Lecture IV.1: 29.06.2026

In this last block we will see in details the cronologically first proof of the sharp quantitative isoperimetric inequality, in the form conjectured by Hall (in 1992).

This version of the proof is based on symmetrization techniques and does not rely on any sofisticated theory. The main idea is that, starting from an initial competitor, to reduce to study more and more symmetric sets, proving that this process does not (significally) increase the isoperimetric deficit but also does not trivializes the problem, i.e. the asymmetry does not de-crease too much.

We consider the qualitative case first, concluding <u>completely</u> the isoperimetric problem.

1. The isoperimetric inequality

Before giving the full (self-contained) proof via symmetrization we recall some partial results that we have seen in the previous blocks.

1.1. Recall of what we have seen so far

We proved that, among regular, bounded, connected sets with vo-lume 1, the ball is the unique minimizer. This was claimed in Lecture I.8 and then proved in Lecture II.1 and III.2.

<u>Lemma</u> (L8, optimality of the ball): for every $R > \omega_n^{-1/n}$, $\mathcal{J}_R^{reg}(1) = n\omega_n^{\frac{1}{n}}$ and equality holds $\iff |E \Delta B_r(x)| = 0$ for some $r > 0$, $x \in B_R$.

<u>proof</u>: e.g. by Theorem II.1.1. of Lecture II.1. $\boxtimes$

In Lecture I.8 we showed that, using the above lemma together with other approximation results, i.e. Lemma 1 and Lemma 2 of Lecture I.7, we can prove the isoperimetric inequality.

<u>Theorem</u> (isoperimetric inequality): if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $|E| < +\infty$, then

$$P(E) \geq n\omega_n^{1/n} |E|^{\frac{n-1}{n}}. \qquad (11)$$

Moreover if E is bounded, equality holds true $\iff |E \Delta B_r(x)| = 0$ for some $x \in \mathbb{R}^n$, $r > 0$.

The only "left-over" was to show that no unbounded set can attain the minimal perimeter. This problem can be solved by using symmetrization.

1. Steiner symmetrization

It was expected, and we actually proved it for bounded sets, that the ball was the <u>unique</u> global minimizer of perimeter with the volume constraint and in full space.
As the ball enjoys strong symmetric features, a way to make a set "more symmetric" can help in the prove of the isoperi-metric problem.

Given $\nu \in \mathbb{S}^{n-1}$ and $E \subseteq \mathbb{R}^n$, we recall the notation $\nu^\perp := \{x \in \mathbb{R}^n : x \cdot \nu = 0\}$, $\mathrm{Proj}_{\nu^\perp} : \mathbb{R}^n \to \nu^\perp$ the orthogonal projection onto $\nu^\perp$ defined as $\mathrm{Proj}_{\nu^\perp}(x) := x - (x \cdot \nu)\nu$, and $R_\nu : \mathbb{R}^n \to \mathbb{R}^n$ the reflection through $\nu^\perp$ defined as $R_\nu(x) := \mathrm{Proj}_{\nu^\perp}(x) - (x \cdot \nu)\nu$.

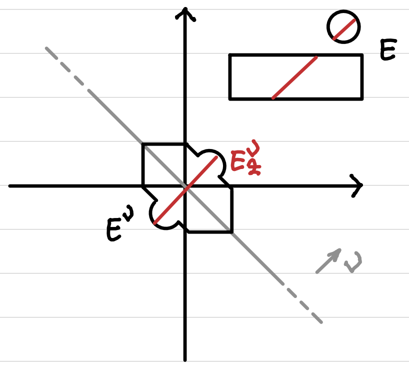
For every $z \in \nu^\perp$ we define the slice of E (in direction ν) at z as

$$E_z^\nu := \{t \in \mathbb{R} : z + t\nu \in E\}.$$

<u>Exc</u>: if E is Lebesgue measurable, for $\mathcal{H}^{n-1}$-a.e. $z \in \nu^\perp$, $E_z^\nu \subseteq \mathbb{R}$ is $\mathcal{L}^1$-measurable.

For $E \subseteq \mathbb{R}^n$ Lebesgue measurable we define its Steiner symmetral (in direction ν) as

$$E^\nu := \left\{ z + t\nu \in \mathbb{R}^n : z \in \nu^\perp,\ |t| < \frac{\mathcal{L}^1(E^\nu_z)}{2} \right\}.$$



Obviously, E^ν is symmetric with respect to reflection across the hyperplane $\nu^\perp$, i.e. $R_\nu(E^\nu) = E^\nu$.
Indeed, if $x \in E^\nu$ by orthogonal decomposition $x = (x\cdot\nu)\nu + \mathrm{Proj}_{\nu^\perp} x$ and, by definition, denoting $z := \mathrm{Proj}_{\nu^\perp} x$, we have $|x \cdot \nu| < \frac{\mathcal{L}^1(E^\nu_z)}{2}$ and hence $R_\nu(x) = z - (x\cdot\nu)\nu$ still belongs to E^ν.

<u>note</u>: another effect of Steiner symmetrization that it is worth noticing is that sets "are translated" towards the origin, which in particular may connect some disconnected components.

<u>note</u>: when $\nu = e_n$ we use the shorthands

$$E_z := \{ t \in \mathbb{R} : (z,t) \in E \},\ z \in \mathbb{R}^{n-1}, \qquad E^s := \{ (z,t) \in \mathbb{R}^n : |t| < m(z),\ z \in \mathbb{R}^{n-1} \}.$$

where we have also denoted $m : \mathbb{R}^n \to [0,+\infty)$ as $m(z) = \frac{\mathcal{L}^1(E_z)}{2}$.

<u>Rmk</u> (volume invariance): as an immediate consequence of Fubini's theorem $|E| = |E^\nu|$. Indeed (for the case $\nu = e_n$ for simplicity)

$$|E| = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_E(z,t)\, dt\, dz$$

but trivially $\chi_E(z,t) = \chi_{E_z}(t)$, hence

$$|E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z)\, dz = \int_{\mathbb{R}^{n-1}} \int_{-m(z)}^{m(z)} 1\, dt\, dz = |E^s|.$$

Rmk (isodiametric inequality): it is not difficult to check that, if $E \subseteq \mathbb{R}^n$ is bounded, then $\operatorname{diam}(E^s) \leq \operatorname{diam}(E)$.
Indeed, among all the bounded sets with the same diameter, the ball is the one that maximizes the volume, i.e.

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2}\right)^n, \quad \forall E \subseteq \mathbb{R}^n.$$

This property, the so-called isodiametric inequality can also be proved exploiting Steiner symmetrization (see the addendum to this lecture) and will turn useful later.

2.1. Steiner inequality

One crucial feature of Steiner symmetrization is that it does not increase the perimeter. To prove this we first need the following result.

Lemma 1: let $E \subseteq \mathbb{R}^n$ be of locally finite perimeter. Then for a.e. $z \in \mathbb{R}^{n-1}$, E_z is of locally finite perimeter and for $I \subseteq \mathbb{R}$ open and bounded, $H \subseteq \mathbb{R}^{n-1}$ compact

$$\int_H P(E_z; I)\, dz \leq P(E; H \times \bar{I}).$$

If E is of finite perimeter, then E_z is of finite perimeter for a.e. $z \in \mathbb{R}^{n-1}$ and

$$\int_{\mathbb{R}^{n-1}} P(E_z)\, dz \leq P(E).$$

note: this result, as well as all the following others, holds identically for E^ν for every $\nu \in \mathbb{S}^{n-1}$.

proof: by regularization $u_\varepsilon := \chi_E * \rho_\varepsilon \longrightarrow \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$. By Fubini's theorem, for every $K \subseteq \mathbb{R}^{n-1}$, $J \subseteq \mathbb{R}$ compacts

$$\int_{K\times J} |u_\varepsilon(x) - \chi_E(x)|\,dx = \int_K \int_J |u_\varepsilon(z,t) - \chi_E(z,t)|\,dz\,dt$$

$$= \int_K \| u_\varepsilon(z,\cdot) - \chi_{E_z} \|_{L^1(J)}\,dz.$$

So, up to subsequences, $\| u_\varepsilon(z,\cdot) - \chi_{E_z}\|_{L^1(J)} \to 0$ for $\mathcal{L}^{n-1}$-a.e. z in $\mathbb{R}^{n-1}$. Also recall that (cf. approximation Proposition in Lecture I.6) $|\nabla u_\varepsilon| \mathcal{L}^n \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \partial^* E$.

Take now $T \in C^1_c(\mathbb{R})$ with $\mathrm{spt}(T) \subseteq J$ and $|T| \leq 1$, then for a.e. $z \in \mathbb{R}^{n-1}$, since $u_\varepsilon(z,\cdot) \to \chi_{E_z}$ in $L^1_{loc}(\mathbb{R})$,

$$\left| \int_{E_z} T'(t)\,dt \right| = \lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}} u_\varepsilon(z,t) T'(t)\,dt \right|$$

$$= \lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}} \partial_n u_\varepsilon(z,t) T(t)\,dt \right|$$

$$\leq \underline{\lim}_{\varepsilon \to 0} \int_J |\partial_n u_\varepsilon(z,t)|\,dt.$$

Taking the sup over T and $J = \bar{I}$ and integrating in $z \in H$ (where I and H are as in the statement) we have by Fatou

$$\int_H P(E_z; I)\,dz = \int_H \sup\left\{ \left| \int_{E_z} T'\,dt \right| : T \in C^1_c(I), |T| \leq 1 \right\} dz$$

$$\leq \int_H \underline{\lim}_{\varepsilon \to 0} \int_{\bar{I}} |\partial_n u_\varepsilon(z,t)|\,dt$$

$$\leq \underline{\lim}_{\varepsilon \to 0} \int_{H\times \bar{I}} |\nabla u_\varepsilon(x)|\,dx$$

$$\leq P(E; H \times \bar{I})$$

which implies that $P(E_z; I) < +\infty$ for a.e. $z \in \mathbb{R}^{n-1}$ and the result is proven. $\boxtimes$

We can now proceed to the proof of Steiner inequality.

__Theorem__ (Steiner inequality): let $E \subseteq \mathbb{R}^n$ be of finite perimeter with $0<|E|<+\infty$. Then E^s is of finite perimeter and $\forall A \subseteq \mathbb{R}^{n-1}$ open

$$P(E^s; A\times\mathbb{R}) \leq P(E; A\times \mathbb{R}),$$

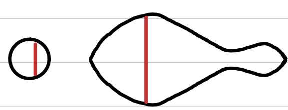
in particular $P(E^s) \leq P(E)$. Moreover

(i) if $P(E^s) = P(E) \Rightarrow$ for $\mathcal{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, E_z is equivalent to an interval

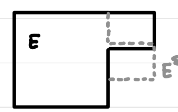
(ii) if E is equivalent to a convex set

$$P(E^s) = P(E) \Longleftrightarrow \exists\, c \in \mathbb{R} \text{ s.t. } |E \,\Delta\, (E^s + c e_n)| = 0.$$

__Rmk__: Steiner inequality tells us that the process of Steiner symmetrization does not increase (even locally) the perimeter. Additionally, point (i) tells us that it "connects" vertical slices, even though E^s can still be non-connected.



(i)



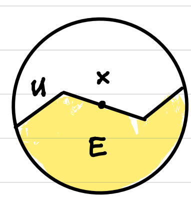
(ii)

The last observation is that, in point (ii), convexity is in general needed to have equivalence between E and E^s (see the picture).

Before turning to the proof we recall the notion of sets with polyhedral boundary.

__Definition__: $E \subseteq \mathbb{R}^n$ has polyhedral boundary if $\forall x \in \partial E$ $\exists r>0$ and $u: \mathbb{R}^{n-1} \to \mathbb{R}$ (finitely) piecewise affine s.t. (up to change the orders of coordinates)

$$B_r(x) \cap E = \{ y \in B_r(x) : y_n < u(y') \}.$$



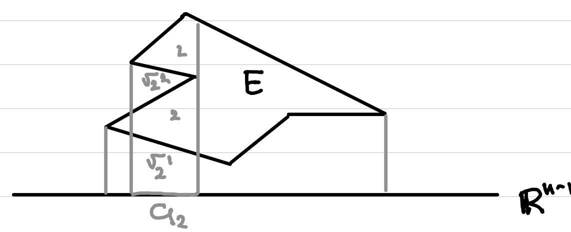
If $x \in \partial^* E$, $\nu_E(y', y_n) = \dfrac{(-\nabla' u(y'), 1)}{\sqrt{1+|\nabla' u(y')|^2}}.$

Proof: for simplicity, we give the proof for the case $A=\mathbb{R}^{n-1}$. The gene-
-ral case is identical.

Step 1: consider first E with polyhedral boundary with $\nu_E(x)\cdot e_n \neq 0$ $\forall x \in \partial E$, and bounded.
Let $G:=\{z\in\mathbb{R}^{n-1}: \mathcal{L}^1(E_z)>0\}$, as E is a polygon, $\forall z\in G$, E_z is a finite union of intervals.
It is easy to prove (see point (i) of the Remark below) that there exist sets $\{G_h\}_{h=1}^M$, $M\in\mathbb{N}$ with the following properties: $\forall h$ $\exists N_h\in\mathbb{N}$ and $v_h^{(k)}, u_h^{(k)}: G_h \to \mathbb{R}$ piecewise affine, $k=1,\dots,N_h$ s.t.

$$G=\bigcup_{h=1}^M G_h, \qquad \partial E=\bigcup_{h=1}^M\bigcup_{k=1}^{N(h)} \Gamma(u_h^{(k)},G_h)\cup\Gamma(v_h^{(k)},G_h),$$

$$E=\bigcup_{h=1}^M\left\{(z,t)\in G_h\times\mathbb{R}:\ t\in\bigcup_{k=1}^{N(h)}(v_h^{(k)}(z),u_h^{(k)}(z))\right\}.$$



We set $m:\mathbb{R}^{n-1}\to\mathbb{R}_+$, $m(z):=\mathcal{L}^1(E_z)$ and

$$m(z)=\sum_{k=1}^{N_h} u_h^{(k)}(z)-v_h^{(k)}(z), \qquad \forall z\in G_h,$$

so m is piecewise affine in G_h. Moreover is continuous, as let $z_0\in\partial G_h\cap\partial G_{h'}$, then by trace of affine functions,

$$\bigcup_{k=1}^{N_{h'}}(v_{h'}^{(k)}(z_0),u_{h'}^{(k)}(z_0))=E_{z_0}=\bigcup_{k=1}^{N_h}(v_h^{(k)}(z_0),u_h^{(k)}(z_0))$$

so m is continuous. As $E^s=\{(z,t)\in\mathbb{R}^n: z\in G, |t|<\frac{m(z)}{2}\}$, then E^s is bounded with polyhedral boundary.

As a consequence of the Area formula, and from the decom-
-position above, we can compute the perimeter of E and E^s.

$$P(E^s)=\mathcal{H}^{n-1}(\partial E^s)=2\mathcal{H}^{n-1}\left(\{(z,\tfrac{m(z)}{2})\in\mathbb{R}^n: z\in G\}\right)$$

Consider $f: G \subseteq \mathbb{R}^{n-1} \longrightarrow \{(z, \frac{m(z)}{2}) \in \mathbb{R}^n : z \in G\} \subseteq \mathbb{R}^n$ defined as $f(z) = (z, \frac{m(z)}{2})$. In the case of graph the Jacobian is

$$Jf(z) = \sqrt{\det(\nabla f(z)^t \nabla f(z))} = \sqrt{1 + 1/4 |\nabla m(z)|^2},$$

see point (ii) of the Remark below. By the Area formula

$$P(E^S) = 2 \int_G Jf(z)\, dz = \int_G \sqrt{4 + |\nabla m(z)|^2}\, dz.$$

Moreover, thanks to the graphical decomposition from above

$$P(E) = \sum_{h=1}^{M} \int_{G_h} \sum_{k=1}^{N_h} \sqrt{1 + |\nabla u_h^{(k)}(z)|^2} + \sqrt{1 + |\nabla v_h^{(k)}(z)|^2}\, dz$$

We now manipulate a bit the expression for $P(E)$ to compare it easily with $P(E^S)$. As the map $x \longmapsto \sqrt{1+|x|^2}$ is convex we have

$$\frac{2N_h}{2N_h} \sum_{k=1}^{N_h} \sqrt{1 + |\nabla u_h^{(k)}|^2} + \sqrt{1 + \nabla v_h^{(k)}|^2} \geqslant 2N_h \left(1 + \left|\frac{1}{2N_h} \sum_{k=1}^{N_h} \nabla(u_h^{(k)} - v_h^{(k)})\right|^2\right)^{\frac{1}{2}}$$

$$= \sqrt{4N_h^2 + |\nabla m|^2}$$

which immediately implies Steiner inequality, namely for E polyhedral it holds that

$$P(E^S) \leqslant P(E).$$

<u>step 2</u>: to prove points (i) and (ii), we will need a stronger estimate than the one proved in the previous step.

For this purpose, let $D := \{z \in G : \mathcal{H}^0(\partial E_z) > 2\}$, namely the set where the slices of E are disconnected. Intuitively, in this area the perimeter of E is a multiple of that of E^S and their difference is comparable to the side of the projection. More precisely by multiplying and dividing by $\sqrt{4N_h^2+|\nabla m(z)|^2}+\sqrt{4+|\nabla m(z)|^2}$ inside the integral

$$
\begin{aligned}
P(E)-P(E^S) &\geq \sum_{h=1}^{M}\int_{G_h}\sqrt{4N_h^2+|\nabla m(z)|^2}-\sqrt{4+|\nabla m(z)|^2}\,dz\\
&=\sum_{h=1}^{M}\int_{G_h}\frac{4(N_h^2-1)}{\sqrt{4N_h^2+|\nabla m(z)|^2}+\sqrt{4+|\nabla m(z)|^2}}\,dz\\
&\geq 2\sum_{h=1}^{M}\int_{G_h\cap D}\frac{1}{\sqrt{4N_h^2+|\nabla m(z)|^2}}\,dz.
\end{aligned}
$$

By Hölder's inequality

$$
\begin{aligned}
(P(E)-P(E^S))\,P(E) &\geq 2\int_D\sum_{h=1}^{M}\frac{\chi_{G_h}(z)}{\sqrt{4N_h^2+|\nabla m(z)|^2}}\,dz\int_D\sum_{h=1}^{M}\chi_{G_h}(z)\sqrt{4N_h^2+|\nabla m(z)|^2}\,dz\\
&\geq 2\left(\int_D\sum_{h=1}^{M}\chi_{G_h}(z)\frac{\sqrt[4]{4N_h^2+|\nabla m(z)|^2}}{\sqrt[4]{4N_h^2+|\nabla m(z)|^2}}\,dz\right)^2=2\mathcal{H}^{n-1}(D)^2.
\end{aligned}
$$

<u>Step 3</u>: given E a general set of finite perimeter, by approximation (cf. Theorem and note of Lecture I.7) there exist E_j bounded with polyhedral boundary s.t. $E_j\to E$ and $P(E_j)\to P(E)$. We can also assume (up to rotations) that $\nu_{E_j}\cdot e_n\neq 0$ see point (iii) of the Remark below.

Note first that, by step 1 $P(E_j^S)\leq P(E_j)$, so Steiner inequality will follow (by lower semicontinuity of perimeter) if we prove that $E_j^S\to E^S$.

By Fubini, and denoting $m_j(z):=\mathcal{L}^1((E_j)_z)$,

$$
\begin{aligned}
|E_j\triangle E| &= \int_{\mathbb{R}^{n-1}}\mathcal{L}^1((E_j)_z\triangle E_z)\,dz=\int_{\mathbb{R}^{n-1}}\mathcal{L}^1((E_j)_z\setminus E_z)+\mathcal{L}^1(E_z\setminus(E_j)_z)\,dz\\
&=\int_{\mathbb{R}^{n-1}}\mathcal{L}^1((E_j)_z)+\mathcal{L}^1(E_z)-2\,\mathcal{L}^1((E_j)_z\cap E_z)\,dz\\
&\geq\int_{\mathbb{R}^{n-1}}\mathcal{L}^1((E_j)_z)+\mathcal{L}^1(E_z)-2\min\{\mathcal{L}^1((E_j)_z),\mathcal{L}^1(E_z)\}\,dz\\
&=\int_{\mathbb{R}^{n-1}}|\mathcal{L}^1((E_j)_z)-\mathcal{L}^1(E_z)|\,dz=2\int_{\mathbb{R}^{n-1}}|m_j(z)-m(z)|\,dz\\
&=|E_j^S\triangle E^S|,
\end{aligned}
$$

where we have used that $\mathcal{L}^1((E_j)_z \cap E_z) \leq \min\{\mathcal{L}^1((E_j)_z), \mathcal{L}^1(E_z)\}$.
So, $E_j^S \to E^S$ and by lower semicontinuity of perimeter

$$P(E^S) \leq \varliminf_{j\to\infty} P(E_j^S) \overset{S1}{\leq} \lim_{j\to\infty} P(E_j) = P(F)$$

which proves Steiner inequality.
As we already commented on, by just integrating on $H \subseteq \mathbb{R}^{n-1}$ we obtain the local version of Steiner inequality as claimed in the statement.

As anticipated, to prove (i) and (ii) we need a stronger inequality. So, by step 2 we note that

$$\mathcal{H}^{n-1}(D_j)^2 \leq P(E_j)(P(E_j) - P(E_j^S)),$$

where $D_j = \{z \in \mathbb{R}^{n-1} : \mathcal{H}^0((E_j)_z) > 2\}$. Taking the limit as $j \to \infty$ we obtain

$$2\varlimsup_{j\to+\infty} \mathcal{H}^{n-1}(D_j)^2 \overset{S2}{\leq} P(E)(P(E) - \varliminf_{j\to+\infty} P(E_j^S)) \leq P(E)(P(E) - P(E^S)).$$

<u>step 4</u> (proof of (i)): if $P(F) = P(E^S)$, the inequality above gives $\mathcal{H}^{n-1}(D_j) \to 0$, that is that a.e. slice is s.t. $\mathcal{H}^0(\partial E_z) = 2$, i.e. an interval. Let us prove this precisely.

Analogously as done in the proof of the previous Lemma, by Fubini $(E_j)_z \to E_z$ for $\mathcal{H}^{n-1}$-a.e. z. By the Lemma a.e. $(E_j)_z$ is of finite perimeter, thus by lower semicontinuity

$$\varliminf_{j\to\infty} \chi_{G\setminus D_j}(z)\, P((E_j)_z) \geq \chi_G(z)\, P(E_z), \quad \text{for a.e. } z \in \mathbb{R}^{n-1}.$$

Hence by Fatou's Lemma

$$\int_G P(E_z)\,dz \leq \varliminf_{j\to\infty} \int_{G\setminus D_j} P((E_j)_z)\,dz = \lim_{j\to\infty} 2\mathcal{H}^{n-1}(G\setminus D_j)$$
$$= 2\mathcal{H}^{n-1}(G) \leq \int_G P(E_z)\,dz$$

where in the last step we used that, if $I\subseteq\mathbb{R}$ is non-trivial and of finite perimeter $\Rightarrow$ it is union of intervals and so $P(I)\geq 2$ (i.e. the one-dimensional isoperimetric inequality). The inequality above implies that $P(E_z)=2$ $\mathcal{H}^{n-1}$-a.e. in z and this proves (i).

<u>Step 5</u> (proof of (ii)): let E be convex and $P(E)=P(E^s)$. We assume E to be open as by Federer's theorem (cf. Lecture I.8) $|E\Delta E^{(1)}|=0$ and $E^{(1)}$ is open. In step 3 we proved $P(E_z)=2\chi_G(z)$. As $G=\mathrm{Proj}_{e_n^\perp}(E)$, G is also open and convex. Define $\psi_1,\psi_2:G\to\mathbb{R}$ as

$$\psi_1(z):=\inf\{t\in\mathbb{R}: t\in E_z\},\qquad \psi_2(z):=\sup\{t\in\mathbb{R}: t\in E_z\}.$$

By convexity of E, ψ_1 is convex and ψ_2 is concave, see point (iv) of the Remark below. Also, by definition

$$E=\{(z,t)\in G\times\mathbb{R}: \psi_1(z)<t<\psi_2(z)\},\quad E^s=\{(z,t)\in G\times\mathbb{R}: |t|<\frac{\psi_2(z)-\psi_1(z)}{2}\}.$$

Again by the Area formula for graphs, for $H\Subset G$

$$P(E;H\times\mathbb{R})=\int_H\sqrt{1+|\nabla\psi_1(z)|^2}+\sqrt{1+|\nabla\psi_2|^2}\,dz,\quad P(E^s;H\times\mathbb{R})=2\int_H\sqrt{1+\left|\frac{\nabla(\psi_2(z)-\psi_1(z))}{2}\right|^2}dz.$$

Taking $\mathcal{H}^{n-1}(\partial H)=0$ we have by step 3 and the above formulas

$$\begin{aligned}P(E^s)=P(E)&=P(E;H\times\mathbb{R})+P(E;\mathbb{R}^n\setminus(H\times\mathbb{R}))\\&\overset{S3}{\geq} P(E^s;H\times\mathbb{R})+P(E^s;\mathbb{R}^n\setminus(H\times\mathbb{R}))=P(E^s)\end{aligned}$$

which yields $P(E;H\times\mathbb{R})=P(E^s;H\times\mathbb{R})$. Hence, by strict convexity of $x\mapsto\sqrt{1+|x|^2}$ it holds a.e. on G that

$$\sqrt{1+|\nabla\psi_1|^2}+\sqrt{1+|\nabla\psi_2|^2}=2\sqrt{1+\left|\frac{\nabla(\psi_2-\psi_1)}{2}\right|^2}\iff\nabla\psi_2=-\nabla\psi_1.$$

Hence $\psi_2 = c - \psi_1$ for some $c \in \mathbb{R}$ and this concludes the proof since

$$E = \{(z,t) \in G \times \mathbb{R} : c - \psi_2(z) < t < \psi_2(z)\}$$

$$E^s = \{(z,t) \in G \times \mathbb{R} : -(\psi_2(z) - c/2) < t < (\psi_2(z) - c/2)\}$$

so $E = \{(z, t + c/2) \in G \times \mathbb{R} : (z,t) \in E^s\} = c/2\, e_n + E^s$. $\boxtimes$

In the next remark we expand some techical arguments we exploited in the above proof.

Rmk: we see some more detailed points of the proof.

(i) we can construct the sets G_h as in step 1 of the proof. For every $k \in \mathbb{N}$, $G^{(k)} := \{z \in G : \mathcal{H}^0(\partial E_z) = 2k\}$. There are only finetly many non-empty, say $M' \in \mathbb{N}$. As E is polyhedral $G = \bigcup_{k=1}^{M'} G^{(k)} \cup N$ with $\mathcal{H}^{n-1}(N) = 0$.

Exc: prove that $\partial G^{(k)} \subseteq \mathrm{Proj}_{e_n}(\bigcup_j \partial S_j)$ where $S_j = \{x \in \partial E : \nu_E \equiv \nu_j\}$, namely the boundaries of $G^{(k)}$ are contained in the projec- - tions of the boundaries of the sides of E.

By the excercise and by linearity, $G^{(k)}$ are $(n-1)$-dimensional polyhedral sets. Denote now $\{G_h\}_{h=1}^M$ the connected components of $\{G^{(k)}\}_{k=1}^{M'}$, for some $M \in \mathbb{N}$, $M \geq M'$.
By definition of polydral sets it is immediate to check that $u_h^{(k)}$ and $v_h^{(k)}$ defining the boundary of E exist.

(ii) given $f(z) = (z, \frac{m(z)}{2})$, we compute its Jacobian. For any $j = 1 - n$ we get $\partial_j f(z) = (e_j, \frac{\partial_j m(z)}{2})$, hence

$$\nabla f(z) = \begin{pmatrix} I_{n-1} \\ \frac{1}{2} \nabla m(z)^t \end{pmatrix} \in \mathbb{R}^{n \times (n-1)}.$$

By row-column multiplication

$$\nabla f(z)^t \nabla f(z) = \left(I_{n-1}, \tfrac{1}{2}\nabla m(z)\right)\begin{pmatrix} I_{n-1} \\ \tfrac{1}{2}\nabla m(z)^t \end{pmatrix} = I_{n-1} + \tfrac{1}{4}\nabla m(z) \otimes \nabla m(z).$$

To compute its determinant it is convenient to compute its eigenvalues. Indeed, let $w_1, \dots, w_{n-2} \in S^{n-2}$ orthogonal base of $\nabla m(z)^\perp$, so that $\{\nabla m(z)/|\nabla m(z)|, w_1, \dots, w_{n-2}\} \subseteq \mathbb{R}^{n-1}$ is an orthonormal base. Now

$$\left(I_{n-1} + \tfrac{1}{4}\nabla m(z) \otimes \nabla m(z)\right)\nabla m(z) = \nabla m(z) + \tfrac{1}{4}\nabla m(z)\,|\nabla m(z)|^2$$

$$= \left(1 + \tfrac{1}{4}|\nabla m(z)|^2\right)\nabla m(z),$$

so $\nabla m(z)$ is an eigenvector with eigenvalue $(1+\frac{1}{4}|\nabla m(z)|^2)$. Whereas, as $(\nabla m(z) \otimes \nabla m(z))w_j = \nabla m(z)(\nabla m(z)\cdot w_j) = 0$ we have

$$\left(I_{n-1} + \tfrac{1}{4}\nabla m(z) \otimes \nabla m(z)\right)w_j = w_j, \qquad \forall\, j=1,\dots,n-2,$$

so w_j are eigenvector with eigenvalues 1. We then immediately have

$$\det(\nabla f(z)^t \nabla f(z)) = 1 + \tfrac{1}{4}|\nabla m(z)|^2$$

as determinant is the product of the eigenvalues.

(iii) given E_j a sequence of polyhedral sets s.t. $E_j \to E$ and $P(E_j) \to P(E)$. Since $\{\nu_{E_j}(x) : x \in \partial^* E_j\}$ is discrete we can find $R_j \in SO(n)$ s.t. $\nu_{R_j E_j}\cdot e_n \neq 0$ and $R_j E_j \to E$.
Indeed, for instance take $R_h \in SO(n)$ the rotation that rotates the h-th and n-th components by an angle $\theta_j \in [0, 2\pi)$, that is $R_1 v = (\cos\theta_j v_1 - \sin\theta_j v_n, v_2, \dots, v_{n-1}, \sin\theta_j v_1 + \cos\theta_j v_n)$ and so on, and take

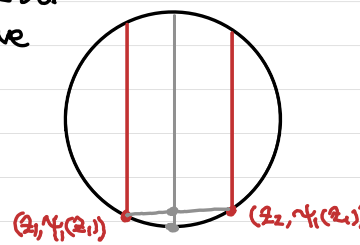
$$R_j = \prod_{h=1}^{n-1} R_h \in SO(n).$$

For θ_j small enough $\nu_{R_jE}\cdot e_n \neq 0$. Moreover, as

$$|R_jE_j \Delta E_j| = \int_{\mathbb{R}^n} |\chi_{E_j}(R_j^t x) - \chi_{E_j}(x)|\,dx \leq P(E_j)\sup_{x\in E_j}|R_j^t x - x|$$
$$\leq C_1|R_j^t - I_n| \leq C_1\,\theta_j^{n-1},$$

by triangle inequality $|R_jE_j\Delta E_j| \leq |E_j\Delta E| + C_1\theta_j^{n-1} \to 0$ taking $\theta_j \to 0$.

(iv) given $z_1, z_2 \in C_1$, $(z_1,\psi_1(z_1)),(z_2,\psi_1(z_2)) \in \partial E$.
By convexity, for every $\lambda\in(0,1)$ we have



$$(\lambda z_1+(1-\lambda)z_2,\ \lambda\psi_1(z_1)+(1-\lambda)\psi_1(z_2)) \in \overline{E}$$

hence $\lambda\psi_1(z_1)+(1-\lambda)\psi_1(z_2) \in \overline{E_{\lambda z_1+(1-\lambda)z_2}}$ that is, as ψ_1 is defined as the inf, that $\psi_1(\lambda z_1+(1-\lambda)z_2) \leq \lambda\psi_1(z_1)+(1-\lambda)\psi_1(z_2)$. Analogous-ly for ψ_2.