As we already commented on, the Hausdorff measure is a generalization of the notion of k-dimensional area.
It is natural to expect that when $k=n$, the Hausdorff measure coincides with Lebesgue's, which generalizes the volume.
This is indeed true.

<u>Theorem</u>: given $E \subseteq \mathbb{R}^n$, $\forall \delta \in (0,\infty]$, $|E| = \mathcal{H}^n_\delta(E) = \mathcal{H}^n(E)$.

To prove it, we will need two tools:

(i) Vitali's property of $\mathcal{L}^n$: given $A \subseteq \mathbb{R}^n$ open, $\delta > 0$ $\exists\, \mathcal{F}_\delta$ countable family of closed, disjoint balls of diameter smaller than δ s.t.

$$\Big| A \setminus \bigcup_{B \in \mathcal{F}_\delta} \overline{B} \Big| = 0.$$

(ii) isodiametric inequality: $\forall E \subseteq \mathbb{R}^n$

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2} \right)^n$$

Property (i) is a standard result in measure theory, and can be seen as a consequence of Besicovitch covering (cf. Lecture 3), so we do not see its proof [Corollary 5.5, Maggi (2012)].

Property (ii) can be proved using Steiner symmetrization, that we will see further in this course.

proof: we work in three steps.

step 1: given a covering $E \subseteq \bigcup_j Q_{r_j}$ of cubes with side-length r_j, hence $\operatorname{diam}(Q_{r_j}) = \sqrt{n}\, r_j$, we have

$$\mathcal{H}^n_\infty(E) \leq \omega_n \sum_j \left(\frac{\sqrt{n}\, r_j}{2}\right)^n = \omega_n \left(\frac{\sqrt{n}}{2}\right)^n \sum_j r_j^n.$$

Infimizing over all the coverings we get

$$\mathcal{H}^n_\infty(E) \leq \omega_n \left(\frac{\sqrt{n}}{2}\right)^n |E|$$

which is an apriori estimate that will be useful later.

step 2: by outer regularity of $\mathcal{L}^n$, $\forall \varepsilon\ \exists A_\varepsilon \supseteq E$ open s.t. $|A_\varepsilon| < |E| + \varepsilon$.
By Vitali's property (i), $\exists \mathcal{F}_\delta$ (as above) s.t.

$$\left| A_\varepsilon \setminus \bigcup_{\bar{B} \in \mathcal{F}_\delta} \bar{B} \right| = 0.$$

Then, by σ-additivity, setting $F = \bigcup_{\bar{B} \in \mathcal{F}_\delta} \bar{B}$,

$$|E| + \varepsilon > |A_\varepsilon| = |F| = \sum_{\bar{B} \in \mathcal{F}_\delta} |\bar{B}| = \omega_n \sum_{\bar{B} \in \mathcal{F}_\delta} \left(\frac{\operatorname{diam}(\bar{B})}{2}\right)^n \geq \mathcal{H}^n_\delta(F).$$

Moreover, by step 1, and since $A_\varepsilon \supseteq E$, we have

$$\mathcal{H}^n_\infty(E \setminus F) \leq \mathcal{H}^n_\infty(A_\varepsilon \setminus F) \leq \omega_n \left(\frac{\sqrt{n}}{2}\right)^n |A_\varepsilon \setminus F| = 0$$

which implies that $\mathcal{H}^n_\delta(E \setminus F) = 0\ \forall \delta$, by a remark we saw in Lecture 2.
Hence by additivity

$$\mathcal{H}^n_\delta(E) = \mathcal{H}_\delta(F) \leq |E| + \varepsilon, \quad \forall \varepsilon > 0$$

which by arbitrariness of $\varepsilon \Rightarrow \mathcal{H}^n(E) \leq |E|$.

Step 3: We are left to prove the opposite inequality whose verification is direct.

Let $\{F_j\}$ be a covering of E with $\operatorname{diam}(F_j) \leq \delta$, then by subadditivity and the isodiametric inequality (ii) we get

$$\omega_n \sum_j \left(\frac{\operatorname{diam}(F_j)}{2} \right)^n \overset{(ii)}{\geq} \sum_j |F_j| \geq \left| \bigcup_j F_j \right| \geq |E|$$

which concludes the proof by taking the inf over the covering and sending $\delta \to 0^+$. $\square$