Γ-convergence for the Cahn-Hilliard Functional.

Recap on BV-functions.

A function u belongs to $BV(\Omega)$ if $\int_\Omega |u|\,dx < \infty$ and $\exists$ a (vector-valued) measure μ with finite total variation such that

$$\int_\Omega \operatorname{div}\phi \, u \, dx = -\int_\Omega \phi \cdot d\mu \quad \forall \phi \in C_c^1(\Omega;\mathbb{R}^N)$$

In other words $Du = \mu \in \mathcal{M}(\Omega;\mathbb{R}^N)$

Examples of BV functions:

1. $u \in W^{1,1}(\Omega)$ satisfies the definition with $\mu = \nabla u \mathcal{L}^N \llcorner \Omega$.

2. Let A be a set compactly contained in Ω with C^2 bdry. Then $\chi_A \in BV(\Omega)$. To see this, we can apply the divergence theorem:

$$\int_\Omega \operatorname{div}\phi \, \chi_A \, dx = \int_A \operatorname{div}\phi \, dx = -\int \phi \cdot \nu \, d\mathcal{H}^{N-1} \llcorner \partial A$$

$$= -\int \phi \cdot d\mu$$

where $\mu = \nu \mathcal{H}^{N-1} \llcorner \partial A$.

In other words $D\chi_A = \underset{\substack{\uparrow \\ \text{Inner normal}}}{\nu} \mathcal{H}^{N-1} \llcorner \partial A$

Sets of Finite Perimeter

We say that $A \subseteq \Omega$ is a set of finite Perimeter if $\chi_A \in BV(\Omega)$.

Ex. ~ C^2 set from before.

It turns out the above example is typical.

In general, if A is a set of finite perimeter, we may define the reduced boundary $\partial^* A$ by the points x_0 such that $\exists\, \nu \in S^{N-1}$ such that

$$\cancel{\lim_{r\to 0} \frac{\mathcal{L}^N(B(x_0,r)\cap A)}{\mathcal{L}^N(B(x_0,r))} = 1/2}$$

~~and~~

$$\lim \fint_{B(x,r)} \frac{d D\chi_A}{d|D\chi_A|}\, d|D\chi_A| = \nu.$$

~~If~~ Thm $x_0 \in \partial^* A$, then $\chi_A(x_0 + ry) \xrightarrow[L^1]{} \chi_{H_\nu}$

Further $D\chi_A = \nu \mathcal{H}^{N-1} \llcorner \partial^* A$. $\longrightarrow$ Define $Per(A)$

We will need some basic properties of BV functions.

Prop. If $u_i \in BV(\Omega)$ and $u_i \xrightarrow[L^1]{} u$, then

$$|Du|(\Omega) \leq \varliminf_{i\to\infty} |Du_i|(\Omega)$$

Prop. If $u_i \in BV(\Omega)$ is uniformly bdd ($\|u_i\|_{L^1} + |Du_i|(\Omega) \leq C < \infty$), then $\exists$ a subsequence such that

$$u_i \xrightarrow[L^1]{} u$$

Theorem. Fix $m \in (0,1)$. Define the energy

$$E_\varepsilon[u] := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon \|\nabla u\|^2 dx, & u \in H^1(\Omega) \text{ \& } \fint u = m \\ +\infty & \text{else in } L^1(\Omega) \end{cases}$$

Then E_ε Γ-converges to $E_0[u] := \begin{cases} c_0 |Du|(\Omega) & \text{if } u \in BV(\Omega; \{0,1\}) \text{ and } \fint u = m \\ +\infty & \text{else} \end{cases}$

$\uparrow$ in $L^1(\Omega)$ $\qquad c_0 = 2\int_0^1 \sqrt{W(s)}\, ds$

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where $W : \mathbb{R} \to \mathbb{R}_+$ is assumed to satisfy

$$\frac{1}{C}|s| - C \le W(s) \quad \forall s$$
$$W(s) = 0 \quad \text{iff } s = 0 \text{ or } 1$$
$$W \ge 0 \quad \forall s$$

Theorem. Compactness. If u_ε is of bounded energy then up to a subsequence, $u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega; \{0,1\})$.

Proof. We have $\sup_{\varepsilon > 0} E_\varepsilon[u_\varepsilon] < C < \infty$.

Define the function $f(t) = \int_0^t 2\sqrt{\min\{W(s), K\}}\, ds$

where $K > \max_{s \in (0,1)} \{W(s)\}$.

Define $w_\varepsilon := f \circ u_\varepsilon \in W^{1,1}(\Omega)$.

We compute $\nabla w_\varepsilon = f'(u_\varepsilon) \nabla u_\varepsilon$

Consequently,

$$\int_\Omega |Dw_\varepsilon| = \int_\Omega |f'(u_\varepsilon)||Du_\varepsilon|\,dx$$

$$\leq \int_\Omega 2\,|\min\{W(u_\varepsilon), K\}|\,|Du_\varepsilon|\,dx$$

$$\leq \int_\Omega \frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon|Du_\varepsilon|^2 = E_\varepsilon[u_\varepsilon] \leq C < \infty$$

Further

$$\int |w_\varepsilon| = \int_\Omega |f\circ u_\varepsilon| \leq C_K \int_\Omega (|u_\varepsilon| + 1)\,dx$$

$$\leq C\int_\Omega W(u_\varepsilon) + C \leq C < \infty$$

Thus, w_ε is bounded in $BV(\Omega)$, and up to a subsequence, we have that $w_\varepsilon \xrightarrow[L^1]{} w \in BV(\Omega)$

Note that f is monotone increasing, continuous, and has linear growth at $|\infty|$. Consequently, f^{-1} is monotone increasing, continuous, and has linear growth.

By dominated convergence theorem it follows that

$$u_\varepsilon = f^{-1}(w_\varepsilon) \xrightarrow[L^1]{} f^{-1}(w) =: u$$

By Fatou's Lemma

$$\int W(u) \leq \lim_{\varepsilon\to 0}\int W(u_\varepsilon) \leq C\cdot\varepsilon = 0$$

$\Rightarrow$ $u = 0$ or 1 a.e.

Consequently $u = \chi_A$ for some $A \subseteq \Omega$.

But this implies $w = f(\chi_A) = f(1)\chi_A \in BV(\Omega)$.

$\Rightarrow \chi_A \in BV(\Omega)$, concluding the theorem.

Prop. If $u_\varepsilon \xrightarrow[L^1]{} u$ then $\underline{\lim}_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] \geq E_0[u]$.

Proof.

Up to a subsequence, we may assume $\underline{\lim}_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] = \lim_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] < \infty$.

Applying compactness, we have that $u \in BV(\Omega; \{0,1\})$. As before we have $f(t) := 2\int_0^t \sqrt{\min\{W(s), k\}}\, ds$ with $k > \max_{s \in (0,1)} W(s)$.

and $w_\varepsilon := f \circ u_\varepsilon \to f \circ u = w$.

$$E_0[u] = f(1)|Du|(\Omega) = |Dw|(\Omega) \leq \underline{\lim} \int |\nabla w_\varepsilon|\, dx$$

$$= \underline{\lim} \int |f'(u_\varepsilon)||\nabla u_\varepsilon|\, dx$$

$$\leq \underline{\lim} \int 2|\sqrt{W(u_\varepsilon)}|\,|\nabla u_\varepsilon|\, dx$$

$$\leq \underline{\lim} \int_\Omega \frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon |\nabla u_\varepsilon|^2\, dx$$

$$\leq \underline{\lim}\, E_\varepsilon[u_\varepsilon]$$ as desired.

<u>Prop.</u> Fix $u \in BV(\Omega;\{0,1\})$ with $⨍_\Omega u = m$. Then $\exists\, u_\varepsilon$ with

$u_\varepsilon \xrightarrow[L^1]{} u$ such that $\overline{\lim_{\varepsilon \to 0}}\, E_\varepsilon[u_\varepsilon] \le E_0[u]$.

Proof. Step 1 - Construction of near optimal transition.

Consider E_ε on the 1-D interval $(0,1)$. Suppose we wished to connect the 2-states $\{0\}$ and $\{1\}$ by an optimal transition ~~near optim~~

In principle, the energy would be a local minimizer, and we can compute the first variation

$$E_\varepsilon[g_\varepsilon] = \int \frac{1}{\varepsilon} W(g_\varepsilon) + \underbrace{\varepsilon \|\nabla g_\varepsilon\|^2}_{\varepsilon |g_\varepsilon'|^2}$$

First variation implies

$$0 = \frac{1}{\varepsilon} W'(g_\varepsilon) - 2\varepsilon g_\varepsilon'' \qquad g_\varepsilon(0) = 0,\ g_\varepsilon(1) = 1$$

Multiply by g_ε' and integrate to find

$$C = \frac{1}{\varepsilon} W(g_\varepsilon) - \varepsilon (g_\varepsilon')^2$$

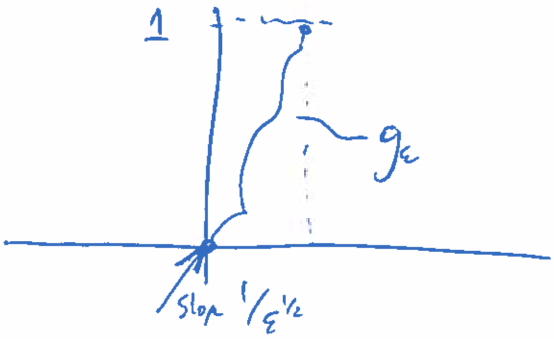
$$\rightsquigarrow \quad \varepsilon (g_\varepsilon')^2 = C + \frac{1}{\varepsilon} W(g_\varepsilon)$$

$$\rightsquigarrow \quad g_\varepsilon' = \frac{\sqrt{C\varepsilon + W(g_\varepsilon)}}{\varepsilon} \qquad (\text{take } C \equiv 1)$$

So we are interested in the ODE.

$$g_\varepsilon' = \frac{\sqrt{\varepsilon + W(g_\varepsilon)}}{\varepsilon} \qquad g_\varepsilon(0) = 0$$

Note $\quad g_\varepsilon'(0) = \dfrac{1}{\varepsilon^{1/2}} >> 1$ for small ε.



We solve the ODE explicitly. Define

$$\phi_\varepsilon(t) = \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}}\, ds .$$

As the integrand is strictly positive, we can invert ϕ_ε on the interval $[0, \phi_\varepsilon(1)]$ to find

$$\frac{d}{dt}\phi_\varepsilon^{-1}(t) = \frac{1}{\frac{d}{dt}\phi_\varepsilon(\phi_\varepsilon^{-1}(t))} = \frac{\sqrt{\varepsilon + W(\phi_\varepsilon^{-1}(t))}}{\varepsilon}$$

Note also $\phi_\varepsilon(1) \leq \int_0^1 \sqrt{\varepsilon} = \varepsilon^{1/2}$.

We define the "optimal" transition by

$$g_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \phi_\varepsilon^{-1}(t) & \text{if } 0 < t < \phi_\varepsilon(1) \\ 1 & \text{if } t > \phi_\varepsilon(1) \end{cases}$$

Step 2: Construction of an optimal Recovery Sequence for Smooth Sets. (8)

Assume $A \subseteq \Omega$ has C^2 boundary and $\underbrace{\mathcal{H}^{N-1}(\partial A \cap \Omega \cap \partial\Omega) = 0}_{\text{transversal intersection.}}$

Then $\exists\ u_\varepsilon \to \chi_A$ such that

$$\overline{\lim_{\varepsilon \to 0}} E_\varepsilon[u_\varepsilon] \leq E_0[\chi_A].$$

Define the signed distance function

$$d_A(x) := \begin{cases} \operatorname{dist}(x, A^c) & \text{in } A \\ -\operatorname{dist}(x, A) & \text{in } A^c \end{cases}$$

Then $|\nabla d_A| = 1$ a.e, and is a Lipschitz function.

Defining the function $g_0(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$

We have that $\chi_A = g_0 \circ d_A$

We would like to approximate χ_A by $g_\varepsilon \circ d_A$, but need to make sure the mass constraint is satisfied.

For this, ~~we draw a picture~~

Note that $g_\varepsilon \leq g_0 \leq g_\varepsilon(\cdot + \phi_\varepsilon(1))$

So $\displaystyle\int_\Omega g_\varepsilon \circ d_A \leq \int_\Omega g_0^{\circ d_A} = m \leq \int g_\varepsilon(d_A(x) + \phi_\varepsilon(1))$

Thus $\exists\ s_\varepsilon \in [0, \phi_\varepsilon(1)]$ such that $u_\varepsilon := g_\varepsilon(d_A(x) + s_\varepsilon)$ has $\int u_\varepsilon = m$.