Gamma Convergence

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Improved Mass Constraint Argument

Proof. As in class, we showed that for any A with $\chi_A \in BV(\Omega)$ we can construct a sequence of sets A_δ such that

$$\begin{aligned} \mathcal{L}^N(A\Delta A_\delta) \to 0 \quad &\text{as } \delta \to 0, \\ \mathrm{Per}(A_\delta;\Omega) \to \mathrm{Per}(A;\Omega) \quad &\text{as } \delta \to 0, \\ A_\delta = \tilde{A}_\delta \cap \Omega, & \end{aligned} \tag{0.1}$$

where $\tilde{A}_\delta \subset \mathbb{R}^N$ is a set with smooth boundary and $\mathcal{H}^{N-1}(\partial\tilde{A}_\delta \cap \partial\Omega) = 0$.

We now want to show that the same result holds with the mass constraint $⨍ \chi_{A_\delta} = m$. To do this, let x_1 be in the measure theoretic interior and let x_2 be in the measure theoretic exterior of A, meaning

$$\frac{\mathcal{L}^N(B(x_1,r)\cap A)}{\mathcal{L}^N(B(x_1,r))} \to 1 \quad \text{and} \quad \frac{\mathcal{L}^N(B(x_2,r)\cap A)}{\mathcal{L}^N(B(x_2,r))} \to 0 \quad \text{as } r \to 0.$$

Define the new set

$$A_k = A \cup B(x_1,1/k) \setminus B(x_2,1/k),$$

which essentially fills in the region $B(x_1,1/k)$ that was already almost all a part of A; likewise for the ball centered at x_2. It is clear that the A_k converges to A in L^1 and that the perimeter converges.

Using the explicit construction for (0.1), we know that we can choose sets $A_{k,\delta}$ (δ to remind us it came from the super level set of a mollifier at level δ) such that

$$\begin{aligned} \mathcal{L}^N(A_{k,\delta}\Delta A_k) &\le \tau\frac{1}{k^N}, \\ |\mathrm{Per}(A_{k,\delta};\Omega) - \mathrm{Per}(A_k;\Omega)| &\le \frac{1}{k} \\ B\left(x_1,\gamma\frac{1}{k}\right) &\subset A_{k,\delta} \\ B\left(x_2,\gamma\frac{1}{k}\right) &\subset A_{k,\delta}^c \end{aligned} \tag{0.2}$$

for any fixed $\gamma,\tau \in (0,1)$.

Now we wish to correct the mass of $A_{k,\delta}$. If $\mathcal{L}^N(A_{k,\delta}) > \mathcal{L}^N(A)$, we proceed as follows (the other case follows similarly). Let $r_{k,\delta}$ be such that $\mathcal{L}^N(B(0,r_{k,\delta})) = \mathcal{L}^N(A_{k,\delta}) - \mathcal{L}^N(A)$. Define $\bar{A}_k := A_{k,\delta} \setminus B(x_1,r_{k,\delta})$. As $A_{k,\delta}$ converges to A in L^1 (as $k \to 0$) we have that $r_{k,\delta} \to 0$, and consequently, $\bar{A}_k$ satisfies (0.1) with $k \to \infty$ instead of $\delta \to 0$. It remains to show that $\mathcal{L}^N(\bar{A}_k) = \mathcal{L}^N(A)$ and that $\bar{A}_k$ has smooth boundary. By the third line of (0.2), this will be true if

$$r_{k,\delta} < \gamma\frac{1}{k}. \tag{0.3}$$

To do this, we fix $\alpha > 0$ and estimate for sufficiently large k using that x_1 is in the measure theoretic interior:

$$\begin{aligned}
\mathcal{L}^N(A_{k,\delta}) \leq & \mathcal{L}^N(A_k) + \tau\frac{1}{k} \\
\leq & \mathcal{L}^N(A) + \mathcal{L}^N(B(x_1, 1/k) \setminus A) - \mathcal{L}^N(B(x_2, 1/k) \cap A) + \tau\frac{1}{k^N} \\
\leq & \mathcal{L}^N(A) + \alpha\mathcal{L}^N(B(x_1, 1/k)) + \tau\frac{1}{k^N} \\
\leq & \mathcal{L}^N(A) + (\alpha + \tau)\mathcal{L}^N(B(0, 1/k)).
\end{aligned}$$

Rearranging the two sides of the above inequality, it follows that $r_{k,\delta} < (\alpha + \tau)^{1/N}\frac{1}{k}$, and consequently to satisfy (0.3) it suffices to choose $(\alpha + \tau)^{1/N} < \gamma$.

□