Let us Recall what we were doing. We want to show that

$$E_\delta[u_\delta, z_\delta] = \int_\Omega (z^2+\delta)\|\nabla u\|^2 + \frac{1}{\delta}V(z) + \delta\|\nabla z\|^2 + \|u-g\|^2 \, dx$$

Γ-converges to the functional

$$E[u] = \int_\Omega \|\nabla u\|^2 + \alpha \mathcal{H}^{N-1}(J_u) + \|u-g\|_{L^2}^2$$

where $\alpha = 4\int_0^1 \sqrt{V(s)}\, ds$.

We are focused on the ($\underline{\lim}$) bound in the 1-dimensional case

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Step 1: So far we have shown that

$$\varliminf_{\delta\to 0} E_\delta[u_\delta, z_\delta; I] \geq \alpha \mathcal{H}^0(J_u \cap I) \text{ for any (union of) interval}$$

in Ω. We now want to recover the gradient term in the energy.

Let I be open interval such that $\bar{I} \cap J_u = \emptyset$.
WLOG, we let $I = (0,1)$ and break it into n segments:

$$I_n^k = \left(\tfrac{k-1}{n}, k/n\right).$$

Letting $S_n = S_n(\gamma)$ be defined as

$$S_n := \left\{ k : \varliminf_{\delta\to 0} \left[\inf_{x\in I_n^k} z_\delta(x)\right] < \gamma \right\}$$

By the same argument used to recover the jump, we have that every time z_δ drops down to γ, we pay a price on the order of $2\int_\gamma^1 2\sqrt{V(s)}\,ds$,

~~IIAND~~ Recalling that $\varlimsup_{\delta\to 0} E_\delta[u_\delta, z_\delta] \le C < \infty$ WLOG, we have that

$$4\int_\gamma^1 \sqrt{V(s)}\,ds\,|S_n| \le C \Rightarrow |S_n| \le \frac{C}{4\int_\gamma^1 \sqrt{V}\,ds} = C(\gamma),$$

which is independent of n. The set of indices in S_n are bad ...

$$\varliminf_{\delta\to 0} E_\delta[u_\delta, z_\delta) \ge \varliminf_{\delta\to 0} \int_{\bigcup_{k\notin S_n} I_n^k} (z_\delta^2 + \delta)\|\nabla u_\delta\|^2\,dx$$

$$\ge \gamma^2 \varliminf_{\delta\to 0} \int_{\bigcup_{k\notin S_n} I_n^k} \|\nabla u_\delta\|^2 dx \ge \gamma^2 \int_{\bigcup_{k\notin S_n} I_n^k} \|\nabla u\|^2 dx$$

Letting $n\to\infty$, $\bigcup_{k\notin S_n} I_n^k \xrightarrow[\text{meas}]{} \Omega$, so

$$\varliminf_{\delta\to 0} E_\delta[u_\delta, z_\delta] \ge \gamma^2 \int_I \|\nabla u\|^2 dx.$$ Now let $\gamma\to 1$, to find

$$\varliminf_{\delta\to 0} E_\delta[u_\delta, z_\delta; I] \ge \int_I \|\nabla u\|^2 dx$$

(3)

Putting things together, let $\eta << 1$ and define $I_\eta = J_u + (-\eta, \eta)$.

Then we have that

$$\underline{\lim} E_\delta[u_\delta, z_\delta; \Omega] \geq \underline{\lim} E_\delta[u_\delta, z_\delta; I_\eta] + \underline{\lim} E_\delta[u_\delta, z_\delta; \Omega \setminus I_\eta]$$
$$\alpha \mathcal{H}^0(J_u) + \int_{\Omega \setminus I_\eta} \|\nabla u\|^2 + \|u - g\|^2 \, dx$$

Letting $\eta \to 0$ gives the result.

For the next step, we need to introduce tools for slicing

We start just with the Sobolev Case. $(u \in H^1(\Omega))$

Let ~~us~~ us denote $x \in \mathbb{R}^N$ by

$$x = (x', x_N) \text{ or more generally,}$$
$$x = x_i' + x_i e_i$$

Then we can write $u_{x_i'}(x_i) = u(x_i' + x_i e_i) = u(x)$,

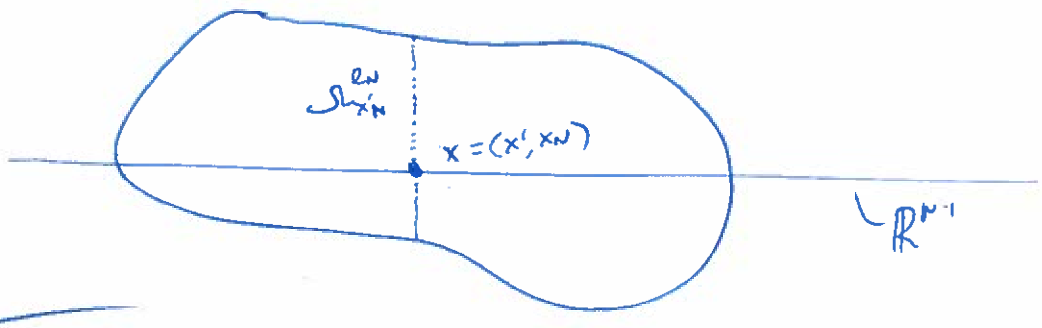
which is a function of 1 variable for fixed x_i'. With domain

$$\Omega_{x_i'}^{e_i} = \{ x_i \in \mathbb{R} \text{ s.t } x_i' + x_i e_i \in \Omega \}$$

$$u_{x_i'} : \Omega_{x_i'}^{e_i} \to \mathbb{R}$$

Thm: $u \in H^1(\Omega)$ iff, $u \in L^2(\Omega)$ and

$$\int_{\mathbb{R}^{N-1}} \left[\int_{\Omega_{x_i'}^{e_i}} \| (u_{x_i})' \|^2 \, dx_i \right] dx_i' < \infty \quad \forall i$$



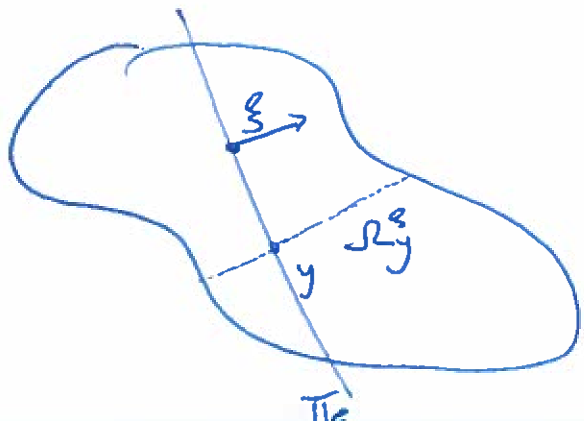
Generally the result holds for BV and slicing in arbitrary directions as follows. For $\xi \in S^{N-1}$, define the associated Hyperplane

$$\Pi_\xi = \{ y \in \mathbb{R}^N : \langle y, \xi \rangle = 0 \}$$

For fixed y, we define the 1-D slice passing through y in Ω as Ω_y^ξ:

$$\Omega_y^\xi = \{ t \in \mathbb{R} : y + t\xi \in \Omega \}$$

Likewise $u_y^\xi(t) = u(y + t\xi)$, $u_y^\xi : \Omega_y^\xi \to \mathbb{R}$.



Thm: $u \in BV(\Omega)$ iff

$$\int_{\Pi_\xi} \| u_y^\xi \|_{BV(\Omega_y^\xi)} \, d\mathcal{H}^{N-1}(y) < \infty \quad \forall \xi \in S^{N-1}$$

Further

$$J_{u_y^\xi} = \{ t \in \mathbb{R} \text{ s.t. } y + t\xi \in J_u \} \quad y\text{-a.e.}$$

$$\langle D_a u, \xi \rangle = D_a u_y^\xi \otimes \mathcal{H}^{N-1} \llcorner \Pi_\xi$$

$$\langle D_j u, \xi \rangle = D_j u_y^\xi \otimes \mathcal{H}^{N-1} \llcorner \Pi_\xi$$

$$\langle D_c u, \xi \rangle = D_c u_y^\xi \otimes \mathcal{H}^{N-1} \llcorner \Pi_\xi$$

How is this relevant to ($\underline{\lim}$) bounds? Suppose we want to show LSC of $\int \|\nabla u\|$ under weak convergence with $u_i \to u$ in L^2 having 1 result.

$$\underline{\lim}_{i\to\infty} \int_\Omega \|\nabla u_i\|^2 \geq \underline{\lim}_{i\to\infty} \int_\Omega \| \langle \nabla u_i, \xi \rangle \|^2 \, dx$$

slicing

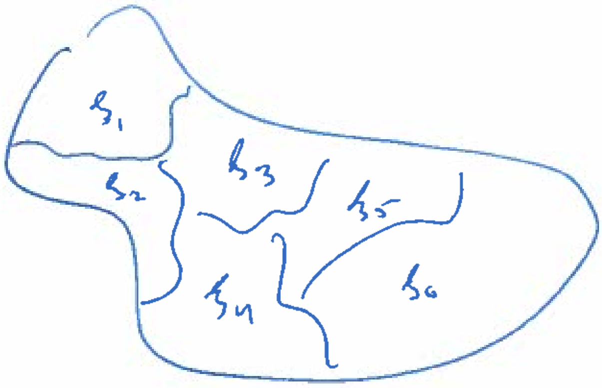
$$= \underline{\lim}_{i\to\infty} \int_{\Pi_\xi} \left[\int_{\Omega_y^\xi} |(u_{i,y}^\xi)'|^2 \, dt \right] d\mathcal{H}^{N-1}(y)$$

Fatou's

$$\geq \int_{\Pi_\xi} \underline{\lim}_{\delta\to 0} \left[\int_{\Omega_y^\xi} |(u_{i,y}^\xi)'|^2 \, dt \right] d\mathcal{H}^{N-1}(y)$$

$$\geq \int_{\Pi_\xi} \int_{\Omega_y^\xi} |(u_y^\xi)'|^2 \, dt \, dy \underset{\text{slicing}}{=} \int_\Omega \| \langle \nabla u, \xi \rangle \|^2 \, dx$$

Noting that $\|Du\| = \sup_{\xi \in S^{N-1}} |\langle Du, \xi \rangle|$, if we could take the supremum over ξ, we would have $\underline{\lim} \int \|Du_j\|^2 \geq \int \|Du\|^2$,



(use ($\underline{\lim}$) locally).

$$\underline{\lim} \int \|Du\|^2 \geq \sum_i \int_{\Omega_i} |\langle Du, \xi_i \rangle|^2 \, dx$$

$\rightsquigarrow$ then optimize.

Thm. Suppose $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ is an increasing set-function, superadditive on disjoint open sets: $\mu(A \cup B) \geq \mu(A) + \mu(B)$.

If ψ_i is a countable collection of Borel functions and λ radon measure st

$$\mu(A) \geq \int_A \psi_i \, d\lambda \quad \forall i \Rightarrow$$

$$\mu(A) \geq \int_A \sup_i \psi_i \, d\lambda.$$

Proof:

$$\int_A \sup \psi_i \, d\lambda = \sup \left\{ \sum_{i=1}^N \int_{B_i} \psi_i \, d\lambda : B_i \text{ disjoint Borel} \right\}$$

Inner Regularity

$$= \sup \left\{ \sum_{i=1}^N \int_{K_i} \psi_i \, d\lambda : K_i \text{ compact disjoint} \right\}$$

outer regularity.

$$= \sup \left\{ \sum_{i=1}^N \int_{A_i} \psi_i \, d\lambda : A_i \text{ open disjoint} \right\}$$

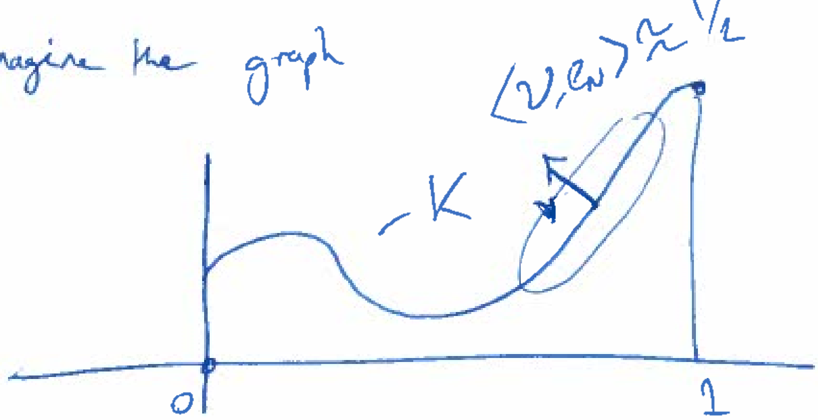
$$\leq \sum \mu(A_i) \leq \mu(\cup A_i) \leq \mu(A).$$

One last fact for K and $N-1$-rectifiable subset.

$$\int_{\mathbb{R}^{N-1}} \mathcal{H}^0(\{x : x' = y\} \cap K\}\, dy = \int_K |\langle \nu, e_N \rangle| \, d\mathcal{H}^{N-1}$$

(Coarea formula with $\Pi(x) = (x', 0)$)

For intuition, imagine the graph



$\int_0^1 \mathcal{H}^0(\{x; x' = y\} \cap K\}\, dy = 1$, So if we integrate over surface, the surface area is to much. $|\langle \nu, e_N \rangle|$ is the appropriate factor scaling area down.

Proof of Step 2 $(N > 1)$. Define $\mu(A) = \underline{\lim}\, E_\delta[u_\delta, z_\delta; A]$

Slicing in the N^{th} direction, we have (Drop L^2-term as it is easy)

$$E_\delta[u_\delta, z_\delta; A] \geq \int_\Omega (z_\delta^2 + \delta) |\langle \nabla u_\delta, e_N \rangle|^2 + \frac{V(z_\delta)}{\delta} + \delta |\langle \nabla z_\delta, e_N \rangle|^2$$

$$= \int_{\Pi_{e_N}} \left[\int_{\Omega_{x'}^{e_N}} (z_\delta^2 + \delta) |u_{\delta x'}'|^2 + \frac{V(z_\delta)}{\delta} + \delta |z_{\delta, x'}'|^2 \, dx_N \right] dx'$$

$$\Rightarrow \underline{\lim}_{\delta \to 0} E_\delta \geq \int_{\Pi_{e_N}} \left[\int_{\Omega_{x'}^{e_N}} |\langle \nabla u, e_N \rangle|^2 + \alpha \mathcal{H}^0(J_{u_{x'}}) \right] dx'$$

Remember that $J_{u_{x'}} = \{t : (x', t) \in J_u\}$

So

$$\int_{\Pi_{e_N}} \alpha \mathcal{H}^0(J_{u_{x'}})\, dx' = \int_{\Pi_{e_N}} \alpha\, \mathcal{H}^0(\{x : x' = y\} \cap J_u)\, dx'$$

$$= \alpha \int_{J_u} |\langle \nu, e_N \rangle|\, d\mathcal{H}^{N-1}$$

$$\underline{\lim}\, E_\delta[u_\delta, z_\delta] \geq \int_\Omega |\langle \nabla u, e_N \rangle|^2 + \alpha \int_{J_u} |\langle \nu, e_N \rangle|\, d\mathcal{H}^{N-1}.$$

Same holds for e_N with $\xi \in S^{N-1}$. Taking supremum with Lemma concludes the result.