(1)

Today we introduce the Mumford-Shah functional and the associated Ambrosio-Tortorelli Functional.

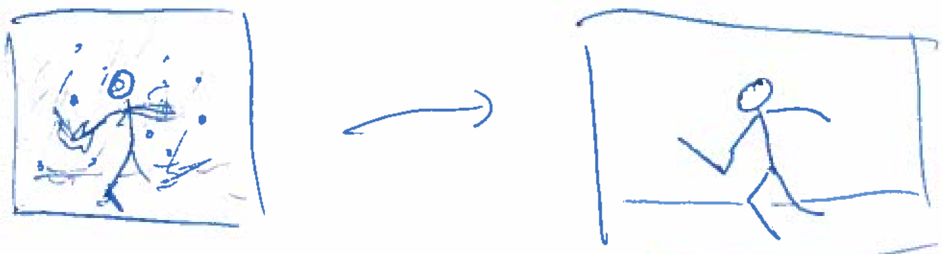
Mumford-Shah was introduced in the context of Image Regularization.

Here an image (in gray-scale) is represented by a function $g:\Omega \to [0,1]$ 0 = white, 1 = Black

but perhaps the way g was obtained lead to errors in the measurement and the image is noisy. One way to recover a cleaner image is Regularization. For ~~this~~ this we introduce

$$\tilde{E}[u,K] = \int_{\Omega\setminus K} \|\nabla u\|^2 dx + \alpha \mathcal{H}^{N-1}(K) + \|u-g\|^2_{L^2}$$

$$K \subseteq \mathbb{R}^N \text{ closed}, \quad u \in W^{1,2}(\Omega\setminus K).$$



g (u,K) minimizing $\tilde{E}$.

K allows for contrasting colors in neighbors

From the variational perspective the first challenge is applying the direct method. For which topology does

$$(u_i,k_i) \longrightarrow (u,K)\,???$$

The energy was studied a lot in the late 80's ~~early~~ through 90's (and still). An essential step was the relaxation of the problem to SBV: the space of functions of Special Bounded Variation.

We recall that $u \in BV(\Omega)$ has $Du \in \mathcal{M}(\Omega; \mathbb{R}^N)$, the space of vector valued measures.

One can show that Du has a decomposition as follows:

$$Du = \underbrace{D_a u}_{\text{absolutely continuous}} + \underbrace{D_j u}_{\text{jump}} + \underbrace{D_c u}_{\text{Cantor}}$$

$D_a u = \nabla u \mathcal{L}^N \llcorner_\Omega$ is the absolutely continuous part of the gradient.

$D_j u = (u^+ - u^-)\nu \mathcal{H}^{N-1} \llcorner_{J_u}$ where J_u is an $\mathcal{H}^{N-1}$-rectifiable set (collection of surfaces)

such that for each point $x \in J_u$ $\exists$ $\nu(x) \in S^{N-1}$ such that

$$u^{\pm} = \lim_{r \to 0} \fint_{B(x,r) \cap \{(y-x)\cdot\nu \gtrless 0\}} u(y)\,dy, \quad u^+ \neq u^-$$

[One sided Lebesgue points. Characterizes J_u up to set of $\mathcal{H}^{N-1}$-measure]

$D_c u = Du - D_a u - D_j u$ is the rest.

Clear geometric interpretations for $D_a u$ and $D_j u$. $D_c u$ is <u>weird</u>.

Now that we know the decomposition for BV functions, we define

$$SBV(\Omega) = BV(\Omega) \cap \{u : D_c u \equiv 0\}$$

i.e. $u \in SBV(\Omega)$ iff $u \in L^1(\Omega)$, $Du = D_a u + D_j u$

and change in u comes from a standard gradient or a jump across an interface.

[Immediate Problem: no easy compactness associated w/ norm. Same problem for $W^{1,1}(\Omega)$

De Giorgi et al. introduced

$$E[u] = \int_\Omega \|\nabla u\|^2 dx + \alpha \mathcal{H}^{n-1}(J_u) + \|u-g\|_{L^2}^2$$

as a Relaxation of the problem. Note if $u \in$ ~~$W^{1,1}(\Omega \setminus K) \cap L^\infty$~~ $W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega)$

$\Rightarrow u \in SBV(\Omega)$ with $J_u \subseteq K$. So

$$E[u] \leq \dot{E}[u, K].$$

De Giorgi, Carriero, Leaci Showed that minimizing E is equivalent to minimizing $\hat{E}$ by showing that a minimizer of E has $\overline{J_u} \setminus J_u$ $\mathcal{H}^{n-1}$-null.

Ambrosio, Fusco, Pallara showed even higher Regularity of J_u

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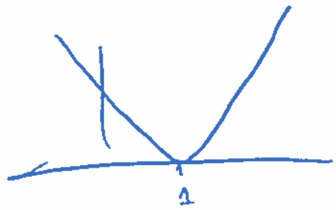
To model minimizers of E is a challenge though as it is a free discontinuity problem. Consequently Ambrosio and Tortorelli proposed an elliptic Regularization

We introduce a second parameter z to capture "damage".

Define
$$E_\delta[u,z] = \int_\Omega (z^2+\delta)\|\nabla u\|^2 + \frac{V(z)}{\delta} + \delta\|\nabla z\|^2 dx + \|u-g\|_{L^2}^2$$

where
$$V(s) \geq 0$$
$$V(s) = 0 \quad \text{iff} \quad s=1$$
$$V(s) \geq \frac{1}{c}|s| - C.$$
$\rightsquigarrow$ No doesn't matter.



$z \in W^{1,2}(\Omega)$, $0 \leq z \leq 1$ a.e.

$u \in W^{1,2}(\Omega)$, $-\|g\|_{L^\infty} \leq u \leq \|g\|_\infty$ a.e.

Note we introduced the last assumption as we are interested in low energy functions (minimizers) and

$$E[\max\{\min\{u, \|g\|_\infty\}, -\|g\|_\infty\}] \leq E[u].$$

(always better to truncate)

Theorem: The functional $E_\delta \xrightarrow{\Gamma} E$ where

$$E[u,z] = \begin{cases} E[u] & z=1, \\ +\infty & \text{else} \end{cases}$$

with $\alpha = 4\int_0^1 \sqrt{V(s)}\,ds$

The topology we use is convergence in measure for u and z. (could do L^1).

Here all u satisfy $-\|g\|_\infty \leq u \leq \|g\|_\infty$ a.e.

With this, we have the naturally associated Compactness Proof (5)

<u>Theorem (Compactness)</u>: If $\underline{\lim}_{\delta \to 0} E_\delta[u_\delta, z_\delta] \le C < \infty$, then up to a subsequence

$$u_\delta \xrightarrow[\text{meas}]{} u \in SBV(\Omega) \text{ and}$$
$$z_\delta \xrightarrow[\text{meas}]{} 1.$$

To prove this, we rely on an SBV compactness theorem of Ambrosio's.

<u>Theorem</u>: Suppose that $u_i \in SBV(\Omega)$ and

$$\overline{\lim} \left[\|u_i\|_{L^\infty} + \int_\Omega \|\nabla u_i\|^2 dx + \mathcal{H}^{N-1}(J_{u_i}) \right] \le C < \infty.$$

Then up to a subsequence $u_i \xrightarrow[L^1]{} u \in SBV(\Omega)$, and $\mathcal{H}^{N-1}(J_u) \le \underline{\lim}\, \mathcal{H}^{N-1}(J_{u_i})$, $\int \|\nabla u\|^2 \le \underline{\lim} \int \|\nabla u_i\|^2$

[The point of this theorem is to stop Cantor parts from forming].

<u>Proof of Compactness</u>: By the Compactness proof for Modica-Mortola, we have that $z_\delta \xrightarrow[L^1]{} 1$. Further defining

$$f(t) = \int_0^t \sqrt{V(s)}\, ds, \text{ we have}$$

$$\|\nabla(f \circ z_\delta)\|_{L^1} \le E_\delta[u_\delta, z_\delta] \le C < \infty$$

Applying the Coarea formula, we find

(6)

$$\int_0^1 \mathrm{Per}(\{z_\delta > t\})\,dt \le C < \infty \quad \forall\, \delta.$$

Thus $\exists\ t_\delta \in (1/4, 1/2)$ such that

$$\mathrm{Per}(\{z_\delta > t_\delta\}) \le C < \infty \quad \forall\, \delta.$$

Note also that as $z_\delta \xrightarrow[L^1]{} 1$,

$$\chi_{\{z_\delta > t_\delta\}} \xrightarrow[L^1]{} \chi_\Omega.$$

Defining $\bar{u}_\delta = u_\delta \chi_{\mathcal{O}_\delta}$ where $\mathcal{O}_\delta = \{z_\delta > t_\delta\}$.

Then

$$D_a \bar{u}_\delta = \nabla u_\delta \mathcal{L}^N \llcorner \mathcal{O}_\delta$$

$$J_{\bar{u}_\delta} \subseteq J_u \cup \partial^* \{z_\delta > t_\delta\}$$

Thus

$$\int_\Omega \|\nabla \bar{u}_\delta\|^2 + \mathcal{H}^{N-1}(J_{\bar{u}_\delta}) \le \frac{1}{t_\delta} \int_\Omega \|\nabla u_\delta\|^2 + \mathcal{H}^{N-1}(J_u) + \mathrm{Per}(\{z_\delta > t_\delta\})$$

$$\le C < \infty.$$

Consequently, we apply the SBV-compactness theorem to conclude that $\bar{u}_\delta \xrightarrow[L^1]{} u \in SBV(\Omega)$.

But of course, this implies that $u_\delta \xrightarrow[L^1]{} u$ as

$$\|\bar{u}_\delta - u_\delta\|_{L^1} \longrightarrow 0.$$

We now turn to the proof of ~~Lower-Semicontinuity~~ the ($\underline{\lim}$) bound

<u>Proposition</u>: Suppose that $(u_\delta, z_\delta) \longrightarrow (u, z)$ then

$$\underline{\lim}\, E_\delta[u_\delta, z_\delta] \geq E[u, z]$$

<u>Proof</u>: Step 1: $N=1$. WLOG (By Compactness), we assume that $z=1$ and that $u \in SBV^2(\Omega)$ where $\Omega \subseteq \mathbb{R}$ is an interval. In this setting J_u is a finite set

$$J_u = \{t_1, \dots, t_n\}.$$

Let I be an interval such that $t_1 \in I$ and $I \cap (J_u \setminus \{t_1\}) = \emptyset$.

We show that

$$\underline{\lim}_{\delta \to 0} E_\delta[u_\delta, z_\delta, I] \geq 4\int_0^1 \sqrt{V(s)}\, ds = \alpha.$$

Suppose that $\displaystyle \underline{\lim}_{\delta \to 0} \inf_{x \in I'} z_\delta = m_0 > 0$ where $t_1 \in I' \subseteq\subseteq I$

Then,

$$\underline{\lim}_{\delta \to 0} \int \|\nabla u_\delta\|^2 \leq \frac{1}{m_0^2} \underline{\lim}_{\delta \to 0} \int (z_\delta^2 + \delta) \|\nabla u_\delta\|^2 \leq C < \infty.$$

Thus, up to a subsequence $\|\nabla u_\delta\|_{L^2} \leq C < \infty \Rightarrow u_\delta \underset{H'}{\rightharpoonup} u$,

but then $t_1 \notin J_u$ as $J_u = \emptyset$ for $u \in H$!

Consequently, we must have that $m_0 = 0$, and $\exists$ $t_\delta \in I'$ such that $z_\delta(t_\delta) \to 0$. (subsequence)

As $z_\delta \to 1$ $\exists$ $t_1 < t_\delta < t_2$ $\forall \delta$ with $t_1, t_2 \in I \setminus I'$ such that

$$z_\delta(t_1), z_\delta(t_2) \to 1.$$

It follows that

$$E_\delta[u_\delta, z_\delta, I] \ge \int_{t_1}^{t_\delta} \frac{V(z_\delta)}{\delta} + \delta \|\nabla z_\delta\|^2 + \underbrace{\int_{t_\delta}^{t_2} \frac{V(z_\delta)}{\delta} + \delta \|\nabla z_\delta\|^2}_{\text{focus here}}$$

$$\int_{t_\delta}^{t_2} \frac{V(z_\delta)}{\delta} + \delta \|\nabla z_\delta\|^2 \ge \int_{t_\delta}^{t_2} 2\sqrt{V(z_\delta)}\, \|\nabla z_\delta\|\, dt$$

Reparametrization

$$\ge \int_{z_\delta(t_\delta)}^{z_\delta(t_2)} 2\sqrt{V(s)}\, ds$$

It follows (doing the same argument for the other term that

$$E_\delta[u_\delta, z_\delta, I] \ge 2\int_{z_\delta(t_\delta)}^{\min\{z_\delta(t_i)\}} 2 \sqrt{V(s)}\, ds$$

taking liminf

$$\liminf_{\delta \to 0} E_\delta[u_\delta, z_\delta, I] \ge 4\int_0^1 \sqrt{V(s)}\, ds = \alpha.$$

Consequently $\liminf_{\delta \to 0} E_\delta[u_\delta, z_\delta, \tilde{I}] \ge \alpha \mathcal{H}^0(J_u)$ for $J_u \subseteq \tilde{I}$ ~ union of intervals.

Let I now be an interval such that $\bar{I} \cap J_u = \phi$. (9)

We show that $\varliminf_{\delta \to 0} E_\delta[u_\delta, z_\delta, I] \geq \int_I \|\nabla u\|^2 dx + L^2\text{-term}$

Let $I = (0,1)$ for convenience. For $n \in \mathbb{N}$ we split I into n subintervals $I_n^k = \left(\frac{k-1}{n}, \frac{k}{n}\right)$. ~~Define~~ For $\gamma \in (0,1)$ define

$$S_n = \{ k : \varliminf_{\delta \to 0} \inf_{x \in I_n^k} z_\delta(x) < \gamma \}$$

By the same argument to control the jumpset, we have

$$|S_n| \leq \frac{\sup E_\delta[u_\delta, z_\delta]}{4 \int_\gamma^1 \sqrt{V(s)}\, ds} \leq C(\gamma) < \infty.$$

It follows that

$$C \geq \varliminf E_\delta[u_\delta, z_\delta, I]$$

$$\geq \varliminf \int_{\bigcup_{k \notin S_n} I_n^k} (z_\delta^2 + \delta) \|\nabla u_\delta\|^2 \geq \gamma^2 \varliminf_{\delta \to 0} \int_{\bigcup_{k \notin S_n} I_n^k} \|\nabla u_\delta\|^2 + L^2$$

Up to a subsequence, we find

$$\varliminf E_\delta[u_\delta, z_\delta, I] \geq \gamma^2 \int_{\bigcup_{k \notin S_n} I_n^k} \|\nabla u\|^2 dx + L^2$$

as $\chi_{\bigcup_{k \notin S_n} I_n^k} - \chi_I \xrightarrow[L^1]{} 0$, we have

$$\varliminf E_\delta[u_\delta, z_\delta, I] \geq \gamma^2 \int_I \|\nabla u\|^2 dx.$$

But now γ is arbitrary and we have

$$\underline{\lim}\, E_\delta[u_\delta, z_\delta, I] \geq \int_I \|\nabla u\|^2 + L^2 \text{ term.}$$

Letting $\eta \ll 1$ with $I_\eta := J_u + (-\eta, \eta)$, we have

$$\begin{aligned}
\underline{\lim}_{\delta}\, E_\delta[u_\delta, z_\delta] &\geq \underline{\lim}_{\delta}\, E_\delta[u_\delta, z_\delta, I_\eta] + \underline{\lim}_{\delta}\, E_\delta[u_\delta, z_\delta, \Omega \setminus I_\eta] \\
&\geq \alpha \mathcal{H}^0(J_u) + \int_{\Omega \setminus I_\eta} \|\nabla u\|^2 dx + L^2
\end{aligned}$$

Letting $\eta \to 0$ concludes Step 1.

Some References: Braides "Approximation of Free Discontinuity Problems"

Focardi "On the variational Approximation of free-discontinuity problems in the vectorial case"