Couple of different approaches:

- Density
- Global method.

Define $\mathcal{E}[u] := \inf\{\underline{\lim}\, E[u_i] : u_i \xrightarrow[L^2]{} u\}$

$$E \xrightarrow[\Gamma]{} E_Q \iff \mathcal{E}[u] = E_Q[u]$$

In other words, the Γ-limit is the LSC relaxation (envelope)

We have shown $\mathcal{E}[u] \geq E_Q[u]$.

Now we show $\mathcal{E}[u] \leq E_Q[u]$.

Argue via density.

Suppose u is affine in $\mathcal{U} \subseteq \Omega$ (open), $\nabla u \equiv \xi$ in $\mathcal{U}$.

We apply the Besicovitch/Morse Covering theorem to find cubes

$$\{Q[x_i, r_i]\} \text{ such that } \mathcal{L}^N\left(\mathcal{U} \setminus \left[\bigcup Q[x_i, r_i]\right]\right) = 0$$

and $Q[x_i, r_i]$ are pairwise disjoint.

Let $\phi \in C_c^1(Q; \mathbb{R}^d)$ be extended periodically.

Define
$$\phi_{i,n} = \begin{cases} \frac{r_i}{n} \phi\left(n\left(\frac{x - x_i}{r_i}\right)\right) & x \in Q[x_i, r_i] \\ 0 & \text{else} \end{cases}$$

$$\phi_n = \sum_{i=1}^{\infty} \phi_{i,n}$$

Note that $u + \phi_n \xrightarrow[L^2]{} u$, thus

$\xi \equiv \nabla u$ in $\mathcal{U}$ (5)

$$\mathcal{E}[u] \leq \varliminf_{n\to\infty} \int_\Omega F(\nabla u_n) = \int_{\Omega\setminus\mathcal{U}} f(\nabla u) + \varliminf_{n\to\infty} \int_{\mathcal{U}} f(\xi + \nabla\phi_n)$$

$$\int_{\mathcal{U}} f(\xi + \nabla\phi_n) = \sum_{i=1}^{\infty} \int_{Q[x_i, r_i]} f\left(\xi + \nabla\phi\left(n\left(\frac{x_i - x}{r_i}\right)\right)\right)$$

$$= \sum_{i=1}^{\infty} r_i^N \int_{Q[0,1]} f(\xi + \nabla\phi(nx))\, dx$$

$$= \sum_{i=1}^{\infty} r_i^N \int_{Q[0,1]} f(\xi + \nabla\phi(x))\, dx$$

$$= |\mathcal{U}| \int_{Q[0,1]} f(\xi + \nabla\phi(x))\, dx.$$

$$\Rightarrow \mathcal{E}[u] \leq \int_{\Omega\setminus\mathcal{U}} f(\nabla u) + |\mathcal{U}| \int_{Q[0,1]} f(\xi + \nabla\phi(x))\, dx.$$

$$\Rightarrow \mathcal{E}[u] \leq \int_{\Omega\setminus\mathcal{U}} f(\nabla u) + \int_{\mathcal{U}} Qf(\nabla u)\, dx$$

Suppose now that $u \in H^1(\Omega; \mathbb{R}^d)$ is piecewise affine.

$\Omega = \bigcup \mathcal{U}_k$, $\nabla u|_{\mathcal{U}_k} \equiv \xi_k$. Can repeat argument in each section simultaneously to find

$$\mathcal{E}[u] \le \sum_{k=1} \int_{U_k} Qf(\nabla u) = \int_\Omega Qf(\nabla u)$$

<u>Finally density</u>: Let $u \in H^1(\Omega;\mathbb{R}^d)$. Then $\exists\ u_i \xrightarrow[H^1]{} u$ for which u_i are piecewise affine. Consequently

$$\mathcal{E}[u] \le \underline{\lim}\, \mathcal{E}[u_i] \le \underline{\lim} \int_\Omega Qf(\nabla u_i) = \int_\Omega Qf(\nabla u) = E_Q[u]$$

Dom. convergence + $\underbrace{\text{continuity of } Qf}_{\text{assumed}}$

A comment on density

To show that piecewise affine are dense in $H^1(\Omega;\mathbb{R}^d)$, it suffices to show they are dense in $C^1(\Omega;\mathbb{R}^d)$

- Need to construct a "fine" simplex covering