Before turning to the proof of the isoperimetric problem we need a last techincal ingredient.

Lemma: if E is of finite perimeter and for a.e. z, E_z is equi-valent to an interval $\Rightarrow$ $\forall z$, $(E^{(1)})_z$ is an interval.

Since this is a very natural property, and its proof is an easy ap-plication of the definiton of density of a point we omit it to shorten the presentation. You find a proof in [Maggi (2012), Lemma 14.5].

1. (Finally) Solving the isoperimetric problem

We are now finally ready to prove the isoperimetric inequality as well as the fact that the ball is the unique minimizer, even including unbounded sets.

Theorem (isoperimetric problem): if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $|E|<+\infty$, then

$$P(E) \geq n\omega_n^{1/n}\,|E|^{\frac{n-1}{n}}, \tag{11}$$

and the equality holds true $\iff$ $|E \Delta B_r(x)|=0$ for some $x\in\mathbb{R}^n$ and $r>0$.

proof: we work into several steps.

step 1 (constraint problem): consider, for $V>0$, $R>\left(\frac{V}{\omega_n}\right)^{\frac{1}{n}}$

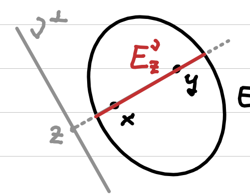
$$\gamma_R(V) = \inf\left\{P(E) : E\subseteq B_R,\ |E|=V\right\}.$$

Using the Direct Method we proved in Lemma 1 of Lecture I.7 that a minimizer E exists.

We prove that E is a ball. As usual we can assume $E=E^{(1)}$. Given $\nu\in S^{n-1}$ and consider the Steiner symmetral with respect to $\nu^\perp$ of E, E^ν. As $|E^\nu|=|E|$ and $\mathrm{diam}(E^\nu)\leq \mathrm{diam}(E)$, then $|E^\nu|=V$ and (up to a translation if needed) $E^\nu\subseteq B_R$. So by minimality of E and Steiner inequality $P(E)\leq P(E^\nu)\leq P(E)$ which yields that $P(E^\nu)=P(E)$.

So, by point (i) of Steiner inequality (and the previous Lemma) we have that, $\forall \nu\in S^{n-1}$, $\forall z\in\nu^\perp$, E_z^ν is an interval.
Now, $\forall x,y\in E$, let $\nu:=\frac{x-y}{|x-y|}$ and $z:=\mathrm{Proj}_{\nu^\perp}(x)=\mathrm{Proj}_{\nu^\perp}(y)$. As $x,y\in E_z^\nu$ and E_z^ν is an interval $\Rightarrow [x,y]\subseteq E_z^\nu$ where $[x,y]$ denotes the segment with endpoints x and y. Hence E is convex.

As E is convex, by point (ii) of Steiner inequality we get that $\forall\nu\in S^{n-1}$, $\exists\, c_\nu\in\mathbb{R}$ s.t. $E=c_\nu\nu+E^\nu$.



<u>step 2</u>: we want to show that E enjoys strong symmetry properties by saying that, up to (one) translation, it coincides with E^ν for any ν.
We first consider $\nu=e_1$. Thus, by step 1, $E-c_{e_1}e_1=E^{e_1}$ which is invariant under reflection through $e_1^\perp$. Translating this set in e_2 does not affect this, so $E-(c_{e_1}e_1+c_{e_2}e_2)$ is still invariant with respect to reflection through $e_1^\perp$. Switching the roles of e_1 and e_2, $E-(c_{e_1}e_1+c_{e_2}e_2)$ is also symmetric with respect to $e_2^\perp$. Iterating this for every coordinate direction,

$$F:=E-(c_{e_1}e_1+\_\_\_+c_{e_n}e_n)$$

is symmetric with respect to every coordinate hyperplane. In particular $F=-F$. As F is a translation of E, for every $\nu\in S^{n-1}$ $\exists\, d_\nu\in\mathbb{R}$ s.t. $F=d_\nu\nu+F^\nu$. Since F is origin-symmetric, this is possible only if $d_\nu=0$.

Indeed, $d_\nu \nu + F^\nu = F = -F = -d_\nu\nu - F^\nu$, but

$$F^\nu_{-z} = \{t\in\mathbb{R} : -z+t\nu \in F\} \overset{F=-F}{=} \{-t\in\mathbb{R} : z+t'\nu\in F\} = -F^\nu_z,$$

$$-F^\nu = \left\{-z-t\nu\in\mathbb{R}^n : |t|<\frac{\mathcal{L}^1(F^\nu_z)}{2}\right\} = \left\{z'+t'\nu\in\mathbb{R}^n : |t'|<\frac{\mathcal{L}^1(F^\nu_{-z'})}{2}\right\} = F^\nu,$$

so $2d_\nu\nu + F^\nu = F^\nu \iff d_\nu = 0$. Hence, $\forall \nu\in\mathbb{S}^{n-1}$, $F=F^\nu$.

<u>Step 3</u>: as in step 2 we proved that $F=F^\nu$ for every $\nu\in\mathbb{S}^{n-1}$, by the property of Steiner symmetral, F is symmetric with respect to every hyperplane $\nu^\perp$. We now prove that the only set enjoying this property is the ball centered in the origin.

We first show that, if $x\in F \Rightarrow \forall y$ s.t. $|y|=|x|$, $y\in F$. Indeed, assuming by contradiction that $\exists y$ s.t. $|y|=|x|$ and $y\notin F$, then let $\nu := \frac{x-y}{|x-y|}$ we have that $R_\nu(x)=y$, and since $R_\nu(F)=F$ we have $y\in F$ which gives a contradiction.
Finally, let $r:=\sup\{|x| : x\in F\}\in(0,R)$, then $F\subseteq B_r$. By definition of sup, $\forall\delta>0\ \exists x_\delta\in F$ s.t. $|x_\delta|>r-\delta \Rightarrow \partial B_{|x_\delta|}\subseteq F$ by what we proved above. By convexity $B_{|x_\delta|}\subseteq F$ so $\forall\delta>0$ $B_{r-\delta}\subseteq F\subseteq B_r$, hence $F=B_r$.

Since this process holds for every minimizer E, then every minimizer is (up to a translation) equivalent to the ball of volume $V \Rightarrow \gamma_R(V) = n\omega_n^{1/n} V^{\frac{n-1}{n}}$.

<u>Step 4</u>: by taking V large enough, and by approximation with bounded set (see Lecture I.8) step 3 ensures the validity of (11) and the optimality of the ball among bounded sets.

Assume now by contradiction that there exists an unbounded set $E\subseteq\mathbb{R}^n$ s.t. $P(E)=n\omega_n^{1/n}|E|^{\frac{n-1}{n}}$. As (11) holds for every set, by Steiner inequality $P(E)\leq P(E^\nu)\leq P(E)$, so proceeding as in step 1 we would obtain that E is a ball which is a contradiction. $\boxtimes$

2. Quantitative isoperimetric inequality via symmetrization

We now turn to the proof of Hall's conjecture from the paper [Fusco - Maggi - Pratelli (2008)]. Namely we will prove

Theorem: for $n \geq 2$, $\exists\ C_n > 0$ s.t. $\forall\ E \subseteq \mathbb{R}^n$ Borel with $0 < |E| < \infty$

$$\alpha(E)^2 \leq C_n\, \delta P(E)$$

recalling that, $\delta P(E) := \dfrac{Per(E)}{n\omega_n^{1/n} |E|^{\frac{n-1}{n}}} - 1$, $\alpha(E) := \min\limits_{x \in \mathbb{R}^n} \dfrac{|E \Delta B_{r_E}(x)|}{|E|}$
where $r_E := (|E|/\omega_n)^{1/n}$.

The main idea of this approach is a rielaboration of a general strategy which is very common in variational problems, that is to replace a candidate minimizer with an energetically better one.

In a certain sense, we used this idea in the previous proof of the qualitative case:

(i) we started from a set E;
(ii) we replaced it with E^ν, which has smaller perimeter;
(iii) we iterated this for every direction, obtaing F which is symmetric with respect to every hyperplanes;
(iv) prove that such an F is a ball.

2.1. Road-map of the proof

The proof strategy to prove the quantitative isoperimetric inequality will be to replace a set E with more and more symmetric sets without increasing (up to constants) the isoperimetric deficit or making the asymmetry degenerate.

We start reminding that, as α and δP are scaling invariant, we will consider $|E| = |B| = \omega_n$, and B will denote the unit ball.
Also, as $\alpha \leq 2$, it will be sufficient to study sets with "small" isoperimetric deficit.

Def (n-symmetric): a Borel set $E \subseteq \mathbb{R}^n$ is n-symmetric iff there exist $\{v_j\}_{j=1}^n \subseteq S^{n-1}$ orthogonal directions s.t. E is symmetric with respect to reflection through $v_j^\perp$, namely

$$R_{v_j^\perp}(E) = E, \qquad \forall\, j = 1, \dots, n.$$

If E is n-symmetric up to a translation we still say that it is n-symmetric.

note: in the sequel, when referring to n-symmetric sets we will always assume that $v_j = e_j$, denoting $\pi_j := e_j^\perp$ and $R_j := R_{\pi_j}$. We can always reduce to this case up to rotations and translations.

Def (axis-symmetric): a Borel set $E \subseteq \mathbb{R}^n$ is axis-symmetric with axis of symmetry $v \in S^{n-1}$ iff $\forall\, R \in SO(n)$ s.t. $Rv = v$, $R(E) = E$.

2.2. Property of n-symmetric sets

We will start our proof by noticing that n-symmetric sets are a particularly good class of sets to work on since they enjoy a convenient property.
Loosely speaking, the optimal balls in Fraenkel asymmetry are close to be centered at the origin, i.e. for every n-symmetric set F (of volume ω_n) it holds

$$\alpha(F) \leq |F \Delta B| \leq C_1 \alpha(F),$$

for some $C_1 > 0$ depending on the dimension.

2.3. Reduction to n-symmetric sets

We will prove that from every set E we can construct an n-symmetric set E' such that

$$\alpha(E) \leq C_2\, \alpha(E'), \qquad \delta P(E') \leq C_2\, \delta P(E).$$

This implies that $\frac{\delta P(E')}{\alpha(E')^2} \leq C^3 \frac{\delta P(E)}{\alpha(E)^2}$. So controlling from below the isoperimetric ratio for n-symmetric sets yields the result for every general set.
Also, from the previous point, it will be sufficient to prove that $|E' \Delta B| \leq C_1 \delta P(E')$.

2.4. Reduction to axis-symmetric sets

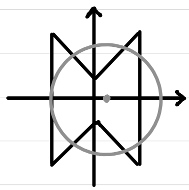
From any n-symmetric set E', we will pass to an axis-symmetric set E^* via Schwartz symmetrization.
By triangle inequality, it will be sufficient to prove the following two estimates

$$|E' \Delta E^*| \leq \sqrt{\delta P(E')} \qquad \text{and} \qquad |E^* \Delta B| \leq \sqrt{\delta P(E')}.$$

This is particularly convenient since the second inequality essentially reduces to a one-dimensional problem (due to the fact that both E^* and B are axis-symmetric).
The first estimate requires some work.

3. <u>Property of n-symmetric sets</u>

We will start working with uniformly bounded sets (we will prove at the end of the whole proof that we can reduce to this case).



We first show that for sets symmetric with respect to reflection through a hyperplane π the center of the optimal ball in the Fraenkel asymmetry is "close" to π.

From now on $\ell = \ell(n) \in (0, +\infty)$ will denote a dimensional constant sufficiently large. We consider the classes

$$X_\ell := \{E \subseteq [-\ell, \ell]^n : \text{Borel}, |E| = \omega_n\}, \quad X_{\ell,\delta} := \{E \in X_\ell : \delta P(E) \leq \delta\}$$

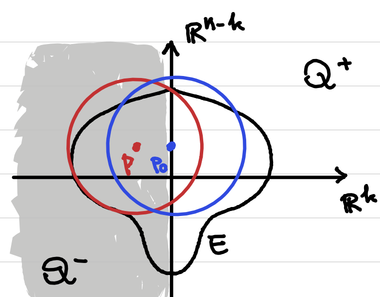
for $\delta > 0$ and sufficiently small.

<u>Lemma 1</u>: let $E \in X_\ell$ and let $\{\nu_j\}_{j=1}^k \subseteq \mathbb{S}^{n-1}$ for $1 \leq k \leq n$ orthogonal. Then

$$\min\left\{ \frac{|E \Delta (x+B)|}{\omega_n} : x \in \bigcap_{j=1}^k \nu_j^\perp \right\} \lesssim 2^k \alpha(E).$$

<u>proof</u>: up to rotations and translations we can assume that $\nu_j = e_j$. Define $Q^\pm := \{x \in \mathbb{R}^n : \pm x_j \geq 0, j = 1, \dots, k\}$ the positive and negative quadrants. By the symmetry of E the center of an optimal ball in Fraenkel asymmetry is in Q^-.

Indeed, let p' such a center, if $p' \notin Q^-$, then the point obtained by switching the sign of all the first k-components that are positive will be optimal for a reflection of E, which equals to E by assumption.



Then clearly $(p+B)\cap Q^+ \subseteq (p_0+B)\cap Q^+$ where $p_0 = \mathrm{Proj}_V(p)$ with $V = \bigcap_{j=1}^{k} \pi_j$, namely $p_0 = (0, \ldots, 0, p_{k+1}, \ldots, p_n)$. Indeed

$$1 > \sum_{j=1}^{n} |x_j - p_j|^2 = \sum_{j=1}^{k} |x_j|^2 + (p_j^2 - 2x_j p_j) + \sum_{j=k+1}^{n} |x_j - p_j|^2$$

$$\geq \sum_{j=1}^{k} |x_j|^2 + \sum_{j=k+1}^{n} |x_j - p_j|^2 = |x - p_0|^2$$

where we used that $p_j \leq 0$ and $x_j \geq 0$ for $j = 1, \ldots, k$. Hence,

$$(E \setminus (p_0 + B)) \cap Q^+ \subseteq (E \setminus (p+B)) \cap Q^+.$$

Again by symmetry, denoting $B(p) = p + B$, $B(p_0) = p_0 + B$, and noticing that $|B(p_0) \setminus E| = |B(p_0)| - |B(p_0) \cap E| = |E| - |B(p_0) \cap E| = |E \setminus B(p_0)|$ since $|E| = |B(p_0)|$, we have

$$|E \triangle B(p_0)| = |E \setminus B(p_0)| + |B(p_0) \setminus E| = 2 |E \setminus B(p_0)|$$

$$\overset{\text{sym}}{=} 2 \cdot 2^k |(E \setminus B(p_0)) \cap Q^+| \leq 2 \cdot 2^k |(E \setminus B(p)) \cap Q^+|$$

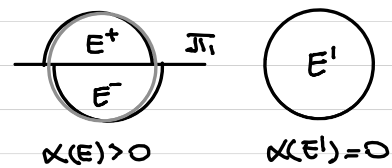
$$\leq 2 \cdot 2^k |E \setminus B(p)| = 2^k |E \triangle B(p)| = 2^k \alpha(E)$$

which concludes the proof. $\boxtimes$

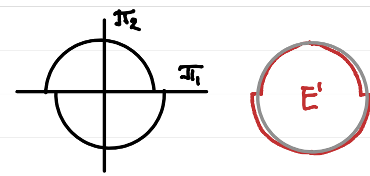
<u>Exc:</u> prove that, for any $\tilde{\pi}_1, \ldots, \tilde{\pi}_k \subseteq \mathbb{R}^n$ orthogonal hyperplanes their intersection is always non-empty. In fact, the intersection is an affine space of dimension $n-k$. In particular, if $k=n$ then the intersection consists of one point.

4. Reduction to n-symmetric sets

We start by reducing ourselves, from a general set E to an n-symmetric set E' so that $\frac{\delta P(E')}{\alpha(E')^2} \leq C \frac{\delta P(E)}{\alpha(E)^2}$.



This is done by cutting E with an hyperplane (say π_1) into two halfs E^+ and E^- having the same volume and reflect one of them. This process trivially controls the deficit but may not control the asymmetry.



This can be avoided by considering another hyperplane orthogonal to π_1. Then at least one of the four sets obtained by reflection has controlled asymmetry. We will then iterate this to finally obtain an n-symmetric set.

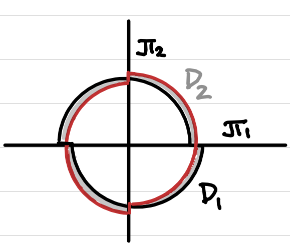
We first need the following technical result. Here and in the sequel we will use the notation $\pi^{\pm} := \{x \in \mathbb{R}^n : \pm \nu \cdot x > 0\}$.

<u>Lemma</u> 2: given $x_1, \sigma_1 \in \pi_1$, $x_2, \sigma_2 \in \pi_2$ and let

$$D_j := (x_j + B) \cap \pi_j^+ \cup (x_j + \sigma_j + B) \cap \pi_j^-, \qquad j = 1,2.$$

Then $\exists\ \varepsilon = \varepsilon(n)$, $C_1 = C_1(n)$ s.t. if $|x_1 - x_2|, |\sigma_1|, |\sigma_2| < \varepsilon$ then

$$|\sigma_1| \vee |\sigma_2| \leq C_1 |D_1 \Delta D_2|.$$

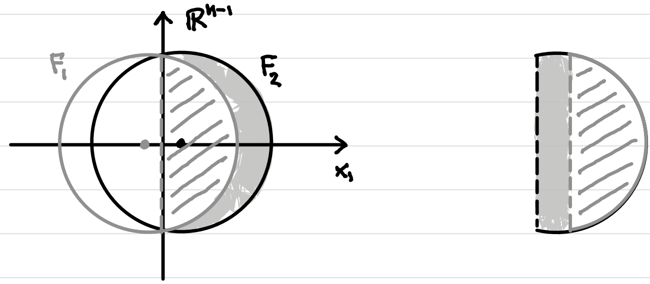


Loosely speaking, this result says that given two half balls slightly shifted across one plane are distant to an orthogonally rotated copy at least linearly in the amplitude of the shift.

Rmk 1: the proof of Lemma 2 is based on a simple geometrical ob-
-servation. Given two unitary balls F_1, F_2 with centers at distance $\eta > 0$ sufficiently small, then $\eta < C_0 |F_1 \Delta F_2|$.
Indeed, up to rotations and translations, we can assume that $F_1 = B - \eta/2\, e_1$, $F_2 = B + \eta/2\, e_1$

$$|F_1 \Delta F_2| = 2|((B+\eta/2\, e_1)\setminus(B-\eta/2\, e_1))\cap \pi_1^+| = 2|\{x\in B_1 : |x_1| < \eta/2\}|.$$

This can be easily controlled from below by $\frac{\omega_{n-1}}{2^{n-2}}\eta$ and from above by $2\omega_{n-1}\eta$, as the set contains the cylinder of base radius $1/2$ and height η, i.e. $B^{n-1}_{1/2}\times(0,\eta)$ and is contained in the cylinder of radius 1 and height η, i.e. $B_1^{n-1}\times(0,\eta)$.



With thi computation we make the following statement that will turn useful many times in the sequel.
There exist $C_1 = C_1(n)$, $C_2 = C_2(n)$, and $\eta_0 = \eta_0(n)$ s.t. for any F_1 and F_2 as above, with $\eta < \eta_0$ it holds that

$$C_1\eta < |F_1 \Delta F_2| < C_2 \eta.$$

With this remark in mind we see the proof of Lemma 2. This proof was not discussed during lecture, so it should be considered a fa-
-cultative topic.

proof (of Lemma 2): let $F_1 = B + x_1$ and $F_2 = B + x_2$ and let $\Omega = \pi_1^+ \cap \pi_2^+$.
By taking ε sufficiently small $\mathrm{dist}(x_1, \pi_2), \mathrm{dist}(x_2, \pi_1) < \varepsilon$ thus clearly $|F_1 \cap \Omega|, |F_2 \cap \Omega| > \frac{\omega_n}{8}$.

By Remark 1, taking $\varepsilon < m_0/2$ we get

$$|D_1 \Delta D_2| \geq |(D_1 \Delta D_2)\cap Q| = |(F_1 \Delta F_2)\cap Q| \geq C_1 |x_1 - x_2|.$$

Working analogously with $F_1 = B + (x_1 + \sigma_1)$ and $F_2 = B + x_2$, taking $Q' = \pi_1^- \cap \pi_2^+$ we get

$$|D_1 \Delta D_2| \geq |(D_1 \Delta D_2)\cap Q'| \geq C_1 |x_1 + \sigma_1 - x_2|.$$

By switching the roles of x_1 and x_2 we conclude. $\square$

With the next result we show how to pass from a general set to one symmetric with respect to π_j with isoperimetric ratio of the same order.
We first introduce some notation. Given $E \subseteq \mathbb{R}^n$, $|E| < +\infty$ and π_j a coordinated hyperplane it is easy to prove that

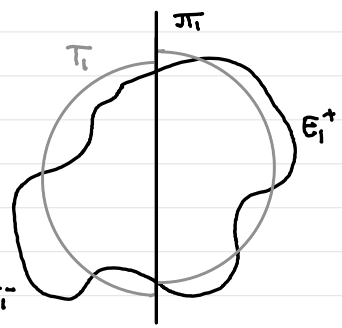
$$\exists\, t \in \mathbb{R} \text{ s.t. } |(E + te_j)\cap \pi_j^+| = \frac{|E|}{2}$$

namely we can cut E with a plane parallel to π_j so that the two halfs have same measure. To prove this is sufficient to show that $t \mapsto |(E+te_j)\cap \pi_j^+|$ is continuous (Exc).
We then write $E_j^\pm := (E + te_j)\cap \pi_j^\pm$.

<u>Lemma</u> 3: there exist $\delta = \delta(n,\ell)$, $C_1 = C_1(n,\ell) > 0$ s.t. $\forall\, E \in X_{\ell,\delta}$ there exists $E' \in \{E_j^\pm \cup R_j(E_j^\pm)\}_{j=1,2}$ s.t.

$$\alpha(E) \leq C_1 \alpha(E'), \qquad \delta P(E') \leq 2\delta P(E).$$

<u>proof</u>: let $p_1^\pm \in \pi_1$ and $p_2^\pm \in \pi_2$ s.t.



$$|((p_j^\pm + B)\cap \pi_j^\pm)\,\Delta\, E_j^\pm| = \min_{p\in \pi_j} |((p+B)\cap \pi_j^\pm)\,\Delta\, E_j^\pm|,$$

and denote $B_j^\pm := (p_j^\pm + B)\cap \pi_j^\pm$. We consider

$$F_j^\pm := E_j^\pm \cup S_j(E_j^\pm) \quad \text{and} \quad T_j := B_j^+ \cup B_j^-$$

Step 1: by qualitative stability of the isoperimetric inequality (cf. Theorem II.4.1 of Lecture II.4), given $\varepsilon(n)$ as in Lemma 2, there exists $\delta(n)$ s.t. if $\delta P(E) < \delta(n) \Rightarrow$ the points $p_1^\pm, p_2^\pm$ are all at mutual distance smaller than ε.
Indeed, assume by contradiction this to be false $\Rightarrow \exists$ a sequence E_h s.t. $\delta P(E_h) \to 0$ and (without loss of generality) $|p_1^+ - p_2^-| > \varepsilon$, where p_1^+ and p_2^- are defined as above for E_h.
By stability, $\alpha(E_h) \to 0$ so up to subsequences $E_h \to B(p)$ for some $p \in \mathbb{R}^n \Rightarrow B(p_1^+), B(p_2^-) \to B(p)$ which gives a contradiction.

Step 2: by construction, $\mathcal{H}^{n-1}(\partial^* F_j^\pm \cap \pi_j) = 0$.
Indeed, assume this is not true, by Exercise III.1.2. of Lecture III.1, $\exists\, x \in \partial^* F_j^\pm \cap \pi_j$ s.t. $\nu_{F_j^\pm}(x) = \pm e_j$. But then, by De Giorgi structure theorem

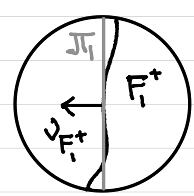
$$\tfrac{1}{\rho}\left(B_\rho(x) \cap F_j^\pm - x\right) \xrightarrow{\rho \to 0^+} B_1 \cap \pi_j^\pm$$

which in particular yields that, for ρ small enough

$$|(F_j^\pm \cap \pi_j^+) \cap B_\rho(x)| > \tfrac{3}{4}\omega_n \rho^n, \quad |(F_j^\pm \cap \pi_j^-) \cap B_\rho(x)| < \tfrac{1}{4}\omega_n \rho^n$$

which contradicts the symmetry of $F_j^\pm$ through π_j.

By this fact we immediately infer that



$$P(F_j^\pm) = P(E;\pi_j^+) + P(E;\pi_j^-) \leq 2P(E)$$

which implies that $\delta P(F_j^\pm) \leq 2\delta P(E)$ $\forall\, j = 1,2$.

Step 3: we prove that for (at least) either $j=1$ or 2 it holds

$$|B_j^- \Delta R_j(B_j^+)| \leq K\left(|E_j^+ \Delta B_j^+| + |E_j^- \Delta B_j^-|\right)$$

for some constant K depending on n.

Indeed, assuming by contradiction this to be false, let $\sigma_1 := p_1^+ - p_1^-$. By Remark 1

$$|B_1^- \Delta R_1(B_1^+)| \leq C_{12}|\sigma_1|.$$

Hence, the contradiction assumption implies that

$$|E \Delta T_1| = |E_1^+ \Delta B_1^+| + |E_1^- \Delta B_1^-| \leq 1/K\, |B_1^- \Delta R_1(B_1^+)| \leq \frac{C_{12}}{K}|\sigma_1|.$$

Working analogously for $j=2$, $|E \Delta T_2| \leq \frac{C_{12}}{K}|\sigma_2|$, which together with Lemma 2 yields

$$\frac{C_{12}}{K}(|\sigma_1| + |\sigma_2|) \geq (|E\Delta T_1| + |E \Delta T_2|) \geq |T_1 \Delta T_2| \geq \tilde{C}_1(|\sigma_1| \vee |\sigma_2|)$$

where $\tilde{C}_1$ is the constant from Lemma 2. This gives a contradi- -ction taking K sufficiently large.

Step 4: let j be the index for which the estimate of step 1 holds true, then

$$\alpha(E) \leq |E \Delta (B_j^+ \cup R_j(B_j^+))| = |E_j^+ \Delta B_j^+| + |E_j^- \Delta R_j(B_j^+)|$$

$$\leq |E_j^+ \Delta B_j^+| + |E_j^- \Delta B_j^-| + |B_j^- \Delta R_j(B_j^+)|$$

$$\overset{S1}{\leq} (1+K)(|E_j^+ \Delta B_j^+| + |E_j^- \Delta B_j^-|) = \frac{1+K}{2}\left(\min_{p \in \pi_j} |F_j^+ \Delta (B+p)| + \min_{p \in \pi_j} |F_j^- \Delta (B+p)|\right).$$

So for (at least) either F_j^+ or F_j^-, say F_j^+, by Lemma 1 of last lec- -ture as F_j^+ is symmetric with respect to π_j

$$2\alpha(F_j^+) \overset{L1}{\geq} \min_{p \in \pi_j} |F_j^+ \Delta (B+p)| \geq \frac{2}{1+K}\alpha(E)$$

which proves the result. $\square$

We iterate this Lemma to obtain an energetically convenient n-sym- -metric set.