Lecture 2: 14.04.2026

Today's reference:

- Maggi (2012), Chapters 1–3

1. Outer measures

We start by recalling some topics on measure theory and fix some notation. In particular we introduce the notion of Radon measure which is central in our cours.

We start by introducing outer measures, which is a slightly more flexible approach with respect to work with measures.

Throughout the course $n \geq 1$ denotes the dimension of the ambient space (unless otherwise specified). When we do not specify when an index j is running, then it is either finite or countable.

Def: a set map $\mu : \mathcal{P}(\mathbb{R}^n) \longrightarrow [0,+\infty]$ is an outer measure iff

(i) $\mu(\emptyset) = 0$

(ii) (σ-subadditivity) $\mu(E) \leq \sum_j \mu(E_j)$ for $E \subseteq \bigcup_j E_j$,

Note: an immediate consequence of (ii) is the monotonicity of μ with respect to set inclusion: $\mu(E) \leq \mu(F)$, $E \subseteq F$.

1.1. Some examples

The main examples of outer measures, which we will often work with are listed below.

- Dirac's delta: $x \in \mathbb{R}^n$, $\delta_x(E) := \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$

- atomic measures: $\{x_j\}_j \subseteq \mathbb{R}^n$, $\{\lambda_j\}_j \subseteq [0,+\infty]$, $\mu(E) = \sum_j \lambda_j \delta_{x_j}(E)$

- counting measure: $\mu(E) = \begin{cases} \# E & \text{if } E \text{ is finite} \\ +\infty & \text{otherwise} \end{cases}$

- Lebesgue (outer) measure: for any $E \subseteq \mathbb{R}^n$

$$\mathcal{L}^n(E) := \inf \left\{ \sum_j (r_j)^n : E \subseteq \bigcup_j Q_j \right\}$$

where Q_j are coordinated n-cubes of side-length r_j. When clear from the context $\mathcal{L}^n(E) = |E|$.

Note: it is immediate to see that $\mathcal{L}^n$ is translation invariant and has a scaling property, namely

$$|E + z| = |E|, \ \forall z \in \mathbb{R}^n, \qquad |\lambda E| = \lambda^n |E|, \ \forall \lambda > 0.$$

1.2. Measurable sets

We recall that $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^n)$ is called a σ-algebra iff

(i) $\emptyset \in \mathcal{M}$
(ii) if $E \in \mathcal{M} \Rightarrow \mathbb{R}^n \setminus E \in \mathcal{M}$
(iii) if $\{E_j\}_j \subseteq \mathcal{M} \Rightarrow \bigcup_j E_j \in \mathcal{M}$.

Given a σ-algebra, a set map $\mu : \mathcal{M} \to [0, +\infty]$ is called a measure iff

(i) $\mu(\emptyset) = 0$
(ii) (σ-additivity) $\mu(E) = \sum_j \mu(E_j)$, $E = \bigcup_j E_j$, $E_j \cap E_{j'} = \emptyset$, $j \neq j'$

Note: as above, (ii) implies monotonicity. It is easy to see that (ii) implies monotone continuity, that is $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mu)$

- $E_j \subseteq E_{j+1} \ \forall j \geq 1 \Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j)$

- $E_{j+1} \subseteq E_j \ \forall j \geq 1, \ \mu(E_1) < +\infty \Rightarrow \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j)$

Thanks to the following result, to each outer measure μ we can associate a σ-algebra $\mathcal{M}(\mu)$ s.t. the μ restricted on $\mathcal{M}(\mu)$ is a measure.

Theorem (Carathéodory): let μ be an outer measure, then the class

$$\mathcal{M}(\mu) := \{E \subseteq \mathbb{R}^n : \mu(F) = \mu(F \setminus E) + \mu(F \cap E), \forall F \subseteq \mathbb{R}^n\}$$

defines a σ-algebra and μ restricted on $\mathcal{M}(\mu)$ is a measure.

note: elements of $\mathcal{M}(\mu)$ are said μ-measurable sets.

Rmk (null sets): if E s.t. $\mu(E) = 0 \Rightarrow E \in \mathcal{M}(\mu)$.

Working identically as for measures, we have a theory of integration with respect to outer measures as well. We quickly recall that

- $u : \mathbb{R}^n \to [-\infty, +\infty]$ is μ-measurable iff $\{u > t\} \in \mathcal{M}(\mu)$ $\forall t$
- linear combinations, products, inf, sup, (pointwise) limits of μ-measurable functions are still μ-measurable
- $s : \mathbb{R}^n \to [0, +\infty]$ is called a μ-measurable simple function if $s(x) = \sum_{j} \lambda_j \chi_{E_j}(x)$, $\{\lambda_j\} \subseteq [0, +\infty]$, $\{E_j\} \subseteq \mathcal{M}(\mu)$. We define

$$\int_{\mathbb{R}^n} s(x)\, d\mu(x) = \sum_{j} \lambda_j \mu(E_j)$$

- given a μ-measurable function $u \geq 0$, we define

$$\int_{\mathbb{R}^n} u(x)\, d\mu(x) = \sup\left\{ \int_{\mathbb{R}^n} s(x)\, d\mu(x) : s \leq u \ \mu\text{-meas. simple} \right\}$$

and we generalize it to every μ-measurable u through their positive and negative parts, $u = u_+ - u_-$.

All the classical results on integrations hold true: linearity, monotonicity, triangle inequality of the integral; monotone and dominated convergence, Fatou's lemmas, Fubini and Egoroff theorems.

2. Borel and Radon measures

A class of sets that we "understand well" are open sets. We then characterize those outer measures the "behave well" on open sets.

We denote by $\mathcal{B}(\mathbb{R}^n)$ the Borel σ-algebra, namely the σ-algebra generated by open subsets of $\mathbb{R}^n$.
An outer measure μ s.t. $\mathcal{M}(\mu) \supseteq \mathcal{B}(\mathbb{R}^n)$ is called a Borel measure.

Theorem (Carathéodory's criterion): an outer measure μ is a Borel measure if and only if

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \qquad \forall E_1, E_2 \subseteq \mathbb{R}^n : \operatorname{dist}(E_1, E_2) > 0.$$

Ex: Lebesgue measure is Borel, as every covering of $E_1 \cup E_2$ can be refined into a covering of cubes of diameter smaller than $\operatorname{dist}(E_1, E_2)$. Hence, it can be split into two disjoint coverings of E_1 and E_2.

Ex: the outer measure $\mu(E) = \begin{cases} 0 & E = \emptyset \\ 1 & \#E < +\infty \\ +\infty & \text{otherwise} \end{cases}$ is not a Borel measure, as $1 = \mu(\{x\} \cup \{y\}) \neq \mu(\{x\}) + \mu(\{y\}) = 2$, $x \neq y$

2.1. Borel regular measures

A Borel measure could still have many measurable sets that are not Borel. It is useful to have some more regularity.

Def: an outer measure μ is a Borel regular measure iff is Borel and $\forall E \subseteq \mathbb{R}^n \ \exists B \in \mathcal{B}(\mathbb{R}^n)$ s.t. $B \supseteq E$ and $\mu(E) = \mu(B)$.

Note: if μ and ν are Borel regular and $\mu(A) = \nu(A)$ for every $A \subseteq \mathbb{R}^n$ open $\Rightarrow \mu \equiv \nu$.

note: Lebesgue and atomic measures are Borel regular

2.2. Radon measures

We now introduce the notion of Radon measures which is a local version of finite, Borel regular measures. In our course, we will use them to generalize the notion of volume and surface area.

Def: an outer measure on $\mathbb{R}^n$ is Radon measure if
(i) is Borel regular
(ii) is locally finite (i.e. $\mu(K) < +\infty$, $\forall K \subseteq \mathbb{R}^n$ compact)

note: Lebesgue measure is a Radon measure

Ex: given $\{x_j\} \subseteq \mathbb{R}^n$ s.t. $x_j \xrightarrow[j\to\infty]{} x_0 \in \mathbb{R}^n$. Then $\mu := \sum_j \delta_{x_j}$ is an example of a Borel regular measure which is not a Radon measure, as $\mu(B_r(x_0)) = +\infty$.

Radon measures inherit, from Borel regular measures, the fact to be completely identified by their values on open and compact sets.

Proposition (inner and outer approximation): let μ be a Radon measure, then

$$(i)\ \mu(E) = \inf \{\mu(A) : A \supseteq E,\ A \text{ open}\} \quad \forall E \subseteq \mathbb{R}^n;$$

$$(ii)\ \mu(E) = \sup \{\mu(K) : K \subset E\} \quad \forall E \in \mathcal{M}(\mu).$$

note: property (i) is called outer regularity, (ii) inner regularity.

It will often useful to restrict the action of an outer measure only on subsets of a certain given set.

<u>Def</u>: μ outer measure, $E \subseteq \mathbb{R}^n$, define the restriction

$$\mu \llcorner E : \mathcal{P}(\mathbb{R}^n) \to [0,+\infty], \quad \mu \llcorner F(E) = \mu(E \cap F).$$

Starting from a Borel regular measure, this is a good way to define a Radon measure (see Hausdorff measure below).

<u>Proposition</u>: let μ be a Borel regular measure, $E \in \mathcal{M}(\mu)$ and $\mu(E \cap K) < +\infty$ $\forall K \subseteq \mathbb{R}^n$ compact. Then $\mu \llcorner E$ is a Radon measure.

To each outer measure we associate a closed set in which it "concentrates", which is called its support.
In this way we can use outer measure as geometrical objects.

<u>Def</u>: let μ be an outer measure, we define $\mathrm{spt}(\mu) \subseteq \mathbb{R}^n$ as the intersection of all closed sets $C \subseteq \mathbb{R}^n$ s.t. $\mu(\mathbb{R}^n \setminus C) = 0$. Equivalently

$$\mathrm{spt}(\mu) := \{x \in \mathbb{R}^n : \mu(B_r(x)) > 0, \ \forall r > 0\}.$$

<u>Rmk</u>: it may exist a set $E \subseteq \mathring{\mathrm{spt}}(\mu)$ s.t. $\mu(E) = 0$.
For example, take $\mu := \sum_{j} 2^{-j} \delta_{x_j}$ where $\{x_j\}_j = \mathbb{Q} \cap [0,1]$. Then $\mathrm{spt}(\mu) = [0,1]$, but $E = (0,1) \setminus \mathbb{Q}$ is even dense in the support but $\mu(E) = 0$.

3. Hausdorff measure

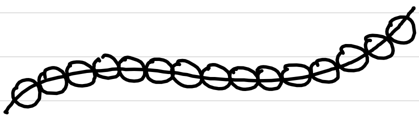
Our main example of outer measure, Lebesgue (outer) measure, cannot capture the behaviour of lower dimen-sional objects, as they have Lebesgue measure zero, think of a curve in $\mathbb{R}^n$, $n \geq 2$.

We introduce Hausdorff measure, which is a refined version of the "area" of lower dimensional sets.

Given $n, k \in \mathbb{N}$ and $\delta > 0$ we define the k-dimensional Hausdorff measure of step δ of $E \subseteq \mathbb{R}^n$ as

$$\mathcal{H}^k_\delta(E) = \inf\left\{ \sum_j \omega_k \left(\frac{\operatorname{diam}(F_j)}{2}\right)^k : E \subseteq \bigcup_j F_j,\ \operatorname{diam}(F_j) \leq \delta \right\}.$$

where $\operatorname{diam}(F) = \sup\{|x-y| : x, y \in F\}$.



<u>Def</u>: the k-dimensional Hausdorff measure of $E \subseteq \mathbb{R}^n$ is

$$\mathcal{H}^k(E) := \sup_{\delta \in (0,+\infty)} \mathcal{H}^k_\delta(E).$$

<u>note</u>: $(0,\infty] \ni \delta \mapsto \mathcal{H}^k_\delta(E)$ is decreasing $\forall E \subseteq \mathbb{R}^n$, so the sup in the definition above can be replaced by $\lim_{\delta \to 0^+}$.

<u>note</u>: it is immediate to see that

$$\mathcal{H}^k(E+z) = \mathcal{H}^k(E)\ \forall z \in \mathbb{R}^n, \quad \mathcal{H}^k(E) = \lambda^k \mathcal{H}^k(E)\ \forall \lambda > 0$$

<u>Rmk</u>: given $s \in [0,\infty)$, we define $\mathcal{H}^s$ as above with

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1+s/2)}, \qquad \Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\,dt.$$

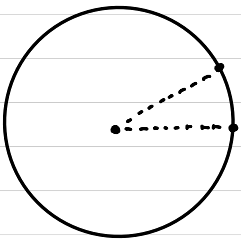
Notice that $\mathcal{H}^0$ is the counting measure.

In the next lecture, by exploiting the Area formula, we see that $\mathcal{H}^k$ coincides with the notion of k-dimensional surface area.

Exc: $\mathcal{H}^s_\delta$ is an outer measure (not Borel), $\mathcal{H}^s$ is Borel regular

It may not be immediately apparent why, in the definition of $\mathcal{H}^s$, we need to take a limit process.

Ex (the need of $\delta \to 0$, squaring the circle): consider $E = \partial B_1$ in $\mathbb{R}^2$. One can prove that, for $\delta \geq 2$, $\mathcal{H}^1_\delta(E) = 2$, namely $\operatorname{diam}(\partial B_1) = 1$.



Let $j \geq 3$ and let δ_j be the side-length of the regular inscribed polygon with j vertices, called P_j.
Then $\mathcal{H}^1_{\delta_j}(E) \leq \operatorname{Per}(P_j)$.

Indeed, we can compute

$$\operatorname{Per}(P_j) = j \left| \left(\cos\left(\tfrac{2\pi}{j}\right), \sin\left(\tfrac{2\pi}{j}\right)\right) - (1,0) \right|$$

$$= j \sqrt{\left(\cos\left(\tfrac{2\pi}{j}\right) - 1\right)^2 + \sin^2\left(\tfrac{2\pi}{j}\right)}$$

$$= j \sqrt{2 - 2\cos\left(\tfrac{2\pi}{j}\right)} = j \sqrt{4 \sin^2\left(\pi/j\right)}$$

$$= 2j \sin(\pi/j) = j\, \delta_j .$$

by choosing j non-overlapping balls with diameter attain on the sizes of P_j. Thus

$$\mathcal{H}^1(E) = \lim_{\delta \to 0^+} \mathcal{H}^1_\delta(E) = \lim_{j \to +\infty} \mathcal{H}^1_{\delta_j}(E) \leq \lim_{j \to +\infty} 2j \sin(\pi/j)$$

$$= 2 \lim_{t \to 0} \frac{\sin(t\pi)}{t} = 2\pi = \operatorname{Per}(\partial B_1).$$

Intuitively, without sending $\delta \to 0$, $\mathcal{H}^1_\delta$ behaves as the diameter, whereas, as $\delta \to 0$, the covering is "forced" to be "close" to ∂B_1, hence $\mathcal{H}^1_\delta$ approximate the length.

3.1. Properties of Hausdorff measure

We collect here some properties and notions that will be useful in the following.

<u>Rmk</u>: $s>n$, $\mathcal{H}^s \equiv 0$.

<u>Indeed</u>: by covering $[0,1]^n = \bigcup \{x+[0,k^{-1}]^n : x \in k^{-1}\mathbb{Z}^n \cap [0,1)\}$ and noticing $\operatorname{diam}([0,1/k]^n) = \sqrt{n}\, 1/k =: \delta$.

$$\mathcal{H}^s_\delta([0,1]^n) \leq \omega_s \left(\frac{\sqrt{n}\, 1/k}{2}\right)^s k^n = \frac{\omega_s n^{s/2}}{2^s} k^{n-s} \xrightarrow[\delta \to 0]{} 0,$$

so $\mathcal{H}^s([0,1]^n) = 0$. By monotone continuity

$$\mathcal{H}^s(\mathbb{R}^n) = \lim_{\lambda \to +\infty} \mathcal{H}^s(\lambda[0,1]^n) = \lim_{\lambda \to +\infty} \lambda^s \mathcal{H}^s([0,1]^n) = 0.$$

Through Hausdorff measure, we can define an "intrinsic" notion of dimension of a subset of $\mathbb{R}^n$, in a measure-theoretical sense.

<u>Def</u>: $E \subseteq \mathbb{R}^n$ its Hausdorff dimension is

$$\dim_{\mathcal{H}}(E) = \inf\{s \in [0,\infty) : \mathcal{H}^s(E) = 0\}$$

The Hausdorff dimension of E, is the last dimension s for which its $\mathcal{H}^s$ measure is not zero.
From the remark below we see that this is actually the unique number for which its $\mathcal{H}^s$ measure is non-trivial.

<u>Rmk</u>: by the previous remark, $\dim_{\mathcal{H}}(E) \in [0,n]$.

Similarly, $\forall s < \dim_{\mathcal{H}}(E) \Rightarrow \mathcal{H}^s(E) = +\infty$

<u>Indeed</u>: let $s>0$ be s.t. $\mathcal{H}^s(E) < +\infty \Rightarrow \forall t > s, \mathcal{H}^t(E) = 0$ as can be seen as follows

$$\mathcal{H}^t_\delta(E) \leq \sum_j \omega_t \left(\frac{\operatorname{diam}(F_j)}{2}\right)^t = \sum_j \omega_t \left(\frac{\operatorname{diam}(F_j)}{2}\right)^s \left(\frac{\operatorname{diam}(F_j)}{2}\right)^{t-s}$$

where $\{F_j\}_j$ is a covering of E with $\operatorname{diam}(F_j) < \delta$, $\forall j$. Since $t > s$ we infer

$$\leq (\delta/2)^{t-s} \frac{\omega_t}{\omega_s} \sum_j \omega_s \left(\frac{\operatorname{diam}(F_j)}{2}\right)^s.$$

Taking the infimum over $\{F_j\}_j$ and $\delta \to 0^+$ we get the claim.

$= 0 \Rightarrow \mathcal{H}^s(E$

Another property that we will often use to prove that a set has zero Housdorff measure is the following

<u>Rmk</u>: if $\mathcal{H}^s_\infty(E)=0 \Rightarrow \mathcal{H}^s_\delta(E)=0 \ \forall \delta>0$. In particular $\mathcal{H}^s(E)=0$.

As Lipschitz functions deform lengths in a controlled way, they "behave well" with diameters, and so with the Housdorff measure.

<u>Rmk</u>: let $f:\mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function of Lipschitz constant $\mathrm{Lip}(f)=L$, namely

$$|f(x)-f(y)| \le L\,|x-y|, \quad \forall x,y \in \mathbb{R}^n.$$

Then $\mathrm{diam}(f(E)) \le L\,\mathrm{diam}(E)$ and so

$$\mathcal{H}^s_{L\delta}(f(E)) \le \omega_s \sum_j \left(\frac{\mathrm{diam}(f(F_j))}{2}\right)^s$$

$$\le L^s \omega_s \sum_j \left(\frac{\mathrm{diam}(F_j)}{2}\right)^s$$

for any covering $\{F_j\}_j$ of E, with $\mathrm{diam}(F_j)<\delta$. Hence $\mathcal{H}^s_{L\delta}(f(E)) \le L^s \mathcal{H}_\delta(E)$ which also yields

$$\mathcal{H}^s(f(E)) \le L^s \mathcal{H}^s(E).$$

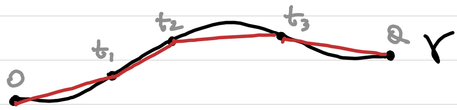
<u>note</u>: if $f:\mathbb{R}^n \to \mathbb{R}^n$ is a projection onto an affine space $H \subseteq \mathbb{R}^n \Rightarrow \mathcal{H}^s(f(E)) \le \mathcal{H}^s(E)$.

3.2. $\mathcal{H}^1$ and the notion of length

One case in which we can easily see "by hands" that $\mathcal{H}^k$ is the k-dimensional surface area is when $k=1$. When $\mathcal{H}^1$ is applied to curves, it coincides with their length.

We say that $\Gamma \subseteq \mathbb{R}^n$ is a is a continuous, simple curve if $\exists\ \gamma : [0,a] \longrightarrow \mathbb{R}^n$, $a>0$, continuous and injective s.t. $\Gamma = \gamma([0,a])$. We define, $\forall\ 0 \leq b < c \leq a$ the length of $\gamma([b,c])$

$$L(\gamma([b,c])) = \sup\left\{ \sum_{h=1}^{N} |\gamma(t_h) - \gamma(t_{h-1})| : b = t_0 < t_1 < \ldots < t_N = c \right\}$$



<u>note</u>: this notion is independent of the parametrization, namely on the function γ.

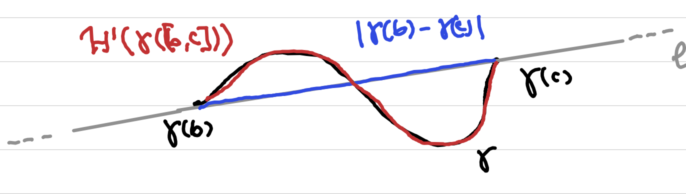
<u>Theorem</u>: let $\Gamma \subseteq \mathbb{R}^n$ be as above such that $L(\Gamma) < +\infty$. Then $\mathcal{H}^1(\Gamma) = L(\Gamma)$.

<u>proof</u>: we work into 3 steps

step 1: let $\ell := \{ \gamma(b) + t(\gamma(c) - \gamma(b)) : t \in \mathbb{R} \}$ be the line passing through $\gamma(b)$ and $\gamma(c)$, and let $p_\ell : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the orthogonal projection onto ℓ.

As p_ℓ is Lipschitz with $\mathrm{Lip}(p_\ell) = 1$, by the previous by the previous remark we have

$$\mathcal{H}^1(\gamma([b,c])) \geq \mathcal{H}^1(p_\ell(\gamma)) \geq \mathcal{H}^1([\gamma(b), \gamma(c)]) = |\gamma(b) - \gamma(c)|.$$



Step 2: let $\{t_k\}$ be a partition of $[0,a]$. And let $F_k := \gamma([t_{k-1}, t_k])$
$\Gamma = \bigcup_{k=1}^{N} F_k$, $F_k \cap F_{k-1} = \{\gamma(t_{k-1})\}$. By step 1

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1\Big(\bigcup_{k=1}^{N} F_k\Big) = \sum_{k=1}^{N} \mathcal{H}^1(F_k) \overset{S1}{\geq} \sum_{k=1}^{N} |\gamma(t_k) - \gamma(t_{k-1})|.$$

Taking the supremum over $\{t_k\}$ we get $\mathcal{H}^1(\Gamma) \geq L(\Gamma)$.

step 3: we now show that Γ can be parametrized by a Lipschitz function γ^* with $\mathrm{Lip}(\gamma^*) \leq 1$ (which is in fact its arch-length parametrization)

Let $v : [0,a] \longrightarrow [0, L(\Gamma)]$, $v(t) := L(\gamma([0,t]))$. Notice is streactley increasing (by γ being injective) and continuous. Let $w : [0, L(\Gamma)] \longrightarrow [0,a]$ be its inverse.

Define $\gamma^* : [0, L(\Gamma)] \longrightarrow \Gamma \subseteq \mathbb{R}^2$ as $\gamma^*(s) := \gamma(w(s))$.
By construction, let $s = v(t)$

$$L(\gamma^*([0,s])) = L(\gamma(w([0,s]))) = L(\gamma([0,t])) = s.$$

By this we get that

$$|\gamma^*(s_1) - \gamma^*(s_2)| \leq L(\gamma^*([s_1, s_2])$$

$$= L(\gamma^*([0,s_2])) - L(\gamma^*([0,s_1]))$$

$$= s_2 - s_1.$$

Hence $\mathrm{Lip}(\gamma^*) \leq 1$ and we obtain

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\gamma^*([0, L(\Gamma)])) \leq \mathcal{H}^1([0, L(\Gamma)]) = L(\Gamma). \boxtimes$$

<u>Exc</u>: let $n \geq 2$, show that $\mathcal{H}^1$ is not a Radon measure.
Given $E \subseteq \mathbb{R}^n$ s.t. $\mathcal{H}^1(E) < +\infty \Rightarrow \mathcal{H}^1 \llcorner E$ is a Radon measure.