I hear Tim is also Doing Γ-convergence of the Modica-Mortola Functional...

We begin today by proving the last detail needed for Γ-convergence of Cahn-Hilliard to the weighted Perimeter Functional.

Lemma: Let $A \subset \Omega$ be such that $\exists\, \tilde{A} \subseteq \mathbb{R}^N$ with

$\partial A = \partial\tilde{A} \cap \Omega$, $\partial\tilde{A}$ is a smooth (C^1) manifold and

relative to Ω

$$\mathcal{H}^{N-1}(\partial\tilde{A} \cap \partial\Omega) = 0. \text{ (transverse intersection)}$$

Then for the distance function

$$d_A(x) := \begin{cases} \operatorname{dist}(x, A^c) & x \in A \\ -\operatorname{dist}(x, A) & x \in A^c \end{cases}$$

we have
$$\lim_{r \to 0} \mathcal{H}^{N-1}(\Omega \cap \{d_A = r\}) = \mathcal{H}^{N-1}(\partial A).$$

Proof: We begin by showing that the statement holds for $\Omega = \mathbb{R}^N$, i.e., $A = \tilde{A}$ a bounded set.

Arguing with the aid of a partition of unity we reduce to the local case that ∂A is given as the subgraph of a ~~Lipschitz~~ C^1 function. Precisely

$$A = \{x : x_N < g(x')\} \text{ for } g : \mathbb{R}^{N-1} \to \mathbb{R} \text{ a } C^1 \text{ function.}$$

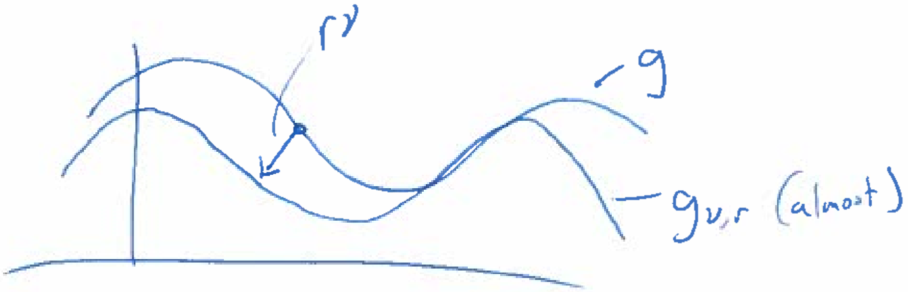
Consider now positive $r \searrow 0$, we prove that

$$\mathcal{H}^{N-1}(Q[0,T]\cap\{d_A=r\}) \xrightarrow[r\to 0]{} \mathcal{H}^{N-1}(Q[0,T]\cap\partial A)$$

We represent $\{d_A=r\}$ as a graph.

For $\nu\in S^{N-1}$, $r\neq 0$, we define

$$g_{\nu,r}(x') = g(x'+r\nu') - r\nu_N$$



We define $g_r := \inf_{\nu\in S^{N-1}}\{g_{\nu,r}\}$, which is a Lipschitz function by Arzela-Ascoli.

One can directly argue that $\{d_A=r\}=\mathrm{Graph}\, g_r$

By the Area formula

$$\mathcal{H}^{N-1}(Q[0,T]\cap \mathrm{Graph}\, g_r) = \int_{Q'[0,T]} \sqrt{1+\|\nabla g_r\|^2}\,dx'$$

Consequently our claim boils down to (3)

$$\mathcal{H}^{N-1}(Q[0,T] \cap \operatorname{Graph} g_r) = \int_{Q'} \sqrt{1+\|\nabla g_r\|^2}\, dx'$$
$$\longrightarrow \int_{Q'} \sqrt{1+\|\nabla g\|^2}\, dx' = \mathcal{H}^{N-1}(Q[0,T] \cap \operatorname{Graph} g)$$
$$= \mathcal{H}^{N-1}(Q[0,T] \cap \partial A)$$

For this, we show that $\nabla g_r \to \nabla g$ in L^∞.

By geometric considerations, $\nabla g_r(x') \in \operatorname{conv}_{\nu \in S^{N-1}} \{\nabla g_{\nu,r}(x')\}$
$$\subseteq \operatorname{conv}_{\nu \in B[0,r]} \{\nabla g(x'+\nu)\}$$

Letting ω be the modulus of continuity for ∇g, it follows that

$$\|\nabla g_r(x') - \nabla g(x')\| \leq \omega(r)$$

and $\nabla g_r \to \nabla g$ in L^∞, concluding the claim

(More Generally?)

To prove in general we argue as follows.

Define $V_r = \{x \in \mathbb{R}^N : |d_A(x)| < r\}$

then $A_r := \tilde{A} \setminus V_r$ so that

$$\partial A_r = \{d_A = r\}.$$

Clearly $\chi_{A_r} \xrightarrow[L^1]{} \chi_A$ so that we can apply BV-LSC to find that

$$\mathcal{H}^{N-1}(\partial A) = |D\chi_A|(\Omega) \leq \varliminf_{r\to 0} |D\chi_{A_r}|(\Omega)$$
$$\leq \varliminf_{r\to 0} \mathcal{H}^{N-1}(\Omega \cap \{d_A = r\})$$

To prove the opposite claim, we have that

$$\cancel{\mathcal{H}^{N-1}(\Omega \cap \{d_A = r\}) \leq \mathcal{H}^{N-1}(\partial A_r) - \mathcal{H}^{N-1}(\partial A_r \cap [\mathbb{R}^N \setminus \overline{\Omega}])}$$

$$\mathcal{H}^{N-1}(\Omega \cap \{d_A = r\}) \leq |D\chi_{A_r}|(\overline{\Omega})$$

Thus $\overline{\lim_{r\to 0}} \mathcal{H}^{N-1}(\Omega \cap \{d_A = r\}) \leq \overline{\lim_{r\to 0}} |D\chi_{A_r}|(\overline{\Omega}) \leq |D\chi_A|(\overline{\Omega})$
$$= \mathcal{H}^{N-1}(\overline{\Omega} \cap \partial \overline{A})$$
$$= \mathcal{H}^{N-1}(\partial A).$$

With this we have properly concluded the proof that.

$$\int \frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon \|\nabla u_\varepsilon\|^2 \xrightarrow[\Gamma]{} c_0 \mathrm{Per}[\cdot]$$

$$c_0 = \int_0^1 2\sqrt{W(s)}\, ds.$$

A note on the Historical Context.

De Giorgi introduced Γ-convergence in '75

And first used in a related problem by Modica & Mortola in '77.

Using Γ-convergence, Modica '87 and Sternberg '88 proved

$$E_\varepsilon \xrightarrow[\Gamma]{} c_0 \mathrm{Per}[\cdot] \quad \text{(up to modifications)}$$

The reason this was essentially of interest was a conjecture by M.E. Gurtin stating that ~~...~~ for minimizers u_ε of the energy E_ε,

$$E_\varepsilon[u_\varepsilon] \sim \mathrm{Per}[A], \quad \text{where}$$

$$\fint \chi_A = \fint u_\varepsilon \quad \text{and} \quad A \text{ minimizes the perimeter among}$$

fixed mass sets.

Given the Γ-convergence result, we know that $u_\varepsilon \to$ minimizer of $c_0\mathrm{Per}$

Consequently, $E_\varepsilon[u_\varepsilon] \longrightarrow c_0 \mathrm{Per}[A]$ as desired.

We now discuss a variety of related models

Example 1: Nonisothermal Phase transitions

Chemistry is dependent on temperature, and W is basically a mixing energy describing chemical incompatibility. It is natural for this relation to change for different temperatures, so

$$\int \frac{1}{\varepsilon} W(x,u) + \varepsilon \|\nabla u\|^2$$ is considered a

nonisothermal model in the static setting. Well of W change with x.

This model has been studied by (Gravina, Cristoferi) Bouchitté, and in the context of vector valued homogenization by (~~CtA~~ Fonseca, Cristoferi, Gravina).

Example 2: Anisotropy of material.

For many crystals, there are preferred directions of orientation, and this is phenomenologically captured by the model

$$\int \frac{1}{\varepsilon} W(u) + \varepsilon \phi(\nabla u)\, dx,$$ where

in simple settings ϕ is basically $a\left(\frac{\nabla u}{|\nabla u|}\right) \|\nabla u\|^2$

Here the Γ-limit is given by

$$\int_{\partial^* A} a(\nu)\, d\mathcal{H}^{n-1}$$

Example 3: Multiphase

When there may be multiple phase that are admissible in crystal Growth, a natural associated energy is

$$\int \frac{1}{\varepsilon} W(u) + \varepsilon \|\nabla u\|^2 dx \quad \text{where } u : \Omega \to \mathbb{R}^d$$

is vector valued and W takes 0's on a compact set (think a finite number of choices). This was studied by (Fonseca, Tartar), (Baldo), (Ambrosio) where the Γ-limit is described in terms of a Caccioppoli partition

Recently the dynamics of L^2 gradient flow have been shown to converge to multiphase MCF by (Laux and Simon '18).

Example 4: Biological Membranes.

As proposed for unilameller membranes with phase separation, the energy

$$\int \frac{1}{\varepsilon} W(u) - q\varepsilon |\nabla u|^2 + \varepsilon^3 \|\nabla^2 u\|^2$$

is a local model describing phase separation. It Γ-converges to a perimeter functional, and a principal challenge is interpolation to control the middle term.

The original model is nonlocal, and is related to the Ohta–Kawasaki model for phase separation in diblock polymers

$$\int \frac{1}{\varepsilon} W(u) + \varepsilon \|\nabla u\|^2 + \int_\Omega \int_\Omega G(x,y)\,(u(x)-m)\cdot(u(y)-m)\,dx\,dy$$

See Zwicknagl and Collaborators. (8)

Nonlocal Diffusion

Example 5: Solid-Solid phase transitions

Let Ω be the ~~too~~ reference configuration of an elastic material and

$u:\Omega \to \mathbb{R}^N$, $N=2,3$ the material displacement.

The energy $\displaystyle\int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon \|\nabla^2 u\|^2$ has been

used to model phase transitions in Solid-Materials.

For $W(S)=0$ iff $S=A$ or B, to have nontrivial

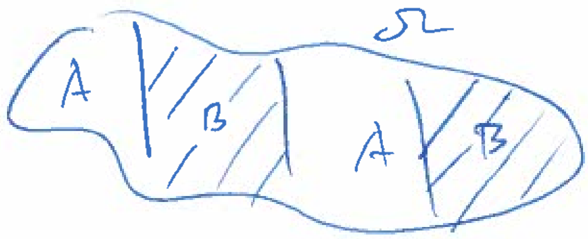
displacements in the limit, one must satisfy the compatibility.

$$A-B = \text{rank-one matrix} = a\otimes e, \quad a,e\in\mathbb{R}^N.$$

A classic result of Ball and James shows that for $\nabla u\in BV(\Omega;\mathbb{R}^N)$

and $\nabla u\in\{A,B\}$, $J_{\nabla u}$ consists of hyperplanes with normal

e.



Generally things are in a frame invariant setting

Γ-convergence was proven by (Conti, Schweizer) see also (Davoli, Friedrich), (Simeon)

Connecting to evolution Problems

The L^2 gradient flow of E_ε is the Allen-Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon^2}$$
$$\partial_\nu u_\varepsilon = 0$$

with the location of ε being movable by appropriate (dilations) scalings in time

For sufficiently smooth solutions u_ε, they can be characterized by the dissipation inequality:

$$E_\varepsilon[u_\varepsilon(T)] + \frac{1}{2}\int_0^T \varepsilon\|\partial_t u_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\|^2_{L^2(\Omega)}\,dt$$
$$\leq E_\varepsilon[u_\varepsilon(0)].$$

This is essentially

$$E_\varepsilon[T] + \frac{1}{2}\int_0^T \left[\|\partial_t u_\varepsilon\|^2_{X_\varepsilon} + \|\nabla E_\varepsilon[u]\|^2_{X_\varepsilon}\right]dt \leq E_\varepsilon[0]$$

Evolutionary Γ-convergence asks when is the above inequality stable as $\varepsilon \to 0$.
$u_\varepsilon \to u$ and

$$\hookrightarrow E_0[T] + \frac{1}{2}\int_0^T \left[\|\partial_t u\|^2_{X_0} + \|\nabla E_0[u]\|^2_{X_0}\right]dt \leq E_0[0].$$

In the case of Allen-Cahn, one recovers Mean Curvature Flow.