Lemma 3 of last lecture provided us with a way to "mirror" any set across some coordinate hyperplane keeping the isoperimetric ratio controlled.
We now iterate this procedure to produce an n-symmetric set, starting from any test set in X_ℓ.

<u>Theorem 1</u>: there exists a constant $C_1(n,\ell)$ such that, $\forall E \in X_\ell$ $\exists F \in X_{3\ell}$, n-symmetric and such that

$$\alpha(E) \leq C_1(n,\ell)\,\alpha(F), \qquad \delta P(F) \leq 2^n \delta P(E).$$

<u>proof</u>: assume first $\delta P(E) \leq \delta \cdot 2^{2-n}$ where δ is given by Lemma 3.
By Lemma 3 (up to switch the orders of the coordinates) $\exists$ F_1 symmetric with respect to π_1 s.t.

$$\alpha(E) \leq \tilde{C}_1 \alpha(F_1), \qquad \delta P(F_1) \leq 2\,\delta P(E),$$

where $\tilde{C}_1$ is the constant given by Lemma 3 (in there is denoted C_1).
Applying Lemma 3 to F_1 and with respect to π_2 and π_3, again up to change the order of the coordinates, $\exists$ F_2 symmetric with respect to π_2 (and π_1 by construction) s.t.

$$\alpha(E) \leq \tilde{C}_1 \alpha(F_1) \leq \tilde{C}_1^2 \alpha(F_2), \quad \delta P(F_2) \leq 2\,\delta P(F_1) \leq 4\,\delta P(E).$$

By iteration, $\exists$ F_{n-1}, symmetric with respect to $\pi_1, \dots, \pi_{n-1}$ and s.t.

$$\alpha(E) \leq \tilde{C}_1^{\,n-1} \alpha(F_{n-1}), \qquad \delta P(F_{n-1}) \leq 2^{n-1} \delta P(E).$$

We cannot simply apply Lemma 3 for the last step since we do not have the freedom to switch coordinates.

Let $F_n^\pm := F_{n-1}^\pm \cup R_n(F_{n-1}^\pm)$, where $F_{n-1}^\pm := F_{n-1} \cap \Pi_n^\pm$. Clearly $\delta P(F_n^\pm) \leq 2\delta P(F_{n-1})$. Moreover, since $F_n^\pm$ are n-symmetric by Lemma 1

$$\alpha(F_{n-1}) \leq |F_{n-1} \Delta B| = |F_{n-1}^+ \Delta (B \cap \Pi_n^+)| + |F_{n-1}^- \Delta (B \cap \Pi_n^-)|$$

$$= \frac{|F_n^+ \Delta B| + |F_n^- \Delta B|}{2} \overset{L1}{\leq} 2^{n-1}(\alpha(F_n^+) + \alpha(F_n^-))$$

$$\leq 2^n(\alpha(F_n^+) \vee \alpha(F_n^-)).$$

This gives the result choosing as F the set between F_n^+ and F_n^- which has the larger asymmetry and taking $C_1(n,\ell) = 2^n \tilde{C}_1(n,\ell)^{n-1}$. By construction it is immediate to check that $F \in X_{3\ell}$.

Finally, if $\delta P(E) > \delta 2^{2-n}$ it is sufficient to take any n-symmetric set $F \in X_{3\ell, 4\delta}$ and $\alpha(F) \geq 2/C_1(n,\ell)$.
For instance, let

$$F := B_r \cup \bigcup_{j=1}^{n} (B_{r'} \pm \ell e_j)$$

where $r' = \frac{(1-r^n)^{1/n}}{2}$. This is n-symmetric and, by a simple computation

$$\alpha(F) = |F \Delta B| = 2(1-r^n), \qquad \delta P(F) = r^{n-1} + 2(1-r^n)^{\frac{n-1}{n}} - 1.$$

Taking $r = (1 - 2/C_1(n,\ell))^{1/n}$ then $\alpha(F) = 4/C_1(n,\ell)$ and $\delta P(F) \leq \frac{4}{C_1(n,\ell)}$ which is smaller then 4δ up to increase the value of the constant $C_1(n,\ell)$. $\boxtimes$

<u>Rmk 2</u>: note that, if E is such that $\nu_E \neq \pm e_1$ $\mathcal{H}^{n-1} \llcorner \partial^* E$-a.e. then by construction also F has this property. This will be useful in the sequel.

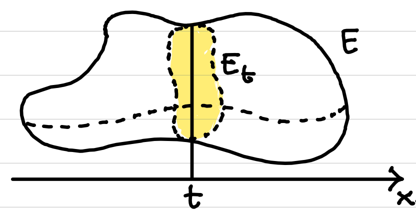
2. Reduction to axis-symmetric sets

Once we reduced to n-symmetric sets, thanks to Theorem 1, we can pass to sets that are symmetric with respect to rotation around one axis keeping the isoperimetric ratio of the same order. This will be done with a finer version of Steiner symmetrization, namely Schwarz symmetrization.
Without loss of generality, we consider x_1 as our axis of symmetry.

2.1. Schwarz symmetrization

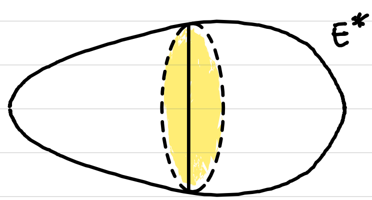
Given $E \subseteq \mathbb{R}^n$ a Borel set, $t \in \mathbb{R}$, we define $E_t \subseteq \mathbb{R}^n$ as

$$E_t := \{x \in E : x_1 = t\}.$$



We define the Schwarz symmetral $E^* \subseteq \mathbb{R}^n$ of E with respect to the axis e_1 as

$$E^* := \left\{x \in \mathbb{R}^n : |x'| < \left(\frac{\mathcal{H}^{n-1}(E_{x_1})}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\right\},$$



namely the set for which every slice $(E^*)_t$ is the $(n-1)$-dimensional ball (centered in 0) with volume the same of E_{x_1}.
We also define $v_E, p_E : \mathbb{R} \to [0, +\infty]$ as $v_E(t) := \mathcal{H}^{n-1}(E_t)$ and $p_E(t) := \mathcal{H}^{n-2}(\partial^* E_t)$.

<u>note</u>: if $n=2$ Schwarz symmetrization coincides with Steiner's.

<u>note</u>: by definition $v_E = v_{E^*}$. As a consequence of the isoperimetric inequality, since E^*_t is a $(n-1)$-dimensional ball of volume $\mathcal{H}^{n-1}(E_t)$, we have that $p_E(t) \geq p_{E^*}(t)$.

As for Steiner symmetrization, Schwarz's does not change the volume and does not increase the perimeter.

<u>Exc</u>: as a direct consequence of Fubini, prove that $|E^*|=|E|$.

To prove that Schwarz symmetrization does not increase the perimeter we need a generalization of the coarea formula.
As this is a standard result, we do not see the proof. You find it for instance in [Maggi (2012), Chapter 18].

<u>Rmk</u> (coarea formula on rectifiable sets): we have seen in Lecture 7 two versions of the coarea formula

$$\int_A |\nabla u(x)|\,dx = \int_{\mathbb{R}} P(\{u>t\};A)\,dt,$$

$$\int_A g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}}\int_{A\cap\{u=t\}} g(y)\,d\mathcal{H}^{n-1}(y)\,dt,$$

under suitable assumptions.
In the next argument, we need to write a surface measure (rather than a volume one) via slicing.
Given $M\subseteq\mathbb{R}^n$, $\mathcal{H}^{n-1}$-rectifiable, $g: M\longrightarrow[0,+\infty]$ Borel function and $u:\mathbb{R}^n\longrightarrow\mathbb{R}$ Lipschitz, then

$$\int_M g(x)\,|\nabla^M u(x)|\,d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}}\int_{M\cap\{u=t\}} g(y)\,d\mathcal{H}^{n-2}(y)\,dt$$

where $\nabla^M u(x) := \nabla u(x) - (\nabla u(x)\cdot\nu_M(x))\,\nu_M(x)$ is the tangential (to M) gradient of u.

<u>Rmk</u> (some facts): we collect some facts that will be useful in the sequel.

(i) if $\nu_E(x)\neq\pm e_1$ $\mathcal{H}^{n-1}\llcorner\partial^*E$-a.e. then $v\in W^{1,1}(\mathbb{R})$ and

$$v_E'(t) = \int_{(\partial^*E)_t} \frac{-(\nu_E(t,z))_1}{\sqrt{1-(\nu_E(t,z))_1^2}}\,d\mathcal{H}^{n-2}(z).$$

Indeed, for any test function $\varphi \in C^1_c(\mathbb{R})$ we get

$$\int_{\mathbb{R}} v_E(t)\varphi'(t)\,dt = \int_{\mathbb{R}} \varphi'(t)\int_{E_t} dz\,dt$$

$$= \int_{\mathbb{R}^n} \varphi'(t)\chi_E(t,z)\,dt\,dz$$

$$= -\int_{\mathbb{R}^n} \varphi(t)\,\partial_1\chi_E(t,z)\,dt\,dz$$

$$= \int_{\partial^* E} \varphi(t)(\nu_E(t,z))_1\,d\mathcal{H}^{n-1}(t,z).$$

By the coarea formula for rectifiable sets (aplied as in the proof of Lemma 4 in the sequel), we get

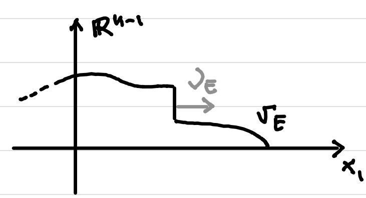
$$\int_{\mathbb{R}} v_E(t)\varphi'(t)\,dt = \int_{\partial^* E}\varphi(t)(\nu_E(t,z))_1\,d\mathcal{H}^{n-1}(t,z)$$

$$= \int_{\partial^* E}\varphi(t)\frac{(\nu_E(t,z))_1}{\sqrt{1-(\nu_E(t,z))_1^2}}\sqrt{1-(\nu_E(t,z))_1^2}\,d\mathcal{H}^{n-1}(t,z)$$

$$= \int_{\mathbb{R}}\varphi(t)\int_{(\partial^* E)_t}\frac{(\nu_E(t,z))_1}{\sqrt{1-(\nu_E(t,z))_1^2}}\,d\mathcal{H}^{n-2}(z)\,dt.$$

This indirectly shows that $v_E' \in L^1(\mathbb{R})$ as well.

Ex ($n=2$): the previous point is very intuitive in dimension 2. In this case $v_E(t)$ defines the boundary of E^*, i.e. $E^* = \{(t,z) : |z| \le \frac{v_E(t)}{2}\}$.



Jumps of v_E corresponds to parts of $\partial^* E^*$ orthogonal to e_1.

$\frac{v_E'(t)}{2}$ is the slope of the tangent line to E^* in $(t, \frac{v_E(t)}{2})$ namely, given $z = \frac{v_E(t)}{2}$,

$$\frac{v_E'(t)}{2} = -\frac{(\nu_E(t,z))_1}{|(\nu_E(t,z))_2|}$$

which coincides with the formula from above for $n=2$.

(ii) if E is axis-symmetric (with axis e_1) then $(\nu_E(t,y))_1$ is constant in y.
Indeed, let us first notice that, for any $R \in SO(n)$, by the area formula and De Giorgi's theorem we have

$$\int_{\partial^* RE} T(x)\cdot \nu_{RE}(x)\, d\mathcal{H}^{n-1}(x) = \int_{RE} \operatorname{div}(T(x))\, dx$$
$$= \int_E \operatorname{div}(T(R^{-1}y))\, dy$$
$$= \int_{\partial^* E} T(R^{-1}y)\cdot \nu_E(y)\, d\mathcal{H}^{n-1}(y)$$
$$= \int_{R(\partial^* E)} T(x)\cdot \nu_E(Rx)\, d\mathcal{H}^{n-1}(x),$$

hence $\partial^* RE = R(\partial^* E)$ and $\nu_{RE}(x) = \nu_E(Rx)$ for ever $x \in \partial^* RE$.
If $Re_1 = e_1$ and E is axis-symmetric with axis e_1 then $RE = E \Rightarrow R\partial^* E = \partial^* E$ hence $\nu_E(x) = \nu_E(Rx)$.
This implies the claim as, for $t \in \mathbb{R}$, $(t,y) \in \partial^* E \Leftrightarrow (t,y') \in \partial^* E$ $\forall |y'| = |y|$, thus $\exists$ R as above s.t. $y' = Ry$.

(iii) E^* is symmetric with respect to $\pi_2, \dots, \pi_n$.
This is immediate by its definition. Indeed, $x \in E^*$ if and only if

$$|(x_2,\dots,x_j,\dots,x_n)| \leq \left(\frac{\mathcal{H}^{n-1}(E_{x_1})}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \Leftrightarrow |(x_2,\dots,-x_j,\dots,x_n)| \leq \left(\frac{\mathcal{H}^{n-1}(E_{x_1})}{\omega_{n-1}}\right)^{\frac{1}{n-1}}$$

thus $R_j x \in E^*$ $\forall$ $j = 2, \dots, n$.
If E is symmetric with respect to $\pi_1 \Rightarrow E^*$ is n-symmetric since, obviously $x \in E_{x_1} \Leftrightarrow R_1 x \in E_{-x_1}$ so $\mathcal{H}^{n-1}(E_{x_1}) = \mathcal{H}^{n-1}(E_{-x_1})$, and so $x \in E^*$ if and only if

$$|x'| \leq \left(\frac{\mathcal{H}^{n-1}(E_{x_1})}{\omega_{n-1}}\right)^{\frac{1}{n-1}} = \left(\frac{\mathcal{H}^{n-1}(E_{-x_1})}{\omega_{n-1}}\right)^{\frac{1}{n-1}}$$

thus $R_1 x \in E^*$.
In particular, if E is n-symmetric $\Rightarrow E^*$ is n-symmetric.

<u>Lemma 4</u> (Schwarz inequality) $E \subseteq \mathbb{R}^n$, $|E| = \omega_n$, of finite perimeter s.t.

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_E(x) = \pm e_1\}) = 0.$$

Then $v_E \in W^{1,1}(\mathbb{R})$ and

$$P(E) \geq \int_{-\infty}^{+\infty} \sqrt{p_E(t)^2 + (v_E'(t))^2}\, dt$$

Moreover, if E is axially-symmetric $p_E(t) = (n-1)\,\omega_{n-1}^{\frac{1}{n-1}}\, v_E(t)^{\frac{n-2}{n-1}}$ and

$$P(E) = \int_{-\infty}^{+\infty} \sqrt{p_E(t)^2 + (v_E'(t))^2}\, dt.$$

This in particular implies that $P(E) \geq P(E^*)$.

<u>proof</u>: $v_E \in W^{1,1}(\mathbb{R})$ by point (i) of the previous Remark.
Notice first that, taking $u(x) = x_1$ in the coarea formula for rectifiable sets we get

$$\nabla u(x) \equiv e_1, \quad \nabla^{\partial^* E} u(x) = e_1 - (\nu_E(x))_1\, \nu_E(x), \quad \partial^* E \cap \{u = t\} = (\partial^* E)_t = \partial^* E_t$$

where the last identity holds up to a set of $\mathcal{H}^{n-2}$-measure zero
Applying the coarea formula as above and Jensen's inequality exploiting the (strict) convexity of $x \mapsto \sqrt{1+|x|^2}$ we obtain

$$\begin{aligned}
P(E) = \mathcal{H}^{n-1}(\partial^* E) &= \int_{\partial^* E} \frac{\sqrt{1 - |(\nu_E(x))_1|^2}}{\sqrt{1 - |(\nu_E(x))_1|^2}}\, dx \\
&= \int_{\mathbb{R}} \int_{\partial^* E_t} \frac{1}{\sqrt{1 - |(\nu_E(t,y))_1|^2}}\, d\mathcal{H}^{n-2}(y)\, dt \\
&= \int_{\mathbb{R}} \int_{\partial^* E_t} \sqrt{1 + \left(\frac{-(\nu_E(t,y))_1}{\sqrt{1 - (\nu_E(t,y))_1^2}}\right)^2}\, d\mathcal{H}^{n-2}(y)\, dt \\
&\geq \int_{\mathbb{R}} \mathcal{H}^{n-2}(\partial^* E_t) \left(1 + \left(\fint_{\partial^* E_t} \frac{-(\nu_E(t,y))_1}{\sqrt{1 - (\nu_E(t,y))_1^2}}\, d\mathcal{H}^{n-2}(y)\right)^2\right)^{\frac{1}{2}} dt.
\end{aligned}$$

Recalling that $p_E(t) = \mathcal{H}^{n-2}(\partial^* E_t)$ and $v_E'(t) = \int_{\partial^* E_t} \frac{-(\nu_E(t,y))_1}{\sqrt{1-(\nu_E(t,y))_1^2}} \, d\mathcal{H}^{n-2}(y)$ by point (i) of the Remark above, we get the desired inequality.

Finally, by point (ii) of the Remark above, if E is axis symmetric $\Rightarrow$ $y \mapsto (\nu_E(t,y))_1$ is constant for $(t,y) \in \partial^* E_t$. Hence, the application of Jensen's inequality in the steps above gives an equality, as claimed.

Since $p_{E^*} \leq p_E$, $v_E = v_{E^*}$, and E^* is axis-symmetric, we have

$$P(E) \geq \int_{\mathbb{R}} \sqrt{p_E(t)^2 + (v_E'(t))^2} \, dt \geq \int_{\mathbb{R}} \sqrt{p_{E^*}(t)^2 + (v_{E^*}'(t))^2} \, dt = P(E^*)$$

which concludes the proof. $\boxed{/}$