We begin by Recalling some technical tools we will need:

Coarea Formula: Let $f:\mathbb{R}^N\to\mathbb{R}$ be a Lipschitz function, and Let $g\in L^1(\mathbb{R}^N)$. Then the equality holds:

$$\int_{\mathbb{R}^N} g|\nabla f|\,dx = \int_{\mathbb{R}}\left[\int_{\{f=r\}} g \, d\mathcal{H}^{N-1}\right] dr$$

Coarea Formula for BV: Let $u\in BV(\Omega)$, then it holds that

$$|Du|(\Omega) = \int_{\mathbb{R}} Per(\{u>r\})\,dr.$$

We now Recall where we left off in the proof of Γ-convergence. ($\underline{\lim}$) bound. Here we used Young's inequality to say that

$$|\nabla(f\circ u_\varepsilon)| = 2|\sqrt{W(u_\varepsilon)}|\,|\nabla u_\varepsilon| \le \left(\frac{1}{\varepsilon}W(u_\varepsilon)+\varepsilon|\nabla u_\varepsilon|^2\right)$$

and BV-lsc.

For the ($\overline{\lim}$) bound, we constructed an optimal profile g_ε:

(step 1)
$$g_\varepsilon(t) = \begin{cases} 0 & t\le 0 \\ \phi_\varepsilon^{-1}(t) & t\in[0,\phi_\varepsilon(1)) \\ 1 & t\ge \phi_\varepsilon(1) \end{cases}$$

with $\phi_\varepsilon(1)\le\varepsilon^{1/2}$ and for $t\in[0,\phi_\varepsilon(1))$ the ODE holds:

$$\frac{d}{dt}g_\varepsilon(t) = \frac{\sqrt{\varepsilon+W(g_\varepsilon)}}{\varepsilon}$$

We now turn to

Step 2: Optimal Recovery sequence for smooth set.

We consider $A \subseteq \Omega$ which is given by $\tilde{A} \cap \Omega$ where $\tilde{A}$ is a set with C^2 bdry such that

$$\mathcal{H}^{N-1}(\partial \tilde{A} \cap \partial \Omega) = 0. \quad (\text{transverse intersection})$$

We introduce the signed distance for $\tilde{A}$ as

$$d_A(x) := \begin{cases} \operatorname{dist}(x, \tilde{A}^c) & x \in A \\ -\operatorname{dist}(x, \tilde{A}) & x \in A^c \end{cases}$$

For a smooth set with transverse intersection with the domain bdry, wrt Ω we have that $\lim_{r \to 0} \mathcal{H}^{N-1}(\Omega \cap \{d_A = r\}) = \mathcal{H}^{N-1}(\partial A)$.

[To be proven next week].

We want to define our optimal profile as $g_\varepsilon \circ d_A$, but we must satisfy the mass constraint. Defining

$$g_0(t) := \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

We have

$$g_\varepsilon \leq g_0 \leq g_\varepsilon(\cdot + \phi_\varepsilon(1))$$

$$\Rightarrow \int g_\varepsilon \circ d_A \leq \int g_0 \circ d_A = m \leq \int g_\varepsilon(d_A(x) + \phi_\varepsilon(1))\, dx$$

(3)

Consequently, by the intermediate value theorem, there is $s_\varepsilon \in [0, \phi_\varepsilon(1)]$ such that $u_\varepsilon(x) := g_\varepsilon(d_A(x) + \phi_\varepsilon(1))$

satisfies $\displaystyle\int_\Omega u_\varepsilon\,dx = m$ as desired.

We now show that $\overline{\lim}\, E_\varepsilon[u_\varepsilon] \leq E_0[u]$

and that $u_\varepsilon \xrightarrow[L^1]{} u = \chi_A$.

For the latter, we have

$$\int_\Omega |u_\varepsilon - u|\,dx = \int \left|g_\varepsilon(d_A(x) + s_\varepsilon) - g_0 \circ d_A\right| dx \quad |\nabla d_A| = 1$$

$$= \cancel{\int_{\mathbb{R}} g_\varepsilon(d_A(x) + s_\varepsilon) - g_0 \circ d_A}$$

$$(\text{coarea formula}) = \int_{\mathbb{R}} |g_\varepsilon(r + s_\varepsilon) - g_0(r)|\, \mathcal{H}^{N-1}(\{d_A = r\})\,dr$$

$$\leq 4\phi_\varepsilon(1) \sup_{|r| \leq \phi_\varepsilon(1)} \mathcal{H}^{N-1}(\{d_A = r\})$$

By the claim which we will show next time

$$\longrightarrow 0$$

Now we prove the ($\overline{\lim}$) relation.

(4)

$$E_\varepsilon[u_\varepsilon] = \int_\Omega \frac{1}{\varepsilon}\{W(u_\varepsilon)\} + \varepsilon|\nabla u_\varepsilon|^2$$

using $|\nabla d_A| = 1$ a.e.

$$= \int_\Omega \frac{1}{\varepsilon}\{W(g_\varepsilon(d_A(x)+s_\varepsilon))\} + \varepsilon|g_\varepsilon'(d_A(x)+s_\varepsilon)|^2 \, dx$$

$$= \int_{\mathbb{R}} \frac{1}{\varepsilon} W(g_\varepsilon(r+s_\varepsilon)) + \varepsilon|g_\varepsilon'(r+s_\varepsilon)|^2 \, \mathcal{H}^{N-1}(\{d_A = r\}) \, dr$$

$$\le \sup_{|r|\le \phi_\varepsilon(1)} \mathcal{H}^{N-1}(\ldots) \int_{\mathbb{R}} \frac{1}{\varepsilon} W(g_\varepsilon(r+s_\varepsilon)) + \varepsilon|g_\varepsilon'(r+s_\varepsilon)|^2 \, dr$$

$$\le \sup_{|r|\le \phi_\varepsilon(1)} \mathcal{H}^{N-1}(\{d_A = r\}) \int_{\mathbb{R}} \frac{1}{\varepsilon} W(g_\varepsilon(r)) + \varepsilon|g_\varepsilon'(r)|^2 \, dr$$

$$\le \text{''} \quad \text{''} \int_0^{\phi_\varepsilon(1)} \frac{\sqrt{\varepsilon + W\circ g_\varepsilon}}{\varepsilon}\sqrt{\varepsilon + W\circ g_\varepsilon} + \varepsilon\left(\frac{\sqrt{\varepsilon+W\circ g_\varepsilon}}{\varepsilon}\right)|g_\varepsilon'(r)|^2 \, dr$$

$$= \text{''} \quad \text{''} \int_0^{\phi_\varepsilon(1)} \sqrt{\varepsilon + W\circ g_\varepsilon}\, |g_\varepsilon'(r)| \, dr$$

$$= \text{''} \int_0^1 \sqrt{\varepsilon + W(s)} \, ds \qquad \text{(reparametrization)}$$

$$\Rightarrow \overline{\lim}\, E_\varepsilon[u_\varepsilon] \le \overline{\lim_{\varepsilon\to 0}} \sup_{|r|\le\phi_\varepsilon(1)} \mathcal{H}^{N-1}(\{d_A = r\}) \cdot \int_0^1 \sqrt{\varepsilon + W(s)} \, ds$$

$$\le \mathcal{H}^{N-1}(\partial A) \int_0^1 \sqrt{W(s)} \, ds$$

$$= E_0[u]$$

That concludes step 2. (5)

Step 3: It suffices to consider smooth sets with transverse intersection.

We show that $\exists$ A_δ such that $\chi_{A_\delta} \xrightarrow[L^1]{} \chi_A$ and

$$(\star) \begin{cases} \lim_{\delta \to 0} \mathrm{Per}(A_\delta) = \mathrm{Per}(A) \\ A_\delta \text{ has associated } \tilde{A} \text{ with transverse intersection.} \end{cases}$$

If we have this, Γ-convergence follows from diagonalization:

For each A_δ, we have Recovery sequence from before

$$u_{\delta,\varepsilon} \longrightarrow \chi_{A_\delta} \text{ as } \varepsilon \to 0 \text{ and}$$

$$\overline{\lim_{\varepsilon \to 0}} E_\varepsilon[u_{\delta,\varepsilon}] \le E_0[\chi_{A_\delta}].$$

Taking $\overline{\lim}_{\delta \to 0}$ we have

$$\overline{\lim_{\delta \to 0}}\, \overline{\lim_{\varepsilon \to 0}} E_\varepsilon[u_{\delta,\varepsilon}] \le E_0[\chi_A] \text{ and}$$

$$\overline{\lim_{\delta \to 0}}\, \overline{\lim_{\varepsilon \to 0}} \| u_{\delta,\varepsilon} - \chi_A \|_{L^1} = 0$$

$\Rightarrow$ (by diag) that $\exists$ $u_\varepsilon \to \chi_A$ such that

$$\overline{\lim}\, E_\varepsilon[u_\varepsilon] \le E_0[\chi_A] \text{ as desired.}$$

We now prove $(\star)$

To prove (A) we first show how to construct a smooth approximation, then we correct for mass.

Extend $\chi_A \in BV(\Omega)$ to a BV function $\tilde{u} \in BV(\mathbb{R}^N)$ such that $|D\tilde{u}|(\partial\Omega) = 0$.

We then define $u_\delta = \tilde{u} * \psi_\delta$ ~ a standard mollifier.

For this we have $Du_\delta \overset{*}{\rightharpoonup} D\tilde{u}$ in $\mathbb{R}^N$ and in particular,

$$|Du|(\Omega) = |D\tilde{u}|(\Omega) \le \underline{\lim} \int_\Omega |\nabla u_\delta| dx \le \overline{\lim} \int_\Omega |\nabla u_\delta| dx \le |D\tilde{u}|(\overline{\Omega}) = |Du|(\Omega).$$

We define $A_{\delta,t} = \{x : u_\delta(x) > t\}$. For $t \in (0,1)$. — Apply Sard's Theorem

To see that $\chi_{A_{\delta t}} \to \chi_A$ ~~for~~ as $\delta \to 0$ we have

$$\int_\Omega |u_\delta - u| \ge \int_{A_{\delta,t} \setminus A} t \, dx + \int_{A \setminus A_{\delta,t}} (1-t) \, dx \ge C(t) \mathcal{L}^N(A_{\delta,t} \Delta A)$$

$\Rightarrow$ L^1-convergence.

To see the convergence of perimeters, we have

$$\begin{aligned} \operatorname{Per}(A) = |Du|(\Omega) &= \lim_{\delta \to 0} \int |\nabla u_\delta| \, dx \\ &= \lim_{\delta \to 0} \int_{\mathbb{R}} \operatorname{Per}(A_{\delta,t}) \, dt \\ &\ge \int_{\mathbb{R}} \underline{\lim}_{\delta \to 0} \operatorname{Per}(A_{\delta,t}) \, dt \end{aligned}$$

(Coarea)

Thus for any $t \in [0,1]$ $\exists$ a subsequence $\delta \to 0$ such that $\lim_{\delta \to 0} \operatorname{Per}(A_{\delta,t}) = \operatorname{Per}(A)$

We use this prior argument to now correct for the (7). mass constraint

[Picture].

Let $x_1, x_2 \in \partial^* A$ such that $x_1 \neq x_2$. Define a modified set $A_k := A \cup B(x_1, 1/k) \setminus B(x_2, 1/k) \to A$ in BV

We proceed as in the previous construction to find $A_{k,\delta}$ such that

$$\| \chi_{A_{k,\delta}} - \chi_{A_k} \|_{L^1} \leq 1/k$$

$$\| Per(A_{k,\delta}) - Per(A_k) \| \leq 1/k^N \cdot \tau$$

$$B(x_1, (4/5)^{1/N} 1/k) \subseteq A_{k,\delta}$$

$$B(x_2, (4/5)^{1/N} 1/k) \subseteq A_{k,\delta}^c.$$

Suppose $\mathcal{L}^N(A_{k,\delta}) > \mathcal{L}^N(A)$

Let $r_{k,\delta}$ be such that $\mathcal{L}^N(B(0, r_{k,\delta})) = \mathcal{L}^N(A_{k,\delta}) - \mathcal{L}^N(A)$.

Then we define $\tilde{A}_k = A_{k,\delta} \setminus B(x_1, r_{k,\delta})$

To see that $\tilde{A}_k$ still has smooth boundary, it suffices to show $r_{k,\delta} < (4/5)^{1/N} 1/k$

$$\mathcal{L}^N(A_{k,\delta}) \leq \mathcal{L}^N(A_k) + \frac{1}{k^N}\tau = \mathcal{L}^N(A) + \mathcal{L}^N(B(x_1, 1/k) \setminus A) - \mathcal{L}^N(B(x_2, 1/k) \cap A) + 1/k^N \tau$$

$$< \mathcal{L}^N(A) + \tfrac{3}{4}\mathcal{L}^N(B(x_1, 1/k)) + 1/k^N \tau \quad \text{k suff small}$$

$$\Rightarrow \mathcal{L}^N(A_{k,\delta}) - \mathcal{L}^N(A) < \left(\tfrac{3}{4} + \tau\right)\mathcal{L}^N(B(x_1, 1/k)) = \mathcal{L}^N\left(B(x_1, (3/4 + \tau)^{1/N} 1/k)\right)$$

Consequently, for $\tau \ll 1$, we see that $r_{k,\delta} < (4/5)^{1/n}\, 1/k$ (8)

and $\tilde{A}_k$ is still a smooth modification.