Lecture 1: 13.04.2026

Quantitative Isoperimetric-Type Inequalities

Main references for the course:

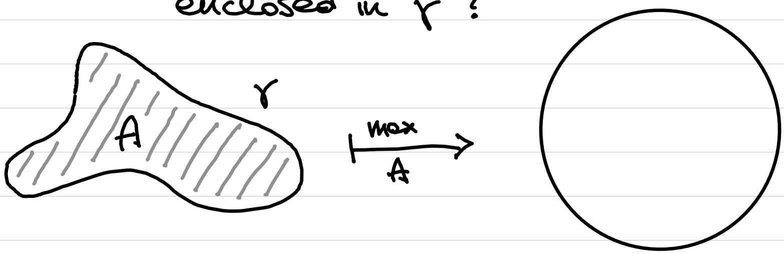
- "The quantitative isoperimetric inequality and related topics", Fusco (2015)
- "Sets of finite perimeter and geometric variational problems", Maggi (2012)
- "The sharp quantitative isoperimetric inequality", Fusco-Maggi-Pratelli (2008)
- "A mass transportation approach to quantitative isoperimetric inequalities", Figalli-Maggi-Pratelli (2010)
- "A selection principle for the sharp quantitative isoperimetric inequality", Cicalese-Leonardi (2012)

The isoperimetric problem

The classical isoperimetric inequality is one of the most well-known and easily understandable mathematical problems.

Its easiest formulation is two-dimensional and dates back to the ancient times, when the legendary founder and queen of Carthage had to face the problem of enclosing the largest area possible with city walls of fixed length.

<u>Dido's Problem:</u> given a simple, closed curve $\gamma \subseteq \mathbb{R}^2$ of length L, what is the maximal area A enclosed in γ?



<u>Folklore solution:</u> $4\pi A \leq L^2$, holds for every γ with the equal sign holding only for circles

This problem (as well as its higher dimensional analog) was not rigorously solved (or even stated) until the 19th century.
There have been partial results by:
Euler (1744), Lagrange (1755), Steiner (1882), Edler (1883);
with the first (not fully rigorous) proof being given by Schwarz (1887)

The first mathematically rigorous proof (in terms of modern mathematical terminologies) is considered the one given by Hurwitz (1901), for dimension 2.

His proof is interesting for two reasons: on the one hand, it shows how a geometrical problem can be stated and solved with analytical methods; on the other hand, it relates the isoperimetric inequality with another functional-analytic one, Wirtinger's inequality (which is a 1D variant, with optimal constant, of Poincaré's inequality).

Theorem (Hurwitz): let $\gamma \subseteq \mathbb{R}^2$ be a simple, closed, absolutely continuous curve of length L enclosing an area A. Then

$$4\pi A \leq L^2,$$

with the equal sign holding only if γ is a circle.

proof: let $E \subseteq \mathbb{R}^2$ be such that $\partial E = \gamma$ (such an E exists by Jordan's Theorem). By Stokes' Theorem

$$A = \int_E dx\,dy = \frac{1}{2}\int_{\partial E} \sigma \cdot \nu(\sigma)\, d\sigma = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

taking γ with counter-clockwise orientation. Let (x_0, y_0) be the barycenter of γ, that is

$$(x_0, y_0) = \left(\frac{1}{L}\int_\gamma x\,dx, \frac{1}{L}\int_\gamma y\,dy\right).$$

Since γ is closed we can write

$$A = \frac{1}{2}\int_\gamma (y_0 - y)\,dx + (x - x_0)\,dy.$$

Let $(x,y) : [0,L] \to \mathbb{R}^2$ be the arclength parametrization of (namely $\dot{x}^2 + \dot{y}^2 \equiv 1$), then

$$A = \frac{1}{2}\int_0^L (y_0 - y(s))\dot{x}(s) + (x(s) - x_0)\dot{y}(s)\,ds$$

$$\overset{C\text{-}S}{\leq} \frac{1}{2}\left(\int_0^L (y_0 - y(s))^2 + (x(s) - x_0)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^L \dot{x}(s)^2 + \dot{y}(s)^2\,ds\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{L}}{2}\left(\int_0^L (y_0 - y(s))^2 + (x(s) - x_0)^2\,ds\right)^{\frac{1}{2}}$$

where we also used Cauchy-Schwarz inequality.
We now use the following lemma

Lemma (Wirtinger's inequality): for any $f:[0,L]\to\mathbb{R}$ absolutely continuous such that $\int_0^L f(t)\,dt=0$, it holds that

$$\int_0^L (f(t))^2\,dt \leq \frac{L^2}{4\pi^2}\int_0^L (f'(t))^2\,dt$$

with the equal sign holding only if $f(t)=\alpha\sin\frac{2\pi t}{L}+\beta\cos\frac{2\pi t}{L}$, for any $\alpha,\beta\in\mathbb{R}$.

Ex: prove the Lemma exploiting Fourier transform

By definition of (x_0,y_0), $\int_0^L (x(s)-x_0)\,ds=\int_0^L x(s)\,ds-Lx_0=0$, and the same holds for $y_0-y(s)$. Applying the lemma

$$A\leq\frac{\sqrt{L}}{2}\left(\int_0^L (y_0-y(s))^2+(x(s)-x_0)^2\,ds\right)^{1/2}$$

$$\leq\frac{\sqrt{L}}{2}\left(\frac{L^2}{4\pi^2}\int_0^L \dot{y}(s)^2+\dot{x}(s)^2\,ds\right)^{1/2}=\frac{\sqrt{L}}{2}\left(\frac{L^3}{4\pi^2}\right)^{1/2}=\frac{L^2}{4\pi},$$

which proves the isoperimetric inequality.
If $4\pi A=L^2$ then

$$\int_0^L (x(s)-x_0)^2\,ds=\frac{L^2}{4\pi}\int_0^L \dot{x}(s)^2\,ds,\quad \int_0^L (y(s)-y_0)^2\,ds=\frac{L^2}{4\pi}\int_0^L \dot{y}(s)^2\,ds$$

and by the lemma

$$x(s)-x_0=\alpha\sin\frac{2\pi s}{L}+\beta\cos\frac{2\pi s}{L},\quad y(s)-y_0=\gamma\sin\frac{2\pi s}{L}+\delta\cos\frac{2\pi s}{L}.$$

Since also Cauchy-Schwartz holds with the equal sign, we infer that (for instance) $y(s)-y_0=\lambda\dot{x}(s)$, for some $\lambda\in\mathbb{R}$. Exploiting the formulas $\sin(\theta_1-\theta_2)=\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1$ and $\cos(\theta_1-\theta_2)=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2$ we got

$$x(s)-x_0=\sqrt{\alpha^2+\beta^2}\cos\left(\frac{2\pi s}{L}-\theta\right),\quad y(s)-y_0=-\frac{2\pi\lambda}{L}\sqrt{\alpha^2+\beta^2}\sin\left(\frac{2\pi s}{L}-\theta\right)$$

with $\theta=\operatorname{arctg}(\alpha/\beta)$. As $\dot{x}^2+\dot{y}^2\equiv1$, $\sqrt{\alpha^2+\beta^2}=|\lambda|=\frac{L}{2\pi}$ and

$$\gamma(s)=(x(s),y(s))=(x_0,y_0)+\left(\frac{L}{2\pi}\cos\left(\frac{2\pi s}{L}-\theta\right),\frac{L}{2\pi}\sin\left(\frac{2\pi s}{L}-\theta\right)\right),$$

which parametrizes a circle. $\boxtimes$

<u>note</u>: by polar coordinates, $\forall\, \alpha,\beta \in \mathbb{R}\ \exists\, \theta \in [0,2\pi)$ s.t. $\alpha = \sqrt{\alpha^2+\beta^2}\cos\theta$ and $\beta = \sqrt{\alpha^2+\beta^2}\sin\theta$.

<u>Rmk</u>: in the second part of the proof, the equal sign in Cauchy-Schwarz inequality is a geometric conditions.
Indeed $(y_0 - y(s))\dot{x}(s) + (x(s)-x_0)\dot{y}(s) = -(x(s)-x_0, y(s)-y_0)\cdot \nu_\gamma(s)$ where $\nu_\gamma(s) = (-\dot{y}(s), \dot{x}(s))$ is the outer normal vector to γ.
As $-(x(s)-x_0, y(s)-y_0)\cdot\nu_\gamma(s) = |(x(s)-x_0, y(s)-y_0)|$, the position vector is normal to γ.

To prove the isoperimetric inequality, in full generality, was not an easy task, as it required a deeper understanding of the concept of surface measure

The most successful approach is variational and is due to De Giorgi (1958). This is so general that can be used to attack different geometric problem analytically.

<u>The isoperimetric inequality</u>: for every measurable set $E \subseteq \mathbb{R}^n$ with finite measure, it holds that

$$n\omega_n^{1/n}\, \mathcal{L}^n(E)^{\frac{n-1}{n}} \leq P(E),$$

where $\mathcal{L}^n$ is the Lebesgue measure, P the Caccioppoli-De Giorgi perimeter, and $\omega_n = \mathcal{L}^n(B_1)$.

In this formulation, the isoperimetric problem can be solved using variational techniques.

<u>Direct Method</u>: given any volume constraint $V>0$, study

$$\min\left\{P(E) : E \subseteq \mathbb{R}^n,\ \mathcal{L}^n(E) = V\right\},$$

with Weierstrass, using coercivity and lower-semicontinuity of P in suitable topologies.
With symmetrization methods (due to Steiner) the unique (up to rigid motions) is the ball.

Rmk: the fact that the ball is the minimizer proves the iso-perimetric inequality.
Indeed, let $r>0$ be s.t. $V=\mathcal{L}^n(B_r)=r^n\mathcal{L}^n(B_1)=r^n\omega_n$, namely $r=V^{1/n}\omega_n^{-1/n}$. The minimality of the ball gives

$$P(E)\geq P(B_r)=r^{n-1}P(B_1)=r^{n-1}n\omega_n=V^{\frac{n-1}{n}}n\,\omega_n^{1/n}=n\omega_n^{1/n}\mathcal{L}^n(E)^{\frac{n-1}{n}}.$$

Ex: use the divergence theorem to prove that the $(d-1)$-dimensional surface area of ∂B_1 (hence $P(B_1)$) is $n\omega_n$.

Rmk (other approaches): – via Brunn-Minkowski inequality
- with the indirect method (variations of surface area) that gives that critical points have constant mean curvature
- via optimal transport (by Gromov)
- countless recent ones (e.g. by Cabré with PDE techniques)

The question of stability

One further question is whether the isoperimetric inequality is "stable" under small perturbations.

<u>Question</u>: if E "almost" minimizes the perimeter is it also "close" to a ball?

<u>Rmk</u>: this question is relevant for all classes of problems with energy of the type

$$F(E) = P(E) + \lambda G(E)$$

with G a term difficult to treat (e.g. nonlocal, anisotropy):
- capillarity (liquid drops);
- nucleation (phase transitions);

This is relevant also for variants of the isoperimetric inequality:
- Wulff's problem;
- Sobolev-type inequality;
- Rayleigh-Faber-Krahn inequality.

Let us formulate rigorously the question of stability.

Let $E \subseteq \mathbb{R}^n$ have positive, finite measure, we define the isoperimetric deficit as

$$\delta(E) := \frac{P(E) - P(B_E)}{P(B_E)} = \frac{P(E)}{n\omega_n^{1/n}|E|^{\frac{n-1}{n}}} - 1,$$

where B_E is the ball with $|B_E| = |E|$.

Quantitative isoperimetric inequalities are those of the form

$$\delta(E) \gtrsim \varphi(E)$$

where φ is called an asymmetry and quantifies "how far" is E from being a ball ($\varphi \geq 0$, $\varphi(E) = 0 \iff E$ is a ball).

These are called Bonnesen-type inequalities, as the first one has been proved (in dimension 2) by Bonnesen (1924).

The first result in higher dimensions is from Fuglede (1989) that used an asymmetry given by the Hausdorff di-stance (a sort of uniform distance for sets):

$$\mathrm{dist}_{\mathcal{H}}(E,F) := \inf\{\rho > 0 : E \subseteq F + B_\rho,\ F \subseteq E + B_\rho\}.$$

<u>note</u>: this is not an asymmetry as $\mathrm{dist}_{\mathcal{H}}(B_1, B_1 + x) = |x|$, whereas an asymmetry should give zero (we need to "quotient" traslations out).

<u>Theorem</u> (Fuglede): $\exists\ \varepsilon, C > 0$ depending on n s.t. for every $K \subseteq \mathbb{R}^n$ convex with $|K| = |B_1|$ and $\delta(K) \leq \varepsilon$ it holds that

$$\min_{x \in \mathbb{R}^n} d_{\mathcal{H}}(E, B_1(x)) \leq C \begin{cases} \delta(K)^{1/2} & n=2 \\ (\delta(K)|\log \delta(K)|)^{1/2} & n=3 \\ \delta(K)^{2/n+1} & n \geq 4. \end{cases}$$

<u>Rmk</u> (ideas): the main idea of Fuglede was to study "nearly spherical" sets, i.e. sets E whose boundary can be written as a graph (of a small function) on the sphere.
In dimension 2, let $E = \{(\rho,\theta) : 0 \leq \rho < 1 + f(\theta),\ \theta \in [0,2\pi)\}$, $\|f\|_{W^{1,\infty}} \ll 1$ and $\int_0^{2\pi} f(\theta)\,d\theta = 0$. Then by Jensen's inequality and the Fundamen-tal Theorem of calculus

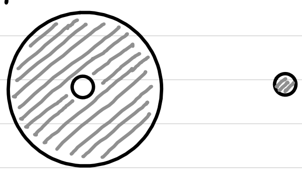
$$P(E) - P(B_1) = \int_0^{2\pi} \sqrt{(1+f(\theta))^2 + f'(\theta)^2} - 1\,d\theta \sim \frac{1}{2}\int_0^{2\pi} (f'(\theta))^2\,d\theta$$

$$\overset{J}{\geq} \frac{1}{4\pi}\left(\int_0^{2\pi} |f'(\theta)|\,d\theta\right)^2 \overset{FT}{\geq} \frac{1}{4\pi}\|f\|_\infty^2 \sim d_{\mathcal{H}}(E,B_1)^2.$$

Unfortunately, dealing with nonconvex sets (not nearly spherical) is more difficult in higher dimensions, e.g. in $n \geq 3$ convexification does not reduce the surface area.

For general sets, the Hausdorff distance is not a good notion to measure the distance between sets.

Ex: $E_\rho := B \setminus B_\rho \cup B_\rho(x)$ with $|x| > 3$, $\rho < 1$.

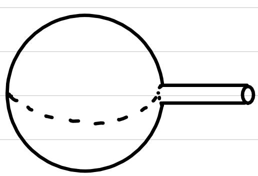


In this case $d_{\mathcal{H}}(E_\rho, B_1) \geq 1$ $\forall\, 0 < \rho < 1$ but $E_\rho \to B_1 \setminus \{0\} \cup \{x\}$ which is the same as B_1 in the sense of Lebesgue.

So $\delta(E_\rho) = 2\pi\rho$, $d_{\mathcal{H}}(E_\rho, B) \geq 1$, so Fuglede's result is violated.

One can have also counter examples with connected sets

Ex: if $n \geq 3$, we can also have tentacles. We can consider



$E_\rho := B_r \cup \{x_1^2 + x_3^2 < \rho^2,\ 0 < x_2 < 2\}$ with $\rho \ll 1$ and $r \sim 1$ so that $|E_\rho| = |B_1|$.
In this case $d_{\mathcal{H}}(E_\rho, B_1) \geq 1$ but $P(E_\rho) \leq P(B_1) + 2\pi\rho + \pi\rho^2$.
So also in this case Fuglede does not hold as $\delta(E_\rho) \leq 2\pi\rho + \pi\rho^2 \to 0$ as $\rho \to 0$.
Also in this case $E_\rho \to B_1 \cup \{(0, x_2, 0) : 0 < x_2 < 1\}$ which is almost everywhere B_1.

Exc: in the two examples above, check that translations do not help

It turns out that, a natural notion of asymmetry is the so-called Fraenkel asymmetry of E

$$\alpha(E) := \inf \left\{ \frac{|E \,\Delta\, (B_E + x)|}{|E|} : x \in \mathbb{R}^n \right\}$$

Exc: check that the inf in α is actually a min.

This was used by Hall (1992) to prove, for smooth sets that

$$\alpha(E)^4 \leq C(n)\,\delta(E).$$

However, he conjectured the sharp exponent to be 2.

<u>Conjecture</u> (the sharp quantitative isoperimetric inequality): for any $n \geq 2$, there exists $C(n) > 0$ such that for every Borel set $E \subseteq \mathbb{R}^n$ with $0 < |E| < +\infty$, it holds that

$$\alpha(E)^2 \leq C(n)\,\delta(E).$$

<u>Rmk</u>: this is the best exponent possible. Indeed, we can consider $E_\varepsilon := \{(1+\varepsilon)^{-1}x_1^2 + (1+\varepsilon)x_2^2 + x_3^2 + \dots + x_n^2 \leq 1\}$.
For this ellipse one can prove that

$$\frac{\delta(E_\varepsilon)}{\alpha(E_\varepsilon)^2} \xrightarrow[\varepsilon \to 0]{} \gamma > 0.$$

<u>Rmk</u>: we can interpret the quantitative isoperimetric inequality (with sharp exponent) as a Taylor expansion of the perimeter around balls (that are minimizers).
Heuristically, for E s.t. $\alpha(E)$ is small (so close to B)

$$\text{"} P(E) - P(B) \gtrsim DP(B)\,\alpha(E) + \tfrac{1}{2}D^2P(B)\,\alpha^2(E) + o(\alpha^2(E)) \text{"}$$

with $DP(B) = 0$ and $D^2P(B) > 0$ by the classical isoperimetric inequality, as balls are unique minimizers.

Hall's conjecture has been proved by three recent contributions

- Fusco–Maggi–Pratelli (2008)
- Figalli–Maggi–Pratelli (2010)
- Cicalese–Leonardi (2012)

These three works not only proved Hall's conjecture but also introduced three methods that can be used to prove other types of quantitative inequalities.

<u>Program</u>: it is divided into 4 blocks

1. preliminaries in GMT
   solve (classical) isoperimetric problem

2. proof via Selection Principle
   refining Fuglede, regularity of minimal surfaces
   application $\mapsto$ quantitative min. of perturbed perimeters

3. proof via optimal transport
   regularity property of the Brenier map
   application $\mapsto$ quantitative Wulff inequality

4. proof via symmetrization
   refining the constructions from Steiner
   application $\mapsto$ quantitative Sobolev inequality