Today we finish the proof of Γ-convergence for the Ambrosio–Tortorelli Functional. The proof works similarly to that for Modica–Mortola.

Prop: $\forall\ (u,z)\in[L^1(\Omega)]^2,\ \exists\ (u_\delta,z_\delta)\to(u,z)$ such that

$$\overline{\lim_{\delta\to 0}}\ E_\delta[u_\delta,z_\delta]=E[u,z].$$

Proof: (step 1: $N=1$). WLOG, we assume $E[u,z]<\infty$, so that $u\in SBV^2(\Omega)$ and $z\equiv 1$.

Here $J_u=\{x_1,\dots,x_K\}$, and we focus in near x_1, the construction can be extended.

Let $I\subseteq\Omega$ be such that $\overline{I}\cap J_u=\{x_1\}$, $x_1=0$.

Let $u_\delta\in H^1(\Omega)$ be such that $u_\delta=u$ in $I\setminus(-\eta_\delta,\eta_\delta)$, where $\eta_\delta\ll 1$ (to be chosen later).

Recall we defined an optimal profile for the phase transition as g_ε:

$$g_\delta(x)=\begin{cases}0 & x\le 0\\ \phi_\delta^{-1}(x) & 0\le x\le\phi_\delta(1)\\ 1 & x\ge\phi_\delta(1)\end{cases}$$

and $|\phi_\delta(1)|\le\delta^{1/2}$ with $\dfrac{d}{dt}g_\delta(x)=\dfrac{\sqrt{\delta+V(g_\delta)}}{\delta}$ for $x\in[0,\phi_\delta(1)]$

We define (2)

$$z_\delta = \begin{cases} g_\delta(|x| - \eta_\delta) & |x| \geq \eta_\delta \\ 0 & \text{else} \end{cases}$$

Now we compute that (dropping δ on the gradient term):

$$E_\delta[u_\delta, z_\delta] = \int_I z_\delta^2 \|\nabla u_\delta\|^2 + \frac{1}{\delta} V(z_\delta) + \delta \|\nabla z_\delta\|^2 + \|u_\delta - g\|^2$$

$$\leq \int_I \|\nabla u\|^2 + 2 \underbrace{\int_0^{\phi_\delta(1)} 2\sqrt{\delta + V(g_\delta)}\, |g_\delta'|}_{= \int_0^1 4\sqrt{V(s)}\, ds} + V(0)\frac{\eta_\delta}{\delta} + L^2\text{-term}$$

$$\Rightarrow \overline{\lim_{\delta \to 0}} E_\delta[u_\delta, z_\delta] \leq \int_I \|\nabla u\|^2 + \alpha + L^2 + V(0) \overline{\lim_{\delta \to 0}} \frac{\eta_\delta}{\delta},$$

so we must choose $\eta_\delta / \delta \to 0$.

Hwk include the δ in the gradient term. (may need δ^2)
(or δ^α with $\alpha > 1$.)

Step 2: $N > 1$.

To prove this, we will need a density Result. The one we use is from De Philippis, Fusco, Pratelli.

Thm: Let $u \in SBV^2(\Omega)$. Then $\exists$ $u_j \in SBV^2(\Omega)$ such that there are C^1 manifolds M_j that are compact, $M_j \subset\subset \Omega$, $\mathcal{H}^{N-1}(M_j \setminus J_{u_j}) = 0$,

$$\|u - u_j\|_{BV(\Omega)} \to 0, \quad u_j \in C^\infty(\Omega \setminus M_j),$$

$$\|\nabla u_j - \nabla u\|_{L^2} \to 0, \quad \mathcal{H}^{N-1}(J_{u_j} \Delta J_u) \to 0.$$

Typical approximation theorems in this setting are much simpler as we only need $\int \|\nabla u_j\|^2 \to \int \|\nabla u\|^2$ and

$$\mathcal{H}^{N-1}(J_{u_j}) \to \mathcal{H}^{N-1}(J_u)$$

+ Regularity properties of u_j.

To get this sort of result, one can define

$$E_k[\tilde{u}] = \int \|\nabla \tilde{u}\|^2 + \mathcal{H}^{N-1}(J_{\tilde{u}}) + k\|u - \tilde{u}\|^2_{L^2}.$$

Let $\tilde{u}_k$ be a minimizer of E_k. Since $\tilde{u}$ is admissible,

$E_k[\tilde{u}_k] \le C < \infty$. It follows that $\tilde{u}_k \to u$ in L^2.

By LSC,
$$\varliminf_k \int \|\nabla \tilde{u}_k\|^2 \ge \int \|\nabla u\|^2,$$
$$\varliminf_k \mathcal{H}^{N-1}(J_{\tilde{u}_k}) \ge \mathcal{H}^{N-1}(J_u).$$

But also $E_k[\tilde{u}_k] \to E[\tilde{u}] \Rightarrow$ the above are limits.

(4)

To get Regularity for $\hat{u}_K$ one uses Regularity of minimizers to the Mumford-Shah functional

Back to Step 2:

By a diagonalization argument, it suffices to prove existence of a Recovery sequence for u smooth as in the density Theorem.

Define $\Omega_\delta = \Omega \setminus \{x : \mathrm{dist}(x, M) \leq \eta_\delta\}$.

Define $z_\delta = g_\delta(\mathrm{dist}(x, M) - \eta_\delta)$.

We let $u_\delta \in H^1(\Omega)$, such that $u_\delta \equiv u$ in Ω_δ.

$$E_\delta[u_\delta, z_\delta] \leq \int_\Omega \|\nabla u\|^2 + \int_{\Omega \setminus \Omega_\delta} \frac{1}{\delta} V(z_\delta) + \delta \|\nabla z_\delta\|^2 + L^2\text{-term}$$

$$\leq \int_\Omega \|\nabla u\|^2 + \int_0^{\phi_\delta(1)} 2\sqrt{\delta + V(g_\delta)}\, |g_\delta'| \cdot \sup_{|r| \leq \eta_\delta} \mathcal{H}^{N-1}(\{\mathrm{dist}(x,M) = r\})$$

$$+ \frac{V(0)}{\delta} \eta_\delta \underbrace{C(M)}_{\text{depends on } u} + L^2\text{-term}$$

$$\Rightarrow \overline{\lim}\, E_\delta[u_\delta, z_\delta] \leq \int_\Omega \|\nabla u\|^2 + \alpha \mathcal{H}^{N-1}(M) + L^2\text{-term}$$

✓

We considered the setting of Image Regularization/Segmentation.

Another similar model is the Griffith energy for fracture:
Griffith

$$\int_\Omega \|e(u)\|^2 + \mathcal{H}^{N-1}(J_u), \text{ where}$$

$u : \Omega \to \mathbb{R}^3$ is subject to Dirichlet B.C.

Here $e(u) = \dfrac{Du + (Du)^T}{2}$, the symmetrized Gradient

A recent Result from A. Chambolle and V. Crismale shows that a related Ambrosio-Tortorelli type approximation holds. The Key tool is an appropriate ~~compactness~~ ~~Result~~ Density Result.

Next week we consider more models in Elasticity. Specifically, we will dive into Kirchoff's plate theory as developed by Friesecke, James and Müller.

Here they consider the bending energy of an elastic material in a thin domain. One of the essential tools of independent interest they develop for the problem is a Rigidity estimate for Nonlinear elasticity.

(6)

Frame invariance says the energy of a material is the same no matter where you observe from, so the energy must be invariant under rotations.

$$E[u] \cong \int \mathrm{dist}^2(\nabla u, SO(3))\, dx,$$

which captures the perturbation from the identity.

If $u \approx Id$, then perhaps, $u = Id + \delta v$, for some small δ, so

$$E[u] = \int \mathrm{dist}^2(Id + \delta \nabla v, SO(3))$$

Rescaling and sending $\delta \to 0$, give

$$\int_\Omega \|e(v)\|^2,$$

Tangent space of $SO(3)$ at Id is the space of skew symmetric matrices.

Korn's inequality says if you control the linearized Energy, you control all of v up to a skew sym. matrix:

$$\int |\nabla v - R|^2 \leq C \int \|e(v)\|^2 \quad \text{for some } R \in Skew$$

$C = C(\Omega)$.

The problem is of course this type of elasticity is only (7).
appropriate for small deformations, and in general we need tools
for the fully nonlinear setting.

FJM show that

$$\int \|\nabla u - R\|^2 \leq c \int \mathrm{dist}^2(\nabla u, SO(3)) \quad \text{for some } R \in SO(3)$$

$$C = C(\Omega).$$

and this is a fundamental estimate used in their proof (many others too).

Should we do the proof in this course?

This Result extends Results by John and Kohn (who assume properties of u).