Mean Curvature Flow (Lecture 2)

We recall from last week that we defined the mean curvature vector as

$$\vec{H}(x) = \Delta d(x) \nabla d(x)$$

and the scalar MC as

$$H(x) = \Delta d(x)$$

and Recall that $d(x) = d(x,E) := \mathrm{dist}(x,E) - \mathrm{dist}(x,E^c)$.

It is of note that $H(x) = \mathrm{div}(\nabla d) = \mathrm{div}(n(x))$, where n is the surface normal.

In fact this may be expressed as the surface divergence. Let $\{\tau_i\}_{i=1}^{N-1}$ (and $\nabla d(x) = \tau_N$) be a basis of $T_{\partial E}(x)$. The surface divergence is denoted by $\mathrm{div}_E = \sum_{i=1}^{N-1} \tau_i \cdot \nabla$ so

$$H(x) = \sum_{i=1}^{N} \tau_i \cdot \nabla(\nabla d) = \mathrm{div}_E(\nabla d) + \nabla d \cdot \nabla^2 d = \mathrm{div}_E(n)$$

Now that we have identified our notion of Mean Curvature, we will want to understand how to express surface velocity. We will do this first with smooth parametrized surfaces, and then put this in terms of the distance function.

(Smooth Flows)

via the distance function: Let $E:[0,T] \to \mathcal{P}(\mathbb{R}^N)$ (or $t \mapsto E(t)$). We say that $t \mapsto E(t)$ is a smooth flow if

- for any $t \in [0,T]$, $E(t)$ is closed
- There is an open neighborhood of $\bigcup_{t\in[0,T]} \partial E(t) \times \{t\} \subseteq \mathbb{R}^N \times [0,T]$ given by $\mathcal{U}$ such that

$$d(z,t) := d(z, E(t)) \in C^\infty(\mathcal{U}).$$

(We will typically assume that $\partial E(t) \subset\subset \mathbb{R}^N$, which is a compact smooth flow)

The normal velocity vector is defined as

$$-\partial_t d(x,t) \nabla d(x,t) \qquad \text{for } x \in \partial E(t)$$

to make sure this is a Reasonable definition consider a surface

We also check consistency with smooth parametrized flows.

Via parametrizations:

Let $S \subseteq \mathbb{R}^N$ be a smooth $(N-1)$-dimensional embedded, oriented, and connected manifold w/o boundary.

We consider $\varphi \in C^\infty(0,T; \mathrm{Emb}(S;\mathbb{R}^N))$, which just means that

- $\varphi(x,t) : S \times [0,T) \to \mathbb{R}^N$ is smooth
- $\varphi(\cdot,t)$ is an embedding for each t $\left(d\varphi(\cdot,t) \text{ injective and } \varphi(S,t) \underset{\text{homeomorphic}}{\sim} S\right)$

(Feel free to think of S as compact).

In most nice cases, these notions of smooth flows are equivalent, in the sense that there is a ~~Let us suppose that~~ parametric description for $\partial E(t)$, i.e., $\varphi(S,t) = \partial E(t)$.

~~The~~ In the case that we may describe the smooth flow via the distance function d or the parametrization φ, we look at the surface velocity.

For a smooth flow with parametrization, we may suppose that $\underset{\text{oriented normal}}{-\vec{n}}(s,t) = \nabla d(\varphi(s,t),t)$

The normal velocity via the parametrization is

$$\vec{V}(s,t) := \langle n(s,t), \partial_t \varphi(s,t)\rangle \, n(s,t)$$

Lemma: For $t \mapsto E(t)$ a smooth flow described via the distance function or parametrization, the notions of velocity coincide:

$$\left[-\partial_t d \, \nabla d(\varphi(s,t),t) = \vec{V}(s,t).\right]$$

Proof:

$$d(\varphi(s,t),t) = 0 \quad \forall\ (s,t) \in S \times [0,T]. \text{ so}$$

$$\langle \nabla d(\varphi(s,t)), \partial_t \varphi(s,t)\rangle = -\partial_t d(\varphi(s,t),t)$$

$$\Rightarrow \langle n(s,t), \partial_t \varphi(s,t)\rangle \, n(s,t) = -\partial_t d(\varphi(s,t),t) \, \nabla d(\varphi(s,t),t)$$

We remark that the difference in notions of velocity is essentially

looking at current configuration $\left[\begin{array}{c}\text{Eulerian} \\ \Updownarrow \\ \text{distance function}\end{array} \quad \text{versus} \quad \begin{array}{c}\text{Lagrangian} \\ \Updownarrow \\ \text{parametrizations}\end{array}\right]$ tracking particles

Finally, we can define Mean curvature flow.

(Mean curvature flow)

We say $t \mapsto E(t)$, a smooth flow, evolves by Mean Curvature flow if

$$\text{Surface Velocity} = (-)\ \text{mean curvature on } \partial E$$

$$-\partial_t d(x,t)\nabla d(x,t) = -\Delta d(x,t)\nabla d(x,t) \quad \forall\ x \in \partial E(t)$$

$$\Updownarrow$$

$$\partial_t d(x,t) = \Delta d(x,t) \quad \forall\ x \in \partial E(t).$$

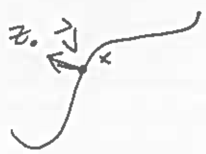
To actually show that there are surfaces evolving by MCF, we will use "typical" PDE methods. Hence, we need to understand what PDE the distance function should solve on a neighborhood of $\left(\bigcup \partial E(t) \times \{t\}\right)$, i.e., on $\mathcal{U}$.

We will often ~~write~~ write $\vec{V}(x,t) = -\partial_t d(x,t)\nabla d(x,t)$ (overwriting our parametric notation)

Lemma: If $t \mapsto E(t)$ is a smooth flow with surface velocity $\vec{V}(x,t)$, then on an open neighborhood $\mathcal{U}$ of $\left(\bigcup_{t\in[0,T]} \partial E(t) \times \{t\}\right)$, we have that

$$\partial_t d(z,t) + \vec{V}(pr(z,t),t)\cdot\nabla d(z,t) \quad \forall\ (z,t) \in \mathcal{U}.$$

Proof: Suppose $z \notin E(t)$ and that $\vec{V}(x,t)$ points in the direction z from $x := pr(z,t)$



Arguing on just the line segment, it is easy to show that

$$\partial_t d(z,t) \le -\vec{V}(x,t)\cdot\nabla d(x,t) = -\vec{V}(x,t)\cdot\nabla d(z,t).$$

Suppose that the inequality is strict, so that there exists an $\eta > 0$ with

$$\partial_t d(z,t) < -\vec{V}(x,t)\cdot\nabla d(x,t) - \eta.$$

$$\Updownarrow$$

$$\frac{d(z,t+\varepsilon) - d(z,t)}{\varepsilon} < -\vec{V}(x,t)\cdot\nabla d(x,t) - \eta. \qquad (A)$$

Let $x_\varepsilon = pr(z,t+\varepsilon)$ be the projection of z onto $\partial E(t+\varepsilon)$ and let $\overline{x}_\varepsilon$ be the projection of x_ε onto $\partial E(t)$, i.e., $\overline{x}_\varepsilon = pr(x_\varepsilon, t)$.

Note by the (Lipschitz, but does not matter) continuity of pr, we have that

$$|x_\varepsilon - x| \text{ and } |\bar{x}_\varepsilon - x| \leq C\varepsilon \quad \text{for some } C \text{ depending on } \partial E.$$

Thus,
$$\underbrace{|x_\varepsilon - \bar{x}_\varepsilon|}_{} \leq \underset{\text{time traveled}}{\varepsilon} \cdot \underbrace{\sup_{\substack{|x_0 - x| \leq C\varepsilon \\ |t_0 - t| \leq \varepsilon}} |\vec{V}(x_0,t_0)|}_{\text{maximum velocity}} \leq \varepsilon\left(|\vec{V}(x,t)| + \eta/2\right)$$
$$= \varepsilon\left(\vec{V}(x,t)\cdot\nabla d(x,t) + \eta/2\right)$$

since we assume $\vec{V}$ points in the direction of z.

Thus, we have

$$|z - \bar{x}_\varepsilon| \leq \overbrace{|z - x_\varepsilon|}^{d(z,t+\varepsilon)} + |x_\varepsilon - \bar{x}_\varepsilon|$$
$$\leq d(z,t) + \varepsilon\left(-\vec{V}(x,t)\cdot\nabla d(x,t) - \eta\right)$$
$$+ \varepsilon\left(\vec{V}(x,t)\cdot\nabla d(x,t) + \eta/2\right)$$
$$= d(z,t) - \varepsilon(\eta/2)$$

Contradicting the definition of $d(z,t)$.

The other cases are proven similarly, potentially reversing the flow in time. $\boxtimes$

In the case that the surface evolves by MCF, we can apply the above Lemma to find that

$$\partial_t d(z,t) = \Delta d(pr(z,t),t) \quad \forall (z,t) \in \mathcal{U}.$$

However, this is not a good PDE to generalize from as it has the nonlinear composition.

<u>Lemma (Expansion of $\nabla^2 d$ in $\mathcal{U}$).</u>

Let ∂E be a smooth surface and $d \in C^\infty(\mathcal{U})$. We let $x := pr(z)$. Let $\{\tau_1, \dots, \tau_{N-1}, \tau_N\}$, $\tau_N = \nabla d(x)$, be an orthonormal basis for which $\nabla^2 d(x)$ diagonalizes: $\nabla^2 d(x)\tau_i = \mathcal{K}_i(x)\tau_i$, $\mathcal{K}_N = 0$.

Then $\{\tau_i\}_{i=1}^N$ also diagonalizes $\nabla^2 d(z)$ and $\nabla^2 d(z)\tau_i = \mu_i(z)\tau_i$

with $\mu_i(z) = \dfrac{\mathcal{K}_i(x)}{1 + d(z)\mathcal{K}_i(x)}$.

Proof: Recall that $\nabla^2 d(z)\nabla d(z) = 0 \quad \forall\, z \in \mathcal{U}$.

Differentiating in the j direction, we have

$$\nabla^2(\partial_j d)\nabla d = -\nabla^2 d\, \nabla(\partial_j d)$$

Component-wise: $$\partial^3_{ijk} d\, \partial_k d = -\partial^2_{ik} d\, \partial^2_{jk} d \quad \forall\, (i,j) \in \{1,\dots,N\}$$

Defining $\Pi(\lambda) := \nabla^2 d(x + \lambda \nabla d(x))$ for $|\lambda| \ll 1$ we have [λ plays the role of $d(z)$]

$$\frac{d}{d\lambda}\Pi_{ij}(\lambda) = \partial^3_{ijk} d(x + \lambda\nabla d(x))\, \partial_k d(x)$$
$$= \partial^3_{ijk} d(x + \lambda\nabla d(x))\, \partial_k d(x + \lambda \nabla d(x))$$
$$= -\partial^2_{ik} d(x + \lambda\nabla d(x))\, \partial^2_{jk} d(x + \lambda \nabla d(x))$$
$$= -(\Pi^2)_{ij}(\lambda)$$

Thus, $\dfrac{d}{d\lambda}\Pi(\lambda) = -\Pi^2(\lambda)$ with $\Pi(0) = \nabla^2 d(x) = \sum_{i=1}^{N-1} \mathcal{K}_i(x)\, \tau_i \otimes \tau_i$

$\Pi(\lambda) := \sum_{i=1}^{N-1} \dfrac{\mathcal{K}_i(x)}{1 + \lambda \mathcal{K}_i(x)}\, \tau_i \otimes \tau_i$ solves the ODE.

As the solution to the ODE is unique, we have

$$\nabla^2 d(z) = \sum_{i=1}^{N-1} \frac{\mathcal{K}_i(x)}{1 + d(z)\mathcal{K}_i(x)}\, \tau_i \otimes \tau_i \quad \text{as desired.}$$

▨

With this we have that

$$\mu_i(z) = \frac{\mathcal{K}_i(x)}{1 + d(z)\mathcal{K}_i(x)} \rightsquigarrow \mathcal{K}_i(x) = \frac{\mu_i(z)}{1 - d(z)\mu_i(z)}$$

But $\dfrac{\mu_i(z)}{1 - d(z)\mu_i(z)}$ are the eigenvalues of $\nabla^2 d(z)\left(\mathbb{I} - d(z)\nabla^2 d(z)\right)^{-1}$. Thus

$$\Delta d(\mathrm{pr}(z)) = \Delta d(x) = \sum \mathcal{K}_i(x) = \mathrm{tr}\left(\nabla^2 d(z)\left(\mathbb{I} - d(z)\nabla^2 d(z)\right)^{-1}\right).$$

Thus if $t \mapsto E(t)$ is a smooth flow

$$\left[\partial_t d = -\vec{V}(x,t)\cdot\nabla d(z,t) = \Delta d(x,t) = \mathrm{tr}\left(\nabla^2 d(z)\left(\mathbb{I} - d(z)\nabla^2 d(z)\right)^{-1}\right)\right]$$