S4B1 - Graduate Seminar on Analysis

Functions of Bounded Variation and Applications

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Homework

Lemma 0.1. *Let* (X, μ, σ) *be a measure space, with* μ *a measure on the space* X *having measurable sets given by* σ*. Suppose* μ *is positive and* $\mu(X) < \infty$*. If* $\{U_i\}_{i\in\mathcal{I}} \subset \sigma$ *is a collection (countable or uncountable) of disjoint subsets of* X *only countably many* U_i *can have nonzero mass, i.e.,* $\mu(U_i) > 0$*.*

Proof. For $n \in \mathbb{N}$, as $\mu(X) < \infty$ and the U_i are disjoint, the collection $\{i \in \mathcal{I} : \mu(U_i) > 1/n\}$ must be finite. Consequently,

$$\{i \in \mathcal{I} : \mu(U_i) > 0\} = \cup_{n\in\mathbb{N}}\{i \in \mathcal{I} : \mu(U_i) > 1/n\}$$

is countable. □

There are surely many tweaks you can make of the above idea. The place this is commonly used, as it was in class, is for a measure μ which is finite on compact sets in $\mathbb{R}^N$. Then we apply the above lemma to the collection $\{\partial B(0,t)\}_{t>0}$ (boundary of ball centered at 0 with radius t) to find that $\mu(\partial B(0,t)) > 0$ only for countably many $t > 0$, which implies (as we had wanted) $\mu(\partial B(0,t)) = 0$ for almost every $t > 0$.