $k=1$, $\nabla f(x) = f'(x) \in \mathbb{R}^n$ so $f'(x)^t f'(x) = f'(x) \cdot f'(x) = |f'(x)|^2$
so $Jf(x) = |f'(x)|$.
If $k=n$, $\nabla f(x) \in \mathbb{R}^{n \times n}$ and by Binet
$Jf(x) = \sqrt{\det(\nabla f(x)^t)\det(\nabla f(x))} = |\det(\nabla f(x))|$.

<u>Theorem</u> (Area formula): let $1 \le k \le n$, $f: \mathbb{R}^k \to \mathbb{R}^n$ be an injective Lipschitz function, $E \subseteq \mathbb{R}^k$ Lebesgue measurable

$$\mathcal{H}^k(f(E)) = \int_E Jf(x)\, dx$$

and $\mathcal{H}^k \llcorner f(\mathbb{R}^k)$ is a Radon measure on $\mathbb{R}^n$.

<u>note</u>: as injective Lipschitz functions map compact sets into compact sets (<u>Exc</u>: show this), by inner regularity, $f(E)$ is $\mathcal{H}^k$-measurable (<u>Exc</u>: show also this).

3.1. Some auxiliary results

We list here some results that will be needed for the proof of the Area formula.
The first deals with the case of linear functions

Lemma: let $1 \le k \le n$ and let $T \in \mathbb{R}^{n \times k}$ and let $\mathbb{R}^k \ni x \overset{T}{\mapsto} Tx \in \mathbb{R}^n$. Then $\forall\, E \subseteq \mathbb{R}^k$, and denoting $T(E) := \{Tx : x \in E\}$,

$$\mathcal{H}^k(T(E)) = \kappa_T \, \mathcal{L}^k(E), \quad \text{where } \kappa_T := \frac{\mathcal{H}^k(T(B_1))}{\omega_k}.$$

proof: if $\kappa_T = 0$ the claim is trivial since, by scaling of $\mathcal{H}^k$

$$\mathcal{H}^k(T(E)) \le \lim_{r \to +\infty} \mathcal{H}^k(T(B_r)) = \lim_{r\to+\infty} \mathcal{H}^k(r\,T(B_1))$$
$$= \lim_{r \to +\infty} r^k \mathcal{H}^k(T(B_1)) = 0.$$

If $\kappa_T > 0$, let $\nu(E) := \mathcal{H}^k(T(E))$, $\forall\, E \subseteq \mathbb{R}^k$, which is a Radon measure (as $\nu = (T^{-1})_\# \mathcal{H}^k \llcorner f(\mathbb{R}^k)$, see next lecture). By linearity and scaling

$$\nu(B_r(x)) = \mathcal{H}^k(T(B_r(x))) = \mathcal{H}^k(Tx + rT(B_1))$$
$$= r^k \mathcal{H}^k(T(B_1)) = \frac{\mathcal{H}^k(T(B_1))}{\omega_k} \mathcal{L}^k(B_r(x))$$
$$= \kappa_T \, \mathcal{L}^k(B_r(x)).$$

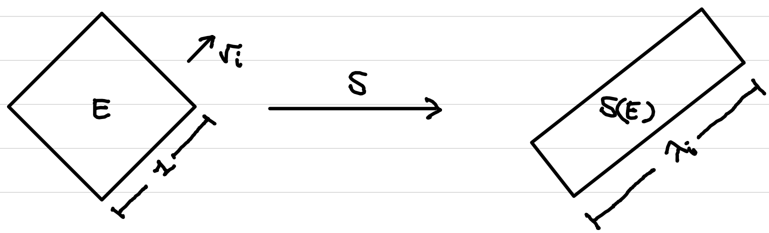
So by Lebesgue-Besicovitch Theorem, $\nu \ll \mathcal{L}^k$ and $D_{\mathcal{L}^k}\nu \equiv \kappa_T$, namely $\nu(E) = \int_E \kappa_T \, dx = \kappa_T \mathcal{L}^k(E)$. $\boxtimes$

Rmk: by polar decomposition (of matrices) we can write $T = PS$ where $P \in \mathbb{R}^{n\times k}$ is orthogonal (i.e. $P^tP = I_k$) and $S \in \mathbb{R}^{k\times k}_{sym}$.
We can show that actually $\kappa_T = JT$.
Indeed: the gradient of $x \mapsto Tx$ is T, thus $JT = \sqrt{\det(T^tT)}$. As symmetric matrices are diagonalizable, $\exists\, Q \in \mathbb{R}^{k\times k}$ orthogonal and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$ s.t. $S = Q^t \Lambda Q$. Then

$$JT = \sqrt{\det(S^t \cancel{P^tP} S)} = \sqrt{\det(Q^t \Lambda \cancel{QQ^t} \Lambda Q)} = \sqrt{\det(\Lambda^2)} = \prod_{i=1}^k |\lambda_i|,$$

where in the last step we applied Binet and used that $\det(Q^t) = \det(Q)^{-1}$!

Moreover, let $v_i \in \mathbb{R}^k$, $i=1,\dots,k$, be the eigenvector of S (associated to the eigenvalue λ_i), and consider the cube oriented into the direction of the eigenspaces of S, namely $E := \{x \in \mathbb{R}^k : |x \cdot v_i| \leq 1/2\}$.



By diagonalization, $S(E) = \{x \in \mathbb{R}^k : |x \cdot v_i| \leq \lambda_i\}$ and hence $|S(E)| = \prod_{i=1}^{k} |\lambda_i| = JT$. As P is a rigid motion $|PS(E)| = JT$. Finally, by the Lemma applied on E we get

$$K_T = \frac{\mathcal{H}^k(T(E))}{\mathcal{L}^k(E)} = JT.$$

This proves the Area formula for linear maps T.

<u>note</u>: from the previous remark, we infer that the Jacobian carries the information on how k-dimensional volume (or area) transforms for linear maps.
This is true also for nonlinear transformation; Jf "measures" how volumes transform infinitesimally.